

# Analogs of Gröbner Bases in Polynomial Rings over a Ring

J. Lyn Miller

*Western Kentucky University*

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In this paper we will define analogs of Gröbner bases for  $R$ -subalgebras and their ideals in a polynomial ring  $R[x_1, \dots, x_n]$  where  $R$  is a noetherian integral domain with multiplicative identity and in which we can determine ideal membership and compute syzygies. The main goal is to present and verify algorithms for constructing these Gröbner basis counterparts. As an application, we will produce a method for computing generators for the first syzygy module of a subset of an  $R$ -subalgebra of  $R[x_1, \dots, x_n]$  where each coordinate of each syzygy must be an element of the subalgebra.

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## 1. Introduction

The concept of Gröbner bases for ideals of a polynomial ring over a field  $k$  can be adapted in a natural way for  $k$ -subalgebras of such a polynomial ring. Robbiano and Sweedler (refer to (RS); see also (KM)) defined a SAGBI basis<sup>†</sup> for a  $k$ -subalgebra  $A$  of  $k[x_1, \dots, x_n]$  to be a subset  $F \subseteq A$  whose leading power products generate the multiplicative monoid of leading power products of  $A$ . The properties and applications of SAGBI bases strongly imitate many of the standard Gröbner basis results when a suitable accompanying reduction algorithm is defined. Sweedler (see (IVR)) went on to extend the theory of Gröbner bases in a way that can be used to define them for ideals of  $k$ -subalgebras of  $k[x_1, \dots, x_n]$ ; this was briefly presented more explicitly by Ollivier (see (Oll)). Based on their work, we define a SAGBI-Gröbner basis for an ideal  $I$  of a  $k$ -subalgebra  $A \subseteq k[x_1, \dots, x_n]$  to be a subset  $G \subseteq I$  whose leading power products generate the monoid-ideal consisting of the leading power products of  $I$  in the monoid of those of  $A$ . Basic properties and applications of SAGBI-Gröbner bases are again straight-forward adaptations of the usual Gröbner basis theory. (See also (Mil).)

Our aim in this paper is to extend the theory of SAGBI and SAGBI-Gröbner bases to the context of a polynomial ring over a noetherian integral domain  $R$  in which we can determine ideal membership and compute syzygies. As we know from the study of this same extension process for Gröbner bases, the leading coefficients of the polynomials now play a large role. The definitions, results, and especially techniques in this new setting are no longer such carbon copies of those for Gröbner bases, although we always attempt to parallel them as much as possible. In particular, the definition of a SAGBI basis

<sup>†</sup> The name SAGBI is an acronym standing for Subalgebra Analog to Gröbner Bases for Ideals.

in  $R[x_1, \dots, x_n]$  must now allow for addition of leading terms, not just multiplication. Therefore, the monoid of leading power products used for SAGBI bases in  $k[x_1, \dots, x_n]$  must be exchanged for a much larger structure, namely, the  $R$ -subalgebra that it generates in  $R[x_1, \dots, x_n]$ . Likewise, for SAGBI-Gröbner bases in  $R[x_1, \dots, x_n]$ , the monoid-ideal in the definition over  $k[x_1, \dots, x_n]$  must be enlarged to an ideal of the new  $R$ -subalgebra just mentioned.

The main goals of this paper are to present and verify algorithms for constructing SAGBI and SAGBI-Gröbner bases in  $R[x_1, \dots, x_n]$ , as well as outlining some of their basic properties. As an application, we will also present a method for computing generators for the first syzygy module of a subset of an  $R$ -subalgebra of  $R[x_1, \dots, x_n]$  where each coordinate of each syzygy must be an element of the subalgebra.

## 2. Notation

Our context is the polynomial ring  $R[x_1, \dots, x_n]$  in  $n$  variables, where  $R$  is a noetherian integral domain in which we can determine ideal membership and compute syzygies. (When the coefficient ring is a field, we use the symbol  $k$  instead of  $R$ .) We abbreviate this polynomial ring as  $R[X]$ . The notation  $R[S]$  stands for the  $R$ -subalgebra generated by the subset  $S \subseteq R[X]$ . Throughout this paper,  $A$  is an  $R$ -subalgebra of  $R[X]$ .

The symbol  $\mathbf{N}$  represents the non-negative integers, and  $\mathbf{T}_X$  represents the set of all power products  $\prod_{i=1}^n x_i^{\beta_i}$  with  $\beta_i \in \mathbf{N}$  of the variables  $x_1, \dots, x_n$ . We will often abbreviate such a power product as  $X^{\vec{\beta}}$  where  $\vec{\beta}$  is the exponent vector  $(\beta_1, \dots, \beta_n)$ . More generally, we have

**DEFINITION 2.1.** Let  $S \subseteq R[X]$ . An  $S$ -power product is a (finite) product of the form  $s_1^{e_1} \cdots s_m^{e_m}$  where  $s_i \in S$  and  $e_i \in \mathbf{N}$  for  $1 \leq i \leq m$ . We usually write this simply as  $S^{\vec{e}}$ , where  $\vec{e}$  represents that vector in  $\oplus_S \mathbf{N}$  whose coordinates are all 0 except for  $e_1, \dots, e_m$  in the positions corresponding to  $s_1, \dots, s_m$ .

**DEFINITION 2.2.** Given a term order on  $R[X]$ ,  $p \in R[X]$ , and  $S \subseteq R[X]$ , we define

$$\begin{aligned} \text{lp}(p) &= \text{the leading } X\text{-power product of } p \\ \text{lc}(p) &= \text{the leading coefficient of } p \\ \text{lt}(p) &= \text{lc}(p)\text{lp}(p) = \text{the leading term of } p \\ \text{Lp}S &= \{\text{lp}(s) : s \in S\} \end{aligned}$$

while  $\text{Lc}S$  and  $\text{Lt}S$  are similarly defined. We also establish the convention that  $\text{lp}(0)$  is undefined while  $\text{lc}(0)$  and  $\text{lt}(0)$  are 0.

We borrow the following terminology from (RS).

**DEFINITION 2.3.** Let  $S \subseteq R[X]$ . Given an expression  $\sum_{i=1}^N r_i s_i$  with  $r_i \in R$  and  $s_i \in S$ , we define its height, written  $\text{ht}(\sum_{i=1}^N r_i s_i)$ , to be  $\max_i \text{lp}(s_i)$ . Moreover, we say that  $s_{i_0}$  contributes to the height of the expression if  $\text{lp}(s_{i_0}) = \max_i \text{lp}(s_i)$ .

We emphasize that the height is defined only for specific representations of an element of  $R[X]$ , not for that element itself. Finally, we establish the following notation:

DEFINITION 2.4. For an  $R$ -subalgebra  $A \subseteq R[X]$  and a subset  $S \subseteq A$ ,

- 1  $\langle S \rangle_A$  represents the ideal of  $A$  generated by  $S$ , omitting the subscript when  $A$  is obvious.
- 2  $\text{Syz}_A(S) = \{\vec{a} = (a_s)_{s \in S} \in \oplus_S A : \sum_{s \in S} a_s s = 0\}$ , the  $A$ -module of syzygies of  $S$  whose coordinates all belong to  $A$ . We call an element of  $\text{Syz}_A(S)$  an  $A$ -syzygy of  $S$ .
- 3 If  $A$  is a graded algebra, and  $\deg(a)$  represents the degree of  $a \in A$ , then  $\text{Syz}_A^*(S) = \{\vec{a} \in \text{Syz}_A(S) : \deg(a_s s) \text{ is the same } \forall a_s s \neq 0\}$ . This common value of  $\deg(a_s s)$  is called the degree of the syzygy, and we write it as  $\deg(\vec{a})$ . The elements of  $\text{Syz}_A^*(S)$  are called homogeneous  $A$ -syzygies of  $S$ .

The subscripts in  $\text{Syz}_A(S)$  and  $\text{Syz}_A^*(S)$  will be omitted when  $A$  is obvious.

### 3. SAGBI Bases in $R[X]$

Our first goal is to define a SAGBI basis and present an algorithm for its construction.

DEFINITION 3.1. Let  $A$  be an  $R$ -subalgebra of  $R[X]$ . We say that  $F \subseteq A$  is a SAGBI basis for  $A$  if  $\text{Lt}F$  generates the  $R$ -subalgebra  $R[\text{Lt}A]$ , i.e.  $R[\text{Lt}A] = R[\text{Lt}F]$ .

We consider an operation which parallels the reduction algorithm used in Gröbner basis theory.

DEFINITION 3.2. Let  $g \in R[X]$ , and let  $F \subseteq R[X]$ . We will say that  $g$  s-reduces to  $h$  via  $F$  in one step, written  $g \xrightarrow{F} h$ , if there exist a non-zero term  $cX^{\vec{\beta}}$  of  $g$  and  $F$ -power products  $F^{\vec{e}_1}, \dots, F^{\vec{e}_N}$  such that

- 1  $\text{lp}(F^{\vec{e}_i}) = X^{\vec{\beta}}$  for  $1 \leq i \leq N$ .
- 2  $c = \sum_{i=1}^N r_i \text{lc}(F^{\vec{e}_i})$  where  $r_i \in R$  for  $1 \leq i \leq N$ .
- 3  $h = g - \sum_{i=1}^N r_i F^{\vec{e}_i}$ .

We also write  $g \xrightarrow{F} h$  if there is a finite chain of 1-step s-reductions leading from  $g$  to  $h$ ; we say that  $g$  s-reduces to  $h$  via  $F$  in this case. If  $h$  cannot be further s-reduced via  $F$ , then we call it a final s-reductum of  $g$ .

It is obvious that if  $g \xrightarrow{F} h$ , then  $g - h \in R[F]$ . Well-ordering of  $\mathbf{T}_X$  implies that any chain of 1-step s-reductions must terminate.

To s-reduce  $g \in R[X]$  via a finite set  $F$  requires us to do two things at each step. After we have chosen the term  $cX^{\vec{\beta}}$  of  $g$  that we wish to eliminate, we must be able to tell

- 1 whether  $X^{\vec{\beta}}$  lies in the multiplicative monoid generated by  $\text{Lp}F$ , and
- 2 whether  $c$  belongs to the ideal of  $R$  generated by  $\{\text{lc}(F^{\vec{e}}) : \text{lp}(F^{\vec{e}}) = X^{\vec{\beta}}\}$ .

To address the first issue, we need to solve the inhomogeneous linear diophantine system arising from the exponents of the variables in  $X^{\vec{\beta}} = \text{lp}(F^{\vec{e}})$  for  $\vec{e} \in \oplus_F \mathbf{N}$ .<sup>†</sup> To address the second point simply requires that we determine ideal membership in  $R$ , which we have assumed is possible.

By a standard proof, we can also show

PROPOSITION 3.3. *The following are equivalent for  $F \subseteq A$ :*

- 1  $F$  is a SAGBI basis for  $A$
- 2 For every  $a \neq 0 \in A$ , the final  $s$ -reductum of  $a$  via  $F$  is always 0.
- 3 Every  $a \in A$  has a SAGBI representation with respect to  $F$ , that is, a representation

$$a = \sum_{i=1}^N r_i F^{\vec{e}_i}, \quad r_i \in R$$

such that  $\max_i \text{lp}(F^{\vec{e}_i}) = \text{lp}(a)$ .

COROLLARY 3.4. *A SAGBI basis for  $A$  generates  $A$  as an  $R$ -subalgebra .*

COROLLARY 3.5. *Suppose  $F$  is a SAGBI basis for  $A$ . An element  $p \in R[X]$  belongs to  $A \iff p \xrightarrow{F} 0$ .*

Now we write  $A = R[F]$ , where  $F = \{f_1, f_2, \dots\}$  is not necessarily finite. To design an algorithm for constructing a SAGBI basis for  $A$ , we intend to determine a collection of polynomials related to  $F$  such that if each of these polynomials  $s$ -reduces to 0 via  $F$ , then  $F$  is a SAGBI basis. These polynomials mimic the  $S$ -polynomials of ordinary Gröbner basis theory, and this desired property will be the basis of our construction algorithm.

Represent  $A = R[F]$  as the homomorphic image of a polynomial ring  $R[Y]$  (where the cardinality of  $Y = \{y_1, y_2, \dots\}$  is the same as that of  $F$ ) via the usual evaluation homomorphism sending each  $y_i \mapsto f_i$ . We will now equip  $R[Y]$  with a graded  $R$ -module structure (which may not be based on any term order in  $R[Y]$ ). Given  $P(Y) \in R[Y]$ , we define

$$\deg P(Y) = \max\{\text{lp}(F^{\vec{a}}) : Y^{\vec{a}} \text{ occurs in } P(Y)\}.$$
<sup>†</sup>

It is easy to check that this degree map from  $R[Y] \rightarrow \mathbf{T}_X$  truly does give a grading on  $R[Y]$ . Notice that the homogeneous elements with respect to this presumed grading will be those polynomials  $P(Y)$  whose terms give rise to  $F$ -monomials all having the same leading  $X$ -power product.

Now define an evaluation map  $\pi : R[Y] \rightarrow R[\text{Lt}F]$  via  $y_i \mapsto \text{lt}(f_i)$ . The ideal  $I(\text{Lt}F) = \{P(Y) : \pi(P(Y)) = P(\text{Lt}F) = 0\}$  is homogeneous with respect to the  $\mathbf{T}_X$ -grading on  $R[Y]$ . Its homogeneous generators take the place in our current theory of the usual  $S$ -polynomials. Recall that such generators may be computed using the familiar tag variable technique of ordinary Gröbner basis theory. (Refer to (AL) et al.)

<sup>†</sup> Refer to (Dach) for a subroutine that can determine such solutions.

<sup>†</sup> It is not necessarily true that  $\deg P(Y) = \text{lp}(P(F))$ . For example, if  $F = \{f_1, f_2\} = \{x^2, x^2 + 1\} \subseteq R[x]$ , then  $\deg(y_2 - y_1) = x^2$ , whereas  $\text{lp}(f_2 - f_1) = 1$ .

We are now in a position to state and prove the main result of this section.

**THEOREM 3.6.** *Let  $F \subseteq R[X]$  have distinct elements, and let  $\{P_j(Y) : j \in J\}$  be a set of  $\mathbf{T}_X$ -homogeneous generators for  $I(\text{Lt}F) \subseteq R[Y]$ .  $F$  is a SAGBI basis for  $R[F] \iff$  for each  $j \in J$ ,  $P_j(F) \xrightarrow{F} 0$ .*

**Proof.**  $\implies$ : This direction is a trivial corollary of Proposition 3.3.

$\impliedby$ : Let  $h \in R[F]$ . We will show that  $\text{lt}(h) = \sum_i r_i \text{lt}(F^{\bar{e}_i}) \in R[\text{Lt}F]$ , which will fulfill Definition 3.1.

Write  $h = \sum_{i=1}^m c_i F^{\bar{e}_i}$ ; furthermore, we will assume that this representation has the smallest possible height  $t_0 = \max_i \text{lp}(F^{\bar{e}_i})$  of all such representations. We know that  $\text{lp}(h) \leq t_0$ . Suppose that  $\text{lp}(h) < t_0$ ; without loss of generality, let the first  $N$  summands be the ones for which  $\text{lp}(F^{\bar{e}_i}) = t_0$ . Then cancellation of their leading  $X$ -power products must occur; i.e.  $\sum_{i=1}^N c_i \text{lt}(F^{\bar{e}_i}) = 0$ . Hence, we obtain an element  $P(Y) = \sum_{i=1}^N c_i Y^{\bar{e}_i} \in I(\text{Lt}F)$ . We can then write

$$\sum_{i=1}^N c_i Y^{\bar{e}_i} = P(Y) = \sum_{j=1}^M g_j(Y) P_j(Y) \quad (3.1)$$

where the elements  $P_j(Y)$  are the stated generators of  $I(\text{Lt}F)$  and the polynomials  $g_j(Y) \in R[Y]$ . Moreover, we may assume that each  $g_j(Y)$  is  $\mathbf{T}_X$ -homogeneous (since  $P(Y)$  and every  $P_j(Y)$  are) and also that  $\deg[g_j(Y)P_j(Y)] = \deg P(Y) = t_0$  for  $1 \leq j \leq M$ .

We have assumed that each  $P_j(F) \xrightarrow{F} 0$ ; therefore, we have SAGBI representations  $P_j(F) = \sum_{k=1}^{n_{kj}} c_{kj} F^{\bar{e}_{kj}}$ . By definition, these sums must have heights  $\max_k \text{lp}(F^{\bar{e}_{kj}}) = \text{lp}(P_j(F)) < \deg P_j(Y)$  for each  $j$ , where the last inequality holds because  $P_j(Y) \in I(\text{Lt}F)$ , so that the highest  $X$ -terms in  $P_j(F)$  cancel. Then for each  $j$ ,  $1 \leq j \leq M$ ,

$$g_j(F)P_j(F) = \sum_{k=1}^{n_j} c_{kj} g_j(F) F^{\bar{e}_{kj}} \quad (3.2)$$

Define  $t_j$  to be the height of the right-hand sum in Equation (3.2), and observe that

$$t_j \leq \deg g_j(Y) \cdot \max_k \text{lp}(F^{\bar{e}_{kj}}) < \deg g_j(Y) \cdot \deg P_j(Y) = t_0.$$

Note that it is impossible for  $F^{\bar{e}_1}$  to occur in the right-hand sum of Equation (3.2) since  $t_0 = \text{lp}(F^{\bar{e}_1})$ .

Returning to our representation in Equation (3.1), we define  $d_j$  to be the coefficient of  $Y^{\bar{e}_1}$  in  $g_j(Y)P_j(Y)$  and assume that  $d_j \neq 0$  for  $1 \leq j \leq M_1$ ,  $d_j = 0$  for  $j > M_1$ . Furthermore, let us define  $U_j(Y) = g_j(Y)P_j(Y) - d_j Y^{\bar{e}_1}$ ; we solve this equation for  $d_j Y^{\bar{e}_1}$ , apply the evaluation map  $R[Y] \leftarrow R[F]$ , and substitute using Equation (3.1) to obtain

$$d_j F^{\bar{e}_1} = -U_j(F) + \sum_{k=1}^{n_j} c_{kj} g_j(F) F^{\bar{e}_{kj}}, \quad 1 \leq j \leq M.$$

Observe that  $F^{\bar{e}_1}$  may not occur on the right-hand side of the equation: it did not appear on the right-hand-side of Equation (3.1), and  $U_j(Y)$  contains no term involving  $Y^{\bar{e}_1}$ , whence  $U_j(F)$  contains no term involving  $F^{\bar{e}_1}$  (This last statement requires our assumption that the members of  $F$  are distinct.)

Our definition of  $d_j$  implies that  $c_1 = \sum_{j=1}^M d_j$ . Therefore,

$$c_1 F^{\vec{e}_1} = \sum_{j=1}^M d_j F^{\vec{e}_1} = \sum_{j=1}^M \left[ -U_j(F) + \sum_{k=1}^{n_j} c_{kj} g_j(F) F^{\vec{e}_{kj}} \right].$$

We can now replace  $c_1 F^{\vec{e}_1}$  by this sum in the original expression for our polynomial  $h$  to get

$$\begin{aligned} h &= \sum_{j=1}^M \left[ -U_j(F) + \sum_{k=1}^{n_j} c_{kj} g_j(F) F^{\vec{e}_{kj}} \right] + \sum_{i=2}^m c_i F^{\vec{e}_i} \\ &= \sum_{j=1}^M [-g_j(F) P_j(F) + d_j F^{\vec{e}_1}] + \sum_{j=1}^M \left[ \sum_{k=1}^{n_j} c_{kj} g_j(F) F^{\vec{e}_{kj}} \right] + \sum_{i=2}^m c_i F^{\vec{e}_i} \\ &= \sum_{i=1}^N [-c_i F^{\vec{e}_i}] + c_1 F^{\vec{e}_1} + \sum_{j=1}^M \left[ \sum_{k=1}^{n_j} c_{kj} g_j(F) F^{\vec{e}_{kj}} \right] + \sum_{i=2}^m c_i F^{\vec{e}_i} \\ &= \sum_{i=N+1}^m c_i F^{\vec{e}_i} + \sum_{j=1}^M \left[ \sum_{k=1}^{n_j} c_{kj} g_j(F) F^{\vec{e}_{kj}} \right]. \end{aligned}$$

If we examine this final expression closely, we see that its height is strictly less than that of our initial representation for  $h$ , for

- 1 The height of  $\sum_{i=N+1}^m c_i F^{\vec{e}_i}$  is strictly less than the old maximum,  $t_0$ , by choice of  $N$ .
- 2 We have already seen that the height of the second sum, which is the maximum of the  $t_j$  we worked with above, is strictly less than  $t_0$ .

But this contradicts our initial assumption that we had chosen a representation for  $h$  that had the smallest possible height. Thus,  $F$  is a SAGBI basis for  $R[F]$ .  $\square$

We may now present an algorithm for computing SAGBI bases. See Algorithm 3.1.

Theoretically, Algorithm 3.1 can be used with an infinite input set  $F$  because all our results so far have been carefully designed not to require any finiteness conditions. Thus, if we assume that we can find generators for  $I(\text{Lt}F)$  when  $F$  is infinite (which may be quite a stretch of imagination!), we shall see that it makes sense to apply the algorithm to any size input set. To this end, we validate that the algorithm produces a SAGBI basis, although it **need not terminate**, even with finite input. (See (RS) for a discussion of infinite SAGBI bases in  $k[X]$ .)

**PROPOSITION 3.7.** *Let  $H_\infty = \cup H$  over all passes of the WHILE loop. Then  $H_\infty$  is a SAGBI basis for  $R[F]$ . Moreover, if  $F$  is finite and  $R[F]$  has a finite SAGBI basis, then Algorithm 3.1 terminates and produces a finite SAGBI basis for  $R[F]$ .*

**Proof.** Set  $\mathcal{P}_\infty = \cup \mathcal{P}$  over all passes of the loop, and let  $Y_\infty$  be a set of variables  $y_i$ , one for each element  $h_i \in H_\infty$ . We will show that  $\mathcal{P}_\infty$  is a set of  $\mathbf{T}_X$ -homogeneous generators for  $I(\text{Lt}H_\infty) \subseteq R[Y_\infty]$ , and then that each element of  $\mathcal{P}_\infty$  s-reduces to 0 via  $H_\infty$ .

$\mathcal{P}_\infty$  is  $\mathbf{T}_X$ -homogeneous since each  $\mathcal{P}$  of each loop is. Now choose  $P(Y_\infty) \in I(\text{Lt}H_\infty)$ . Since only finitely many  $y_i$  can occur in  $P(Y_\infty)$ , only finitely many  $h_i \in H_\infty$  occur

**INPUT:**  $F$

**OUTPUT:** A SAGBI basis for  $R[F]$

**INITIALIZATION:**  $H := F$ ,  $oldH := \emptyset$

**WHILE**  $H \neq oldH$  **DO**

$Y := \{y_i : h_i \in H\}$ , a set of variables

Choose a  $\mathbf{T}_X$ -homog. generating set  $\mathcal{P}$  for  $I(\text{Lt}H)$  in  $R[Y]$ .

$red\mathcal{P} := \{\text{final s-reducta via } H \text{ of } P(H) : P(Y) \in \mathcal{P}\} - \{0\}$

$oldH := H$

$H := H \cup red\mathcal{P}$

ALGORITHM 3.1. : SAGBI Basis Construction Algorithm

in  $P(H_\infty)$ . The sets  $H$  are nested, so these particular  $h_i$ 's must all belong to the set  $H = H_{N_0}$  produced by the end of some finite number  $N_0$  of loops. Let  $\mathcal{P}_{N_0}$  denote the generating set for  $I(\text{Lt}H_{N_0})$ , and let  $Y_{N_0} \subseteq Y_\infty$  be the subset of variables corresponding to  $H_{N_0}$ . Then  $P(Y_\infty) \in I(\text{Lt}H_{N_0}) = \langle \mathcal{P}_{N_0} \rangle_{R[Y_{N_0}]} \subseteq \langle \mathcal{P}_\infty \rangle_{R[Y_\infty]}$ . Hence,  $I(\text{Lt}H_\infty) \subseteq \langle \mathcal{P}_\infty \rangle$ . Conversely, each element  $P(Y_\infty)$  of  $\mathcal{P}_\infty$  belongs to the set  $\mathcal{P}_{N_0}$  of some pass of the WHILE loop; whence,  $P(Y_\infty) \in I(\text{Lt}H_{N_0}) \subseteq I(\text{Lt}H_\infty)$ . Thus,  $\mathcal{P}_\infty \subseteq I(\text{Lt}H_\infty)$ , and  $\langle \mathcal{P}_\infty \rangle = I(\text{Lt}H_\infty)$ .

We have just pointed out that if  $P(Y_\infty) \in \mathcal{P}_\infty$ , then we may assume that  $P(Y_\infty) \in I(\text{Lt}H_{N_0})$  for some pass, in this case the  $N_0$ -th, of the loop. Clearly, either  $P(Y_\infty) \xrightarrow{H_{N_0}} 0$  or  $P(Y_\infty) \xrightarrow{H_{N_0+1}} 0$ . In either case, we have  $P(Y_\infty) \xrightarrow{H_\infty} 0$ . Thus, by Theorem 3.6,  $H_\infty$  is a SAGBI basis for  $R[H_\infty] = R[F]$ .

Now suppose that  $R[F]$  has a finite SAGBI basis  $S$ . Because  $H_\infty$  is also a SAGBI basis, for each  $s \in S$ , we have an expression

$$\text{lt}(s) = \sum_{j=1}^{M_s} r_{j,s} \text{lt}(H_\infty^{\vec{e}_{j,s}}), \quad r_{j,s} \in R.$$

The finite set  $\tilde{H}$  of those elements of  $H_\infty$  for which the corresponding coordinate of some  $\vec{e}_{j,s}$  is non-zero is a SAGBI basis as well since  $R[\text{Lt}\tilde{H}] = R[\text{Lt}S] = R[\text{Lt}R[F]]$ . The set  $\tilde{H}$  must be a subset of the set  $H = H_{N_0}$  produced at the end of some finite number  $N_0$  of

loops, so that  $H_{N_0}$  is also a SAGBI basis for  $R[F]$ , and by Theorem 3.6, we know that the algorithm will terminate after the next loop.

It remains to show that  $H_{N_0}$  is finite. Any loop that begins with a finite input set (as does the very first loop, by assumption on  $F$ ) will create a finite associated variable set  $Y$ . Then the Hilbert Basis Theorem applies to  $R[Y]$  to prove that we can choose the generating set  $\mathcal{P}$  to be finite as well. Hence, the output of that pass of the loop must be finite. Thus, beginning with a finite set  $F$ , Algorithm 3.1 completes a strictly finite number of loops, each of which yields finite output, and we conclude that  $H_{N_0}$  is indeed a finite SAGBI basis for  $R[F]$ .  $\square$

EXAMPLE 3.8. In this example we will compute a SAGBI basis for  $\mathbf{Z}[F] \subseteq \mathbf{Z}[x, y]$  where

$$F = \{f_1 = 4x^2y^2 + 2xy^3 + 3xy, \quad f_2 = 2x^2 + xy, \quad f_3 = 2y^2\}.$$

We use the term order degree lex with  $x > y$ .

Set  $H = F$ . It is evident that the ideal of relations  $I(\text{Lt}H) = I(4x^2y^2, 2x^2, 2y^2)$  in  $\mathbf{Z}[Y_1, Y_2, Y_3]$  is generated by  $P(Y) = Y_1 - Y_2Y_3$ . The polynomial  $P(H) = 3xy$  cannot be s-reduced via  $H$ , so that  $\text{red}\mathcal{P} = \{3xy\}$ . This forces a second pass through the WHILE loop with an additional member  $f_4 = 3xy \in H$ .<sup>†</sup>

On the second pass through the WHILE loop, we calculate generators for the new ideal  $I(\text{Lt}H) \subseteq \mathbf{Z}[Y_1, Y_2, Y_3, Y_4]$ , obtaining the set

$$\mathcal{P} = \{P_1 = Y_1 - Y_2Y_3, \quad P_2 = 9Y_1 - 4Y_4^2, \quad P_3 = 9Y_2Y_3 - 4Y_4^2\}.$$

One can check that  $P_j(H) \xrightarrow{H} 0$  for  $j = 1, 2, 3$ . Thus, the set  $\text{red}\mathcal{P}$  of non-zero s-reducta of  $\mathcal{P}$  is empty, terminating the algorithm. Our SAGBI basis is  $\{4x^2y^2 + 2xy^3 + 3xy, 2x^2 + xy, 2y^2, 3xy\}$ .  $\triangle$

#### 4. SAGBI-Gröbner Bases in $R[X]$

We next address the topic of SAGBI-Gröbner bases in  $R[X]$  and begin by defining the primary objects of study. Then we present an algorithm for their construction. As always,  $A$  is an  $R$ -subalgebra of  $R[X]$ .

DEFINITION 4.1. Let  $I \subseteq A$  be an ideal of  $A$ . A subset  $G \subseteq I$  is a SAGBI-Gröbner basis (SG-basis) for  $I$  if  $\text{Lt}G$  generates  $\langle \text{Lt}I \rangle$  in  $R[\text{Lt}A]$ .

Recall that in ordinary Gröbner basis theory every ideal is assured to have a finite Gröbner basis, due to the Hilbert Basis Theorem. By the same reasoning, we may draw this conclusion about SG-bases for ideals of  $A$  provided that  $A$  has a finite SAGBI basis.

We continue by describing an appropriate reduction theory for the current context.

DEFINITION 4.2. Let  $G \subseteq A, h \in A$ . We say that  $h$  si-reduces to  $h'$  via  $G$  in one step,

<sup>†</sup> The reader may notice that  $f_4$  is actually an s-reductum of  $f_1$  via  $\{f_2, f_3\}$  and that we could therefore have replaced  $f_1$  by  $f_4$  before beginning the computations at all. Such inter-reduction and replacement may well make the algorithm more efficient. However, a serious analysis of improvements is outside the scope of this exposition. The present example is intended merely to illustrate our basic algorithm, so we will avoid introducing any extra techniques at this juncture, tempting and helpful though it may be.

written  $h \xrightarrow{G}_{\text{si}} h'$ , if there exist a non-zero term  $cX^{\vec{\alpha}}$  of  $h$  and elements  $g_1, \dots, g_M \in G$  and  $a_1, \dots, a_M \in A$  for which the following hold:

- 1  $X^{\vec{\alpha}} = \text{lp}(a_i g_i)$  for each  $i$ .
- 2  $cX^{\vec{\alpha}} = \sum_{i=1}^M \text{lt}(a_i g_i)$ .
- 3  $h' = h - \sum_{i=1}^M a_i g_i$ .

We say that  $h$  si-reduces to  $h'$  via  $G$  and again write  $h \xrightarrow{G}_{\text{si}} h'$  if there is a chain of one-step si-reductions as above leading from  $h$  to  $h'$ . If  $h'$  cannot be si-reduced via  $G$ , we call it a final si-reductum of  $h$ .

We point out that  $h \xrightarrow{G}_{\text{si}} h'$  implies that  $h - h' \in \langle G \rangle_A$ . Again, well-ordering of  $\mathbf{T}_X$  implies that every  $h \in A$  must have a final si-reductum via a subset  $G$ ; that is, si-reduction always terminates.

To perform si-reduction, given a term  $cX^{\vec{\alpha}}$  of  $h$ , we must determine

- 1 for each  $g \in G$ , whether  $X^{\vec{\alpha}} = \text{lp}(g)\text{lp}(a)$  for some  $a \in A$ , that is, whether  $X^{\vec{\alpha}} \in \langle \text{lp}(g) \rangle_{\text{Lp}A}$ , and
- 2 whether  $c$  can be expressed as an  $\text{Lc}A$ -linear combination of the appropriate  $\text{lc}(g)$ 's. (This is equivalent to Condition 2 of Definition 4.2 under the homogeneity of Condition 1.)

Given a SAGBI basis  $F$  for  $A$ , answering the monoid-ideal membership question posed first amounts to searching for solutions  $\vec{\eta} \in \oplus_F \mathbf{N}$  to the equation

$$X^{\vec{\alpha}} = \text{lp}(g)\text{lp}(F^{\vec{\eta}})$$

for each  $g \in G$ , which may be converted to an inhomogeneous linear diophantine system in its exponents. We can then check the desired property for the coefficient  $c$ , by our assumption that ideal membership in  $R$  can be determined.

The proofs of the next result and its corollaries again proceed in the standard way.

**PROPOSITION 4.3.** *The following are equivalent for a subset  $G$  of an ideal  $I \subseteq A$ :*

- 1  $G$  is an  $SG$ -basis for  $I$ .
- 2 For every  $h \in I$ , every final si-reductum of  $h$  via  $G$  is 0.
- 3 Every  $h \in I$  has what is called an  $SG$ -representation with respect to  $G$ , that is, a representation

$$h = \sum_{i=1}^M a_i g_i, \quad a_i \in A, g_i \in G$$

such that  $\max_i \text{lp}(a_i g_i) = \text{lp}(h)$ .

**COROLLARY 4.4.** *An  $SG$ -basis for  $I$  generates  $I$  as an ideal of  $A$ .*

**COROLLARY 4.5.** *Suppose that  $G$  is an  $SG$ -basis for  $I \subseteq A$ . Then  $a \in A$  belongs to  $I$   $\iff a \xrightarrow{G}_{\text{si}} 0$ .*

We introduce some basic terminology.

DEFINITION 4.6. For a vector  $\vec{a} \in \oplus_G A$  whose coordinates are denoted by  $a_g$ , we write  $\vec{\text{lt}}(\vec{a})$  for the vector in  $\oplus_G \text{Lt}A$  whose  $g$ -th coordinate is  $\text{lt}(a_g)$ .

DEFINITION 4.7.  $\text{LtSyz}_A^*(G) = \{\vec{a} \in \oplus_G A : \vec{\text{lt}}(\vec{a}) \in \text{Syz}^*(\text{Lt}G) \subseteq \oplus_G R[\text{Lt}A]\}$ . An element of  $\text{LtSyz}_A^*(G)$  is called a homogeneous  $A$ -lt-syzygy for  $G$ .

DEFINITION 4.8. We call  $\mathcal{Q} \subseteq \text{LtSyz}_A^*(G)$  an lt-generating set for  $\text{LtSyz}_A^*(G)$  if  $\{\vec{\text{lt}}(\vec{Q}) : \vec{Q} \in \mathcal{Q}\}$  is a generating set for  $\text{Syz}^*(\text{Lt}G)$ .

For the remainder of this section we assume that  $A$  has a finite SAGBI basis, and that  $G = \{g_1, \dots, g_M\} \subseteq A$  is finite as well; this assures computability. Given an lt-generating set  $\mathcal{Q}$  and writing its elements as  $\vec{Q}_j = (q_{j,1}, \dots, q_{j,M})$ , we shall see that the polynomials  $\sum_{i=1}^M q_{j,i}g_i$  take the place of S-polynomials in our present setting.

THEOREM 4.9. *Let  $G = \{g_1, \dots, g_M\} \subseteq A$ ; let  $\mathcal{Q}$  be an lt-generating set for  $\text{LtSyz}_A^*(G)$ . Then  $G$  is an SG-basis for  $\langle G \rangle_A \iff$  for each  $\vec{Q}_j = (q_{j,1}, \dots, q_{j,M}) \in \mathcal{Q}$ , we have  $\sum_{i=1}^M q_{j,i}g_i \xrightarrow{G}_{\text{si}} 0$ .*

**Proof.**  $\implies$ : The result is a direct consequence of Proposition 4.3.

$\impliedby$ : Let  $h \in \langle G \rangle_A$ ; write  $h = \sum_{i=1}^m a_i g_i$  such that the height  $t_0 = \max_i \text{lp}(a_i g_i)$  of this representation is minimal with respect to all such representations for  $h$ . Now  $\text{lp}(h) \leq t_0$ ; suppose that  $\text{lp}(h) < t_0$ . Without loss of generality, assume that our representation is written such that  $a_1 g_1, \dots, a_{M_0} g_{M_0}$  contribute to the height, in the sense of Definition 2.3. Setting  $\vec{a}' = (a_1, \dots, a_{M_0}, 0, \dots, 0)$ , we see that  $\vec{\text{lt}}(\vec{a}') \in \text{Syz}^*(\text{Lt}G)$ . Thus, there exist  $b_1, \dots, b_N \in A$  and  $\vec{Q}_1, \dots, \vec{Q}_N \in \mathcal{Q}$  such that  $\vec{\text{lt}}(\vec{a}') = \sum_{j=1}^N \text{lt}(b_j) \vec{\text{lt}}(\vec{Q}_j)$ ; also, we may assume that  $\deg(\text{lt}(b_j) \vec{\text{lt}}(\vec{Q}_j)) = \deg(\vec{\text{lt}}(\vec{a}')) = t_0$  for all  $j$  by homogeneity of the syzygies involved. Furthermore, the elements  $b_j$  and  $\vec{Q}_j$  may be chosen so that the expression  $\sum_{j=1}^N \text{lt}(b_j) \text{lt}(q_{j,i})$  is homogeneous in  $R[X]$  for all  $i$  since every non-zero  $\text{lt}(b_j) \text{lt}(q_{j,i}) \text{lt}(g_i) = \deg(\text{lt}(b_j) \vec{\text{lt}}(\vec{Q}_j)) = t_0$ .

Now

$$\begin{aligned} h &= \sum_{i=1}^M a_i g_i - \sum_{i=1}^M \left( \sum_{j=1}^N b_j q_{j,i} \right) g_i + \sum_{j=1}^N b_j \left( \sum_{i=1}^M q_{j,i} g_i \right) \\ &= \sum_{i=1}^M \left( a_i - \sum_{j=1}^N b_j q_{j,i} \right) g_i + \sum_{j=1}^N b_j \left( \sum_{i=1}^M p_{j,i} g_i \right) \end{aligned} \quad (4.1)$$

where  $\sum_{i=1}^M p_{j,i} g_i$  is an SG-representation for  $\sum_{i=1}^M q_{j,i} g_i$ , which exists since we have supposed that every  $\sum_{i=1}^M q_{j,i} g_i \xrightarrow{G}_{\text{si}} 0$ . Furthermore, if we define  $t_j = \text{ht}(\sum_{i=1}^M p_{j,i} g_i)$ , then we have

$$t_j = \text{lp}\left(\sum_{i=1}^M q_{j,i} g_i\right) < \max_i \text{lp}(q_{j,i} g_i) \forall j,$$

where the inequality holds because  $\vec{Q}_j \in \text{LtSyz}_A^*(G)$ .

We proceed to show that the representation for  $h$  in Equation (4.1) has lesser height than our original representation. The height of the first sum (indexed by  $i$ ) is  $\max_i \text{lp}[(a_i - \sum_{j=1}^N b_j q_{j,i})g_i]$ . For  $i \leq M_0$ , we know that  $\text{lt}(a_i) = \text{lt}(\sum_{j=1}^N b_j q_{j,i})$  by homogeneity of  $\sum_{j=1}^N b_j q_{j,i}$  in  $R[X]$ ; therefore, due to cancellation of the highest terms,  $\text{lp}[(a_i - \sum_{j=1}^N b_j q_{j,i})g_i] < \text{lp}(a_i g_i) = t_0$ , our original height. For  $i > M_0$ , we recognize that  $\text{lp}[(a_i - \sum_{j=1}^N b_j q_{j,i})g_i] \leq \max\{\text{lp}(a_i g_i), \text{lp}(\sum_{j=1}^N b_j q_{j,i} g_i)\}$ , for we assume that the expression  $a_i - \sum_{j=1}^N b_j q_{j,i}$  represents a simplified polynomial in  $R[X]$ . Yet  $i > M_0$  implies that  $\text{lp}(a_i g_i) < t_0$  and that  $\sum_{j=1}^N \text{lt}(b_j) \text{lt}(q_{j,i}) = 0$ , which in turn implies that  $\text{lp}(\sum_{j=1}^N b_j q_{j,i} g_i) < \max_j \text{lp}(b_j q_{j,i} g_i) = \deg(\text{lt}(b_j) \vec{\text{lt}}(\vec{Q}_j)) = t_0$ . Thus, the height of the first sum in Equation (4.1) must be less than the original height since for all  $i$ ,  $\text{lp}[(a_i - \sum_{j=1}^N b_j q_{j,i})g_i] < t_0$ .

Now for the second sum, we have the following:

$$\begin{aligned} \text{ht}\left(\sum_{j=1}^N b_j \sum_{i=1}^M p_{j,i} g_i\right) &\leq \max_{i,j} \text{lp}(b_j p_{j,i} g_i) = \max_j [\text{lp}(b_j) \cdot t_j] \\ &< \max_{i,j} \text{lp}(b_j q_{j,i} g_i) = \deg(\vec{\text{lt}}(\vec{a}')) = t_0 \end{aligned}$$

Hence, Equation (4.1) does provide a new representation for  $h \in \langle G \rangle_A$  having smaller height than our assumed minimum. Therefore,  $\text{lp}(h) = t_0$ , the minimum possible height, proving that  $G$  is an SG-basis for  $\langle G \rangle_A$ .  $\square$

We next describe how an lt-generating set for  $\text{LtSyz}_A^*(G)$  may be computed (when  $G = \{g_1, \dots, g_M\}$  is finite). The method is based on the following result, whose proof is straight-forward.

**PROPOSITION 4.10.** *Let  $\pi : \mathcal{R} \rightarrow \mathcal{S}$  be a ring epimorphism. Let  $S' = \{s_1, \dots, s_M\} \subseteq \mathcal{S}$  be given, and choose a set  $\bar{S}' = \{\bar{s}_1, \dots, \bar{s}_M\}$  of pre-images in  $\mathcal{R}$ . Suppose that  $\vec{P}_1, \dots, \vec{P}_L \in \mathcal{R}^M$  with  $\vec{P}_j = (p_{j,1}, \dots, p_{j,M})$  are such that*

$$\vec{P}_1, \dots, \vec{P}_K \text{ generate } \text{Syz}(\bar{s}_1, \dots, \bar{s}_M) \subseteq \mathcal{R}^M$$

while for the remaining  $\{\vec{P}_{K+1}, \dots, \vec{P}_L\}$ ,

$$\sum_{i=1}^M p_{K+1,i} \bar{s}_i, \dots, \sum_{i=1}^M p_{L,i} \bar{s}_i \text{ generate } \ker(\pi) \cap \langle \bar{s}_1, \dots, \bar{s}_M \rangle \subseteq \mathcal{R}.$$

Then  $\text{Syz}(s_1, \dots, s_M)$  is generated by the set  $\{\vec{\pi}(\vec{P}_1), \dots, \vec{\pi}(\vec{P}_L)\}$ , where we define  $\vec{\pi} : \mathcal{R}^M \rightarrow \mathcal{S}^M$  via  $\vec{\pi}(r_1, \dots, r_M) = (\pi(r_1), \dots, \pi(r_M))$  for  $r_1, \dots, r_M \in \mathcal{R}$ .

To apply this result in the desired setting, we take  $\mathcal{S} = R[\text{Lt}A] = R[\text{Lt}F]$  where  $F$  is a finite SAGBI basis for  $A$ , set  $\mathcal{R} = R[Y]$  where  $Y$  is a set of variables of the same cardinality as  $F$ , and take  $\pi$  to be the obvious evaluation map. Proposition 4.10 and ordinary Gröbner basis techniques then allow us to compute generators for  $\text{Syz}(\text{Lt}G)$ , from which we may obtain a homogeneous generating set  $\mathcal{P} = \{\vec{P}_1, \dots, \vec{P}_N\}$  for  $\text{Syz}^*(\text{Lt}G)$ . Furthermore, we may assume that for each generator  $\vec{P}_j(\text{Lt}F) = (P_{j,1}(\text{Lt}F), \dots, P_{j,M}(\text{Lt}F))$ ,

the polynomials  $P_{j,i}(\text{Lt}F)$  are homogeneous in  $R[X]$ . Defining

$$\widetilde{P}_{j,i}(F) = \begin{cases} P_{j,i}(F) & \text{if } P_{j,i}(\text{Lt}F) \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

we see that the set  $\mathcal{Q} = \{(\widetilde{P}_{j,1}(F), \dots, \widetilde{P}_{j,M}(F)) : j = 1, \dots, N\}$  is an lt-generating set for  $\text{LtSyz}_A^*(G)$ , for  $\text{lt}(\widetilde{P}_{j,i}(F)) = P_{j,i}(\text{Lt}F)$  for all  $i$  and  $j$ .

Next we present an algorithm for computing SG-bases. See Algorithm 4.1.

**INPUT:** A finite set  $G \subseteq A$ ,  $F$  a finite SAGBI basis for  $A$

**OUTPUT:** An SG-basis  $H$  for  $\langle G \rangle_A$

**INITIALIZATION:**  $H := G$ ,  $oldH := \emptyset$

**WHILE**  $H \neq oldH$  **DO**

Compute an lt-generating set  $\mathcal{Q}$  for  $\text{LtSyz}_A^*(H)$ .

$\mathcal{P} := \{\sum_{h \in H} q_h h : (q_h)_{h \in H} \in \mathcal{Q}\}$

$red\mathcal{P} := \{\text{final si-reducta via } H \text{ of each element of } \mathcal{P}\} - \{0\}$

$oldH := H$

$H := H \cup red\mathcal{P}$

ALGORITHM 4.1. : SG-Basis Construction Algorithm

**PROPOSITION 4.11.** *Algorithm 4.1 yields a finite SG-basis for  $\langle G \rangle_A$  (when  $G$  is finite and  $A$  has a finite SAGBI basis).*

**Proof.** We first show that the algorithm produces an SG-basis, then that the resulting basis is finite.

Set  $H_\infty = \cup H$  over all passes of the WHILE loop. For each  $\mathcal{Q}$  of each loop, construct a set  $\mathcal{Q}' \subseteq \oplus_{H_\infty} A$  by adding sufficiently many 0 coordinates to each vector in  $\mathcal{Q}$ . We claim that the set  $\mathcal{Q}'_\infty = \cup \mathcal{Q}'$  over all passes of the loop is an lt-generating set for  $\text{LtSyz}_A^*(H_\infty)$ . For choose  $\vec{t} = (t_i)_{h_i \in H_\infty} \in \text{Syz}^*(\text{Lt}H_\infty)$ . Only finitely many coordinates  $t_i$  are non-zero, corresponding to finitely many elements  $h_i \in H_\infty$ . These elements all belong to the set  $H = H_{N_0}$  produced at the end of some finite number of passes of the

WHILE loop. Defining  $\vec{\tau}_0$  to be the vector consisting precisely of the non-zero coordinates of  $\vec{\tau}$ , we note that  $\vec{\tau}_0 \in \text{Syz}^*(H_{N_0})$  and therefore belongs to the  $R[\text{Lt}A]$ -module generated by  $\{\text{lt}(\vec{Q}) : \vec{Q} \in \mathcal{Q}_{N_0}\}$ , where  $\mathcal{Q}_{N_0}$  is the chosen lt-generating set for  $\text{LtSyz}_A^*(H_{N_0})$ . Consequently,  $\vec{\tau}$  belongs to the  $R[\text{Lt}A]$ -module generated by  $\{\text{lt}(\vec{Q}') : \vec{Q}' \in \mathcal{Q}'_\infty\}$ , proving the claim.

We next show that  $H_\infty$  is an SG-basis for  $\langle G \rangle_A$ . Choose  $\vec{Q}' \in \mathcal{Q}'_\infty$ . Again,  $\vec{Q}' = (q'_h)_{h \in H_\infty}$  has only finitely many non-zero coordinates, corresponding to a finite subset of some  $H = H_{N_0} \subseteq H_\infty$ . Clearly,  $\sum_{q'_h \neq 0} q'_h h$  si-reduces to 0 via some subset of  $H_\infty$ , hence via  $H_\infty$  either in the loop in which  $\vec{Q}'$  is created or in the next. Thus,  $H_\infty$  satisfies Theorem 4.9, proving that it is indeed an SG-basis for  $\langle G \rangle_A$ .

Since  $A$  has a finite SAGBI basis, we know that there exists a finite SG-basis  $S$  for  $\langle G \rangle_A$ . We have shown above that  $H_\infty$  is an SG-basis for  $\langle G \rangle_A$ ; therefore, it must be that for each  $s \in S$  there exist  $h_{1,s}, \dots, h_{M_s,s} \in H_\infty$  and  $a_{1,s}, \dots, a_{M_s,s} \in A$  such that

$$\text{lt}(s) = \sum_{i=1}^{M_s} \text{lt}(a_{i,s} h_{i,s}).$$

The set  $\tilde{H} = \cup_{s \in S} \{h_{1,s}, \dots, h_{M_s,s}\}$  is clearly finite, and it is an SG-basis for  $\langle G \rangle_A$  since  $\langle \text{Lt}\tilde{H} \rangle = \langle \text{Lt}S \rangle = \langle \text{Lt}G \rangle \subseteq R[\text{Lt}A]$ . Because  $\tilde{H}$  must be a subset of the set  $H = H_{N_0}$  produced after some finite number of passes of the WHILE loop,  $H_{N_0}$  is also an SG-basis, and the algorithm will terminate at the next loop.

Finally, we show that  $H_{N_0}$  is finite. Our technique for computing an lt-generating set for  $\text{LtSyz}_A^*(H)$  involves calculating a generating set for  $\text{Syz}^*(\text{Lt}H)$ ; these two sets have the same cardinality, according to Definition 4.8. Since  $R[\text{Lt}A]$  is noetherian, we may choose a finite generating set for  $\text{Syz}^*(\text{Lt}H)$  when the input set  $H$  for the loop is finite. Therefore,  $\mathcal{P}$  and consequently the output of such a loop are finite. Then since  $H_{N_0}$  is the result of a finite number of passes of the loop, beginning with finite input  $G$ , it is a finite SG-basis for  $\langle G \rangle_A$ .  $\square$

The example below demonstrates how to compute an SG-basis.

EXAMPLE 4.12. As in Example 3.8, we take  $A = \mathbf{Z}[F] \subseteq \mathbf{Z}[x, y]$  where

$$F = \{f_1 = 2x^2 + xy, \quad f_2 = 2y^2, \quad f_3 = 3xy\},$$

and let  $G \subseteq \mathbf{Z}[F]$  be given by

$$G = \{g_1 = 4x^2y^2 + 2xy^3, \quad g_2 = 18x^2y^4\}.$$

We will again use the term order degree lex with  $x > y$ , with respect to which we have found that  $F$  is a SAGBI basis for  $A$ .

We begin by setting  $H = G$ . Applying the technique described after Proposition 4.10, we calculate<sup>†</sup> an lt-generating set  $\mathcal{Q} = \{(f_3^2, -f_1), (9f_2, -4)\}$  for  $\text{LtSyz}_A^*(H)$ ; we obtain the associated set  $\mathcal{P} = \{0, 36xy^5\}$ . We easily see that  $\text{red}\mathcal{P} = \{36xy^5\}$  since this element cannot be si-reduced via  $H$ . Therefore, we define

$$g_3 = 36xy^5$$

<sup>†</sup> Some of the intermediate computations were performed using the Mathematica sub-package GroebnerZ. See (NG).

and conduct a second pass of the WHILE loop. This time, we construct

$$\mathcal{Q} = \{(f_3^2, -f_1, 0), (3f_2f_3, 0, -f_1), (0, 3f_2, -f_3), (-9f_2, 4, 0)\}.$$

This again yields  $\mathcal{P} = \{0, 36xy^5\}$ , so clearly,  $\text{red}\mathcal{P} = \emptyset$  now, and the stopping criterion  $H = \text{old}H$  is satisfied. We have that

$$\{4x^2y^2 + 2xy^3, 18x^2y^4, 36xy^5\}$$

is an SG-basis for  $\langle G \rangle_A$ .  $\triangle$

### 5. A-syzygies

To conclude, we will present a method for calculating a set of generators for  $\text{Syz}_A(H)$  given a finite subset  $H$  of an  $R$ -subalgebra  $A \subseteq R[X]$ , where we again assume that  $A$  has a finite SAGBI basis. Our technique is based on the following theorem:

**THEOREM 5.1.** *Let  $G = \{g_1, \dots, g_M\} \subseteq A$  be a finite SG-basis for  $\langle G \rangle_A$ . Let  $\mathcal{Q} = \{\vec{Q}_1, \dots, \vec{Q}_N\}$  be an lt-generating set for  $\text{LtSyz}_A^*(G)$ , and write each  $\vec{Q}_j = (q_{j,1}, \dots, q_{j,M})$ . For each  $j$ , let  $\sum_{i=1}^M p_{j,i}g_i$  be an SG-representation for  $\sum_{i=1}^M q_{j,i}g_i$ . Then  $\text{Syz}_A(G)$  is generated as an  $A$ -module by the vectors*

$$\vec{P}_j = (q_{j,1} - p_{j,1}, \dots, q_{j,M} - p_{j,M}), \quad j = 1, \dots, N.$$

**Proof.** Let  $\mathcal{M}$  represent the  $A$ -submodule of  $\text{Syz}_A(G)$  generated by the set  $\{\vec{P}_1, \dots, \vec{P}_N\}$ , and suppose that the conclusion of the theorem is false. Then we can choose  $\vec{h} = (h_1, \dots, h_M) \in \text{Syz}_A(G) - \mathcal{M}$ , such that  $t_0 = \text{ht}(\sum_{i=1}^M h_i g_i)$  as defined in Definition 2.3 is minimal among such elements of  $\text{Syz}_A(G)$ . Without loss of generality, we assume that precisely  $h_1, \dots, h_{M_0}$  contribute to the height of this expression. This implies that  $\sum_{i=1}^{M_0} \text{lt}(h_i)\text{lt}(g_i) = 0$ , i.e., that  $\vec{\text{lt}}(\vec{h}') \in \text{Syz}^*(\text{Lt}G) \subseteq R[\text{Lt}A]$  where  $\vec{h}' = (h_1, \dots, h_{M_0}, 0, \dots, 0)$ . Therefore, we can write

$$\vec{\text{lt}}(\vec{h}') = \sum_{j=1}^N \text{lt}(b_j)\vec{\text{lt}}(\vec{Q}_j)$$

where  $\deg[\text{lt}(b_j)\vec{\text{lt}}(\vec{Q}_j)] = \deg(\vec{\text{lt}}(\vec{h}')) = t_0$  for all  $j$  such that  $b_j \neq 0$ . Also, as we saw in the proof of Theorem 4.9, we may assume that the expression  $\sum_{j=1}^N \text{lt}(b_j)\text{lt}(q_{j,i})$  is homogeneous in  $R[X]$  for all  $i$ .

Now we consider the element  $\vec{s} = \vec{h} - \sum_{j=1}^N b_j \vec{P}_j \in \text{Syz}_A(G) - \mathcal{M}$ . We claim that  $\deg(\vec{s}) < t_0$ . By definition,  $\deg(\vec{s}) = \max_i \text{lp}(s_i g_i)$  where  $s_i$  is the  $i$ -th coordinate of  $\vec{s}$ ; in particular,  $s_i$  is the simplified form of  $h_i - \sum_{j=1}^N b_j p_{j,i}$ . For  $i \leq M_0$ ,  $\text{lt}(\sum_{j=1}^N b_j p_{j,i}) = \text{lt}(\sum_{j=1}^N b_j q_{j,i}) = \text{lt}(h_i)$ ; whence, cancellation of the highest terms yields  $\text{lp}(s_i g_i) < \text{lp}(h_i g_i) = t_0$ . For  $i > M_0$ ,  $\text{lp}(s_i g_i) \leq \max\{\text{lp}(h_i g_i), \text{lp}(\sum_{j=1}^N b_j p_{j,i} g_i)\}$ . By assumption,  $\text{lp}(h_i g_i) < t_0$ , and

$$\text{lp}\left(\sum_{j=1}^N b_j p_{j,i} g_i\right) = \text{lp}\left(\sum_{j=1}^N b_j q_{j,i} g_i\right) < \max_j \text{lp}(b_j q_{j,i} g_i) = t_0$$

where the inequality holds because  $\sum_{j=1}^N \text{lt}(b_j q_{j,i}) = 0$  for  $i > M_0$  and the final equality holds due to our assumption that  $\deg(\text{lt}(b_j)\vec{\text{lt}}(\vec{Q}_j)) = \deg(\vec{\text{lt}}(\vec{h}')) = t_0$ . Hence,  $\text{lp}(s_i g_i) <$

$t_0$  for all  $i$ , and we indeed have  $\deg(\vec{s}) < t_0$ , which contradicts our assumption of the minimality of  $t_0$  for elements of  $\text{Syz}_A(G) - \mathcal{M}$ . Therefore, this difference is empty, and  $\text{Syz}_A(G) = \mathcal{M}$ .  $\square$

We are now prepared to compute the generators for the  $A$ -syzygy module  $\text{Syz}_A(H)$  of an arbitrary finite subset  $H \subseteq A$ . We briefly outline the standard technique, which is described in greater detail in such references as (AL). Specifically, we compute an SG-basis  $G$  for  $\langle H \rangle_A$  and then produce matrices  $\mathcal{W}$  and  $\mathcal{U}$  with entries in  $A$  such that  $H = \mathcal{W}G$  and  $G = \mathcal{U}H$ , where we now view  $G$  and  $H$  as column vectors. The module  $\text{Syz}_A(H)$  is then generated by the vectors  $\vec{P}_j\mathcal{U}$  together with the row vectors of  $\mathcal{I} - \mathcal{W}\mathcal{U}$ , where  $\mathcal{I}$  is the identity matrix of the appropriate size.

EXAMPLE 5.2. Again, we take  $A = \mathbf{Z}[F] \subseteq \mathbf{Z}[x, y]$  where

$$F = \{f_1 = 2x^2 + xy, \quad f_2 = 2y^2, \quad f_3 = 3xy\}$$

is a SAGBI basis for  $A$  with respect to our term order, degree lex with  $x > y$ . Let

$$H = \{h_1 = 4x^2y^2 + 2xy^3, \quad h_2 = 10x^2y^4 + 4xy^5, \quad h_3 = 36xy^5\} \subseteq A.$$

It is apparent that the set

$$G = \{g_1 = 4x^2y^2 + 2xy^3, \quad g_2 = 18x^2y^4, \quad g_3 = 36xy^5\}$$

of Example 4.12 is an SG-basis for  $\langle H \rangle_A$ , for we observe that  $h_1 = g_1$ ,  $h_2 = g_2 - f_2g_1$ , and  $h_3 = g_3$ . Thus, we have the change-of-basis matrices

$$\mathcal{W} = \begin{bmatrix} 1 & 0 & 0 \\ -f_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{U} = \begin{bmatrix} 1 & 0 & 0 \\ f_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

described above. Because  $\mathcal{I} - \mathcal{W}\mathcal{U}$  is the zero-matrix, the only non-trivial generators for  $\text{Syz}_A(H)$  are the vectors  $\vec{P}_j\mathcal{U}$ , which we will now compute.

We recall the lt-generating set

$$\begin{aligned} \mathcal{Q} = \{ & \vec{Q}_1 = (f_3^2, -f_1, 0), \\ & \vec{Q}_2 = (3f_2f_3, 0, -f_1), \\ & \vec{Q}_3 = (0, 3f_2, -f_3), \\ & \vec{Q}_4 = (-9f_2, 4, 0) \} \end{aligned}$$

for  $\text{LtSyz}_A^*(G)$  as described at the end of Example 4.12. For the first three of these vectors, the polynomials  $\sum_{i=1}^3 q_{j,i}g_i = 0$ ; thus,  $\vec{P}_j = \vec{Q}_j$  for  $j = 1, 2, 3$ . However,  $\vec{Q}_4$  gives us the expression  $-9f_2g_1 + 4g_2 = -36xy^5 = -g_3$ , which yields  $\vec{P}_4 = (-9f_2, 4, 1)$ . We conclude that

$$\begin{aligned} \vec{P}_1\mathcal{U} &= (f_3^2 - f_1f_2, -f_1, 0) \\ \vec{P}_2\mathcal{U} &= (3f_2f_3, 0, -f_1) \\ \vec{P}_3\mathcal{U} &= (3f_2^2, 3f_2, -f_3) \\ \text{and } \vec{P}_4\mathcal{U} &= (-5f_2, 4, 1) \end{aligned}$$

generate  $\text{Syz}_A(H)$  as an  $A$ -module.  $\triangle$

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