MS07: Recent trends in asymptotics

Organizer: Gergő Nemes (*Alfréd Rényi Institute of Mathematics, Budapest, Hungary*)

The main goal of the mini-symposium is to bring together researchers from all over the world to discuss their most recent results in the field of asymptotic analysis. The mini-symposium is intended to cover a broad range of subjects in asymptotic analysis, including classical asymptotics, uniform asymptotics, hyperasymptotics, resurgent function theory and exact WKB analysis.

From asymptotics to exact results: unraveling the analytic structure of solutions of Painlevé I

07.01 **Inês Aniceto**
(*University of Southampton, UK*)
Time: Monday 22.07., 10:30 - 11:00, Room HS 5

Abstract: Understanding the asymptotic properties of solutions of the Painlevé I non-linear ODE is of great interest in both mathematics and physics. It is well known that the asymptotic behaviour of these solutions is connected to the existence of exponentially small contributions, directly linked to physical phenomena not captured by a perturbative analysis. The theory of resurgence perfectly captures this perturbative/non-perturbative connection and its consequences. Moreover, it allows us to construct a full non-perturbative solution from perturbative data. In this talk, I will demonstrate the essential role of resurgence theory, coupled to exponentially accurate numerical methods, in going beyond the perturbative results and obtain (analytically and numerically) non-perturbative data. In particular, I will exemplify how these techniques can be applied to to the calculation of poles of Painlevé I solutions.

Invisible catastrophes: when to turn an asymptotic blind eye

07.02 **Christopher J. Howls**
(*University of Southampton, UK*)
Time: Monday 22.07., 11:00 - 11:30, Room HS 5

Abstract: Recent work on high-frequency flow-interaction effects for 3D jet engine noise in a moving media uncovered unusual structures that appeared to violate mathematical aspects of both classical ray analysis and catastrophe theory whereby caustics appeared to end at finite ordinary (as opposed to turning) points in real space. A careful study of the local (complex) ray structure led to the introduction of a novel set of special functions that, a posteriori, resolved these problems. These special functions possess interesting novel properties from the point of view of analysis, with implications for exponential asymptotics. These special functions are now seen to occur in a variety of physical situations, ranging from the original aeroacoustic problem, to black hole event horizons and nonlinear traveling waves such as the Severn Bore. In this talk we will introduce these special functions and give an introduction to their novel properties.

Gaussian unitary ensembles with pole singularities near the soft edge and a system of coupled Painlevé XXXIV equations

07.03 **Dan Dai**
(*City University of Hong Kong*)
Time: Monday 22.07., 11:30 - 12:00, Room HS 5

Abstract: In this paper, we study the singularly perturbed Gaussian unitary ensembles defined by the measure

$$\frac{1}{C_n} e^{-n\mathrm{tr}\, V(M;\lambda,\vec{t})} dM,$$

over the space of $n \times n$ Hermitian matrices M, where $V(x;\lambda,\vec{t}\,) := 2x^2 + \sum_{k=1}^{2m} t_k(x-\lambda)^{-k}$ with $\vec{t} = (t_1, t_2, \ldots, t_{2m}) \in \mathbb{R}^{2m-1} \times (0,\infty)$, in the multiple scaling limit where $\lambda \to 1$ together with $\vec{t} \to \vec{0}$ as $n \to \infty$ at appropriate related rates. We obtain the asymptotics of the partition function, which is described explicitly in terms of an integral involving a smooth solution to a new coupled Painlevé system generalizing the Painlevé XXXIV equation. The large n limit of the correlation kernel is also derived, which leads to a new universal class built out of the Ψ-function associated with the coupled Painlevé system.

This is a joint work with Shuai-Xia Xu and Lun Zhang.

Distribution of the maximal height of N non-intersecting Bessel paths

07.04 **Luming Yao**
(*City University of Hong Kong*)
Time: Monday 22.07., 12:00 - 12:30, Room HS 5

Abstract: Consider N non-intersecting Bessel paths start at a positive position $x = a > 0$ when time $t = 0$ and end at the origin $x = 0$ when time $t = 1$. Using the Karlin–McGregor formula and the Schur function expansion of the corresponding determinants, we derive the exact distribution function for the maximal height of the outermost path, which is given in terms of Hankel determinants of the multiple orthogonal polynomials.

Voros coefficients and the topological recursion for the hypergeometric differential equations associated with the 2-dimensional Garnier system

07.05 **Yumiko Takei**
(*Department of Mathematics, Graduate School of Science, Kobe University, Japan*)
Time: Monday 22.07., 15:30 - 16:00, Room HS 5

Abstract: The N-dimensional Garnier system is a Hamiltonian system with N independent variables obtained through monodromy preserving deformations of second order linear differential equations on $\mathbb{P}^1$ with $N+3$ regular singular points. In the case of $N = 1$, the system reduces to the sixth Painlevé equation P_{VI} and the Gauss hypergeometric function gives a particular solution of P_{VI}. Note that each member of the family of the Gauss hypergeometric equations is obtained from P_{VI} by the so-called confluence process. In the same manner the hypergeometric differential equations with two independent variables are obtained from the 2-dimensional Garnier system. In this talk I consider some relationship between the exact WKB analysis and the topological recursion for the hypergeometric differential equations with two independent variables.

Exact WKB analysis is a powerful tool to study differential equations globally. In particular, Voros coefficients provide important quantities for describing global behavior of solutions of differential equations. On the other hand, the topological recursion introduced by B. Eynard and N. Orantin [2] to study the correlation functions in the random matrix theory gives a generalization of the loop equations for the matrix model. Recently, several surprising connections between exact WKB analysis and topological recursion have been discovered. For example, it is shown that WKB solutions are constructed via the topological recursion [1]. Furthermore, together with Iwaki and Koike I show that in the case of the family of the Gauss hypergeometric equations Voros coefficients are described by the generating functions of free energies defined in terms of the topological recursion [3,4].

In this talk, I would like to discuss a generalization of the above result, that is, I will report that the Voros coefficients for some confluent hypergeometric differential equations of two variables associated with degenerate 2-dimensional Garnier systems are described by the generating functions of free energies defined in terms of the topological recursion. As its application I will also show that the following objects can be computed in an explicit manner: (i) three-term difference equations that the generating function of the free energies satisfies, (ii) explicit form of free energies, and (iii) explicit form of Voros coefficients.

[1] V. Bouchard and B. Eyanard, Reconstructing WKB from topological recursion, *Journal de l'Ecole polytechnique – Mathematiques*, **4** (2017), pp. 845–908.
[2] B. Eynard and N. Orantin, Invariants of algebraic curves and topological expansion, *Communications in Number Theory and Physics*, **1** (2007), pp. 347–452; arXiv:math-ph/0702045.
[3] K. Iwaki, T. Koike, and Y.-M. Takei, Voros coefficients for the hypergeometric differential equations and Eynard–Orantin's topological recursion, part I: for the Weber equation; arXiv:1805.10945.
[4] K. Iwaki, T. Koike, and Y.-M. Takei, Voros coefficients for the hypergeometric differential equations and Eynard–Orantin's topological recursion, part II: for the confluent family of hypergeometric equations, preprint; arXiv:1810.02946.

The hypergeometric function and WKB solutions

07.06 **Takashi Aoki**
(*Kindai University, Japan*)
Time: Monday 22.07., 16:00 - 16:30, Room HS 5

Abstract: We introduce a large parameter in the three parameters of the hypergeometric differential equation and consider exact WKB solutions. These formal solutions are Borel summable with respect to the parameter under suitable conditions. Taking the Borel sum, we obtain analytic solutions of the equation. We give the relations between the hypergeometric function and Borel resummed WKB solutions. As an application, we have asymptotic expansion formulas of the hypergeometric function with respect to the large parameter.

The confluent hypergeometric function and WKB solutions

07.07 **Toshinori Takahashi**
(*Kindai University, Japan*)
Time: Monday 22.07., 16:30 - 17:00, Room HS 5

Abstract: In this talk, we consider the relation between the confluent hypergeometric function and the Borel resummed WKB solutions. We show that the relations and the Voros coefficients and their Borel sums can be obtained by applying the operation of confluent to those in the case of the hypergeometric equation. We also talk about further confluence.

On functions K and E generated by a sequence of moments

07.08 **Avner Kiro**
(*Tel Aviv University, Israel*)
Time: Tuesday 23.07., 15:30 - 16:00, Room HS 5

Abstract: For a class of functions γ analytic in the sector $\{s\colon |\arg(s)| < \alpha_0\}$ with $\frac{\pi}{2} < \alpha_0 < \pi$, we describe the asymptotic behavior of the analytic function

$$K(z) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} z^{-s}\gamma(s)\,\mathrm{d}s,$$

that solves the moment problem

$$\int_0^\infty t^n K(t)\,\mathrm{d}t = \gamma(n+1), \qquad n \geq 0\,,$$

and of the entire function

$$E(z) = \sum_{n\geq 0} \frac{z^n}{\gamma(n+1)}.$$

These two functions naturally appear in various classical problems of analysis. The talk is based on a joint work with M. Sodin.

Transition region expansions

07.09 **Adri B. Olde Daalhuis**
(*University of Edinburgh, UK*)
Time: Tuesday 23.07., 16:00 - 16:30, Room HS 5

Abstract: For the normalised incomplete gamma function $Q(a,z)=\Gamma(a,z)/\Gamma(a)$ we will construct what we call *transition region expansions*, and provide full details of the inversion of these new expansions. These are expansions that are valid in the regions in which $Q(a,z)$ changes dramatically, and their coefficients are polynomials satisfying simple recurrence relations. The region of validity overlaps with those of the non-uniform "outer" expansions. Furthermore, the coefficients of their inversions are simple polynomials, whose computation and implementation are straightforward.

It is surprising to us that these transition region expansions for the normalised incomplete gamma function have not yet been discussed in the literature, given the fact that expansions of similar type for Bessel functions are well known (see §10.19.iii in the DLMF). What our new expansions and the transition region expansions for the Bessel functions have in common is that both mimic the corresponding uniform expansions.

This is a joint work with Gergő Nemes.

On the Borel summability of WKB solutions near a simple pole

07.10 **Gergő Nemes**
(*Alfréd Rényi Institute of Mathematics, Budapest, Hungary*)
Time: Tuesday 23.07., 16:30 - 17:00, Room HS 5

Abstract: We consider the following Schrödinger-type differential equation:

$$\frac{\mathrm{d}^2W(u,\xi)}{\mathrm{d}\xi^2}=(u^2+\psi(\xi))W(u,\xi),$$

where u is a large positive parameter and $\psi(\xi)$ is an analytic function of ξ apart from countably many singularities. It is known that this equation has formal solutions of the form

$$W_{1,2}(u,\xi)=\mathrm{e}^{\pm\xi u}\sum_{n=0}^{\infty}\frac{\mathsf{A}_n(\pm\xi)}{u^n}.$$

These are called the WKB solutions. We study the Borel summability of these WKB solutions near a simple pole of the function $\psi(\xi)$ which we assume to be located at the origin. It is shown that both of the formal series are Borel summable in every closed strip $\{\xi:|\Im\xi|\leq\gamma\}$ contained in the domain of analyticity of $\psi(\xi)$ apart from the Stokes rays $\arg\xi=0$, 2π and $\arg\xi=\pm\pi$ emanating from the origin. We determine the type of singularities of the Borel transforms near the origin when $|\xi|$ is small and also provide global connection formulae between the solutions $W_1(u,\xi)$ and $W_2(u,\xi)$.