A family of entire functions connecting the Bessel function J_1 and the Lambert W function

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Time: Thursday 25.07., 11:00 - 11:30, Room SH 02

Abstract: At the 7th OPSFA, Copenhagen 2003, we posed the problem of determining the largest value $\alpha = \alpha^* > 0$ for which $f_\alpha(x) = e^\alpha - (1 + 1/x)^{\alpha x},\ x > 0$ is a completely monotonic function, and it was noticed that $1 \leq \alpha^* < 3$ and that graphs suggest that $\alpha^* > 2$. Numerical estimates given in [2] showed that $\alpha^* \approx 2.29965\,6443$.

We improve this result by combining Fourier analysis with complex analysis to find a family φ_α, $\alpha > 0$, of entire functions such that $f_\alpha(x) = \int_0^\infty e^{-sx}\varphi_\alpha(s)\,ds$ for $x > 0$.

We show that each function φ_α has an expansion in power series, whose coefficients are determined in terms of Bell polynomials. This expansion leads to several properties of the functions φ_α, which turn out to be related to the well known Bessel function J_1 when α is large, and to the Lambert W function when α is small.

On the other hand, by numerically evaluating the series expansion by using the alternating series test, we are able to show the behavior of φ_α as α increases from 0 to ∞ and to obtain a very precise approximation of α^* such that $\varphi_\alpha(s) \geq 0$, $s > 0$, or equivalently, such that f_α is completely monotonic precisely for $0 < \alpha \leq \alpha^*$. We find $\alpha^* \approx 2.29965\,64432\,53461\,30332$.

The talk is based on the manuscript [1].

[1] C. Berg, E. Massa and A. P. Peron, *A family of entire functions connecting the Bessel function J_1 and the Lambert W function.* ArXiv:1903.07574.

[2] E. Shemyakova, S. I. Khashin and D. J. Jeffrey, *A conjecture concerning a completely monotonic function*, Computers and Mathematics with Applications **60** (2010), 1360–1363.