MS06: Symbolic computation and special functions

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Computer algebra plays an increasingly important role in the investigation of special functions. For large classes of special functions we now have strong algorithmic theories. Software packages based on these theories successfully solve interesting problems that are not accessible by other means and they also routinely and reliably solve tedious subproblems that frequently arise in day-to-day calculations. The purpose of this minisymposium is to join computer algebra people interested in special functions with special functions people interested in computer algebra, in order to share recent trends, new techniques, and open problems at the intersection of these two areas.

Rigorous numerical evaluation of D-finite functions in SageMath

06.01 **Marc Mezzarobba**
(*Sorbonne Université, Campus Pierre et Marie Curie, Paris, France*)
Time: Tuesday 23.07., 10:30 - 11:00, Room HS 5

Abstract: I will give a demo of the symbolic-numeric features available for working with D-finite functions in the Sage package `ore_algebra`.

Recall that a complex analytic function is called D-finite when it satisfies a linear ODE with polynomial coefficients. D-finite functions form a class analogous to that of hypergeometric functions, but more general. They come up in areas such as analytic combinatorics and mathematical physics, and lend themselves well to symbolic manipulation by computer algebra systems.

At the heart of the analytic features of `ore_algebra` is a rigorous implementation of numerical analytic continuation of D-finite functions. Numerical analytic continuation consists in computing numerical approximation of the transition matrices that maps initial values of an ODE somewhere on the complex plane to initial values elsewhere that define the same solution. The implementation is rigorous in the sense that it returns not just an approximation but an enclosure of the exact mathematical result.

Numerical analytic continuation is the basic brick for computing values of D-finite functions anywhere on their Riemann surfaces, rigorous polynomial approximations of D-finite functions on real or complex domains, monodromy matrices of differential operators, and other related objects. The code fully supports the important limit case where the (generalized) initial values are provided at regular singular points of the ODE, making it possible in particular to compute connection constants between regular singularities.

Implementing finite summation identities of polygamma and related functions into Mathematica

06.02 **Lu Wei**
(*University of Michigan, Dearborn, USA*)
Time: Tuesday 23.07., 11:00 - 11:30, Room HS 5

Abstract: Finite sums of polygamma and related functions find diverse applications in mathematical physics and other fields. These types of sums often admit closed-form expressions by exploring recurrence relations of various forms. In this talk, we will first outline some strategies to approach the considered sums. We will then discuss the on-going collaboration with Wolfram Research to possibly implement these sums into future versions of the computer algebra system Mathematica.

Orthogonal polynomials for higher-order Euler polynomials

06.03 **Lin Jiu**
(*Department of Mathematics and Statistics, Dalhousie University, Halifax, Canada*)
Time: Tuesday 23.07., 11:30 - 12:00, Room HS 5

Abstract: Since recent results recognize higher-order Euler polynomials as the moments of certain random variables, it is natural to study the corresponding monic orthogonal polynomials. Based on the orthogonal polynomials with respect to the Euler numbers, obtained by Carlitz and Al-Salam, we identify the orthogonal polynomials with respect to higher-order Euler polynomials are the Meixner-Pollaczek polynomials, with certain arguments and constant factors. Applications, based on the connection to generalized Motzkin numbers, involve matrix and lattice path representations. Analogues for Bernoulli numbers and Bernoulli polynomials are also presented. This is joint work with Diane Y. H. Shi

Positive systems of polynomial equations

06.04 **Michael Drmota**
(*Technische Universität Wien, Austria*)
Time: Tuesday 23.07., 12:00 - 12:30, Room HS 5

Abstract: A positive system of polynomial equations is of the form $y = P(x, y)$, where $y = (y_1, \ldots, y_k)$ is a k-dimensional vector and $P(x, y) = (P_1(x, y), \ldots, P_k(x, y))$ a system of k polynomials with non-negatives coefficients in x and $y = (y_1, \ldots, y_k)$. Under quite natural conditions such systems have a unique solution $y(x) = (y_1(x), \ldots, y_k(x))$ of power series in x that have - by construction - non-negative coefficients and are algebraic functions. Such positive systems of polynomial equations appear naturally in many combinatorial questions. In contrast to arbitrary algebraic functions the Puiseux expansion at the dominant singularity of these functions (that determines the asymptotic behavior of the coefficients) is quite restricted, in particular the exponent can only be a dyadic rational number. This has been shown by Banderier and Drmota in 2015. Since we are in the framework of algebraic functions it is clear that full asymptotics of the coefficients of the functions $y_j(x)$ can be automatically determined. It is, however, a non-trivial problem to make this computation efficient, in particular for large systems of equations. The purpose of this talk to introduce the main results on positive systems of equations and to pose the efficiency computational question as an open problem.

Polynomials from $(0, m, s)$-nets and Walsh functions

06.05 **Elaine Wong**
(*RICAM, Austrian Academy of Sciences, Linz, Austria*)
Time: Wednesday 24.07., 10:30 - 11:00, Room HS 5

Abstract: We consider the problem of integrating a function f over the unit hypercube of dimension s. In practice, a digital net (a discrete breakdown of the continuous interval) can be cast over the unit hypercube in a way such that performing the integration over this net effectively and accurately estimates the integral. In our present work, we consider the integration of the joint probability density function of distinct points randomly chosen from a scrambled $(0, m, s)$-net and multivariate Walsh functions. This idea expands on the work of Wiart and Lemieux (2019). It allows us to simplify the integral into a discrete, symbolic sum containing the parameters m and s. From there, we are able to construct a certain univariate polynomial which can be used to determine how well the integration performs compared to the more widely used Monte-Carlo and other equidistribution methods. The coefficients of this polynomial conveniently contain hypergeometric series in the parameters. In this talk, we illustrate how to use available symbolic computation machinery to simplify such a polynomial into a nice closed form consisting of beta functions, from which we can draw our desired conclusions.

Computer algebra for basic hypergeometric functions

06.06 **Christoph Koutschan**
(*RICAM, Austrian Academy of Sciences, Linz, Austria*)
Time: Wednesday 24.07., 11:00 - 11:30, Room HS 5

Abstract: With the exception of q-hypergeometric summation, the use of computer algebra packages implementing Zeilberger's holonomic systems approach in a broader mathematical sense is less common in the field of q-series and basic hypergeometric functions. As a case study, we look at the celebrated Ismail-Zhang formula, an important q-analog of a classical expansion formula of plane waves in terms of Gegenbauer polynomials, and demonstrate how the Mathematica package HolonomicFunctions can be employed to generate a computer-assisted proof of this identity. The HolonomicFunctions package was originally developed for dealing with classical special function identities (sums, series, integrals), but its range of applicability also includes q-series and q-orthogonal polynomials. This is joint work with Peter Paule.

An algorithmic summation theory for indefinite nested sums and products

06.07 **Carsten Schneider**
(*Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria*)
Time: Wednesday 24.07., 11:30 - 12:00, Room HS 5

Abstract: Inspired by Karr's pioneering work (1981) we developed a summation theory of difference rings that enables one to rephrase special functions in terms of indefinite nested sums in the setting difference rings. Within this representation certain optimality criteria are fulfilled: e.g., the objects represented in the difference ring (except elements such as the alternating sign) do not satisfy any polynomial relations or the nesting depth of the arising sums have minimal nesting depth. Combining such optimal representations of special functions in combination with definite summation algorithms, like creative telescoping and recurrence solving in the setting of difference rings, yield a strong summation machinery for practical problem solving. We will demonstrate these features implemented in the summation package Sigma by concrete examples coming from combinatorics and particle physics.

The mathematical functions grimoire

06.08 **Fredrik Johansson**
(*INRIA Bordeaux and Institut de Mathématiques de Bordeaux, France*)
Time: Wednesday 24.07., 12:00 - 12:30, Room HS 5

Abstract: Fungrim (http://fungrim.org) is a new, open source database of formulas and tables for special functions. All formulas are encoded symbolically and include explicit conditions of validity for the variables, representing rewrite rules that can be applied mechanically without implicit exceptions (for instance, regarding branch cuts).

The immediate goal is to create a web-based special functions reference work that addresses some of the drawbacks of resources such as the NIST Digital Library of Mathematical Functions, the Wolfram Functions site, and Wikipedia. A potential longer-term ambition is to provide a software library for symbolic knowledge about special functions, usable by computer algebra systems and theorem proving software.

This talk will discuss the motivation behind the project, design issues, and possible applications. The project is still in experimental stages, and feedback is invited.

Interpolated sequences and critical L-values of modular forms

06.09 **Armin Straub**
(*University of South Alabama, USA*)
Time: Thursday 25.07., 10:30 - 11:00, Room HS 5

Abstract: It is well-known that the Apéry numbers which arise in the irrationality proof for $\zeta(3)$ satisfy many interesting arithmetic properties and are related to the Fourier coefficients of a weight 4 modular form. Recently, Zagier expressed an interpolated version of these numbers in terms of a critical L-value of the same modular form. We discuss this evaluation as well as extensions, including to interpolations of Zagier's six sporadic sequences. Our focus is on applications of and challenges for computer algebra that come up naturally in the context of these evaluations. This talk is based on joint work with Robert Osburn.

Efficient rational creative telescoping

06.10 **Hui Huang**
(*University of Waterloo, Canada*)
Time: Thursday 25.07., 11:00 - 11:30, Room HS 5

Abstract: Since 1990s, creative telescoping has become the cornerstone for evaluating definite sums of discrete special functions in computer algebra. Various algorithmic generalizations and improvements for this technique have been developed over the past two decades. At the present time, the reduction-based approach has gained the most support as it is both efficient in practice and has the important feature of being flexible to find a telescoper for a given function with or without construction of a certificate. There is, however, one handicap of this approach. That is, the approach can suffer from intermediate expression swell, especially in the part of a certificate, even if the final output ends up to be small.

In this talk, we present a new algorithm to compute minimal telescopers for rational functions in two discrete variables. This is the first step towards the long-term goal of developing fast creative telescoping algorithms for special functions that circumvent intermediate expression swell. As with the reduction-based approach, our algorithm also has the nice feature that the computation of a telescoper is independent of its certificate. Moreover, our algorithm uses a sparse representation of the certificate, which allows to be more easily manipulated and analyzed without knowing the precise expanded form. This sparse representation hides any potential exponential expression swell until the final (and optional) expansion. A complexity analysis, along with a Maple implementation, suggests that our algorithm has better theoretical and practical performances than the reduction-based approach when restricted to the rational case. This is joint work with M. Giesbrecht, G. Labahn and E. Zima.

Inverse Zeilberger's Problem

06.11 **Marko Petkovšek**
(*Faculty of Mathematics and Physics, University of Ljubljana, Slovenia*)
Time: Thursday 25.07., 11:30 - 12:00, Room HS 5

Abstract: Given a proper hypergeometric term $F(n,k)$, Zeilberger's *Creative Telescoping algorithm* finds a linear recurrence with rational coefficients satisfied by the sequence $s_n = \sum_{k=0}^{n} F(n,k)$. In the context of solving recurrence equations, we consider here what might be called the *inverse Zeilberger's problem*: given a homogeneous linear recurrence with polynomial coefficients, find its solutions representable as definite sums of a certain form.

As a first step in this direction, we provide an algorithm which, given a linear recurrence operator L with polynomial coefficients, and a product of binomial coefficients of the form

$$F(n,k) \;=\; \prod_{i=1}^{m} \binom{a_i n + b_i}{k}$$

where a_i are positive integers and b_i are arbitrary constants, returns a linear recurrence operator L' with rational coefficients such that for any sequence y of the form $y_n = \sum_{k=0}^{\infty} F(n,k)h_k$, we have $Ly = 0$ if and only if $L'h = 0$. This enables us to find all such solutions y where h belongs to a class of holonomic sequences with a known algorithm for converting from recursive to explicit representation.

Sparse polynomial interpolation with arbitrary orthogonal polynomial bases

06.12

Erdal Imamoglu
(*Department of Mathematics, North Carolina State University, USA*)
Time: Thursday 25.07., 12:00 - 12:30, Room HS 5

Abstract: An algorithm for interpolating a polynomial f from evaluation points whose running time depends on the sparsity t of the polynomial when it is represented as a sum of t Chebyshev polynomials of the first kind with non-zero scalar coefficients is given by Lakshman and Saunders [SIAM J. Comput., vol. 24, nr. 2 (1995)]; Kaltofen and Lee [JSC, vol. 36, nr. 3–4 (2003)] analyze a randomized early termination version which computes the sparsity t. Those algorithms mirror Prony's algorithm for the standard power basis to the Chebyshev basis of the first kind. An alternate algorithm by Arnold's and Kaltofen's [Proc. ISSAC 2015, Sec. 4] uses Prony's original algorithm for standard power terms. Here we give sparse interpolation algorithms for generalized Chebyshev polynomials, which include the Chebyshev bases of the second, third and fourth kind. Our algorithms also reduce to Prony's algorithm. If given on input an upper bound B for the sparsity t, our new algorithms deterministically recover the sparse representation in the first, second, third and fourth kind Chebyshev representation from exactly $t + B$ evaluations. Finally, we generalize our algorithms to bases whose Chebyshev recurrences have parametric scalars. We also show how to compute those parameter values which optimize the sparsity of the representation in the corresponding basis, similar to computing a sparsest shift.

This is a joint work with Erich L. Kaltofen (North Carolina State University and Duke University) and Zhengfe.