Combinatorial description for the Hall-Littlewood expansion of unicellular LLT and chromatic quasi symmetric polynomials

Joint work with Seung Jin Lee

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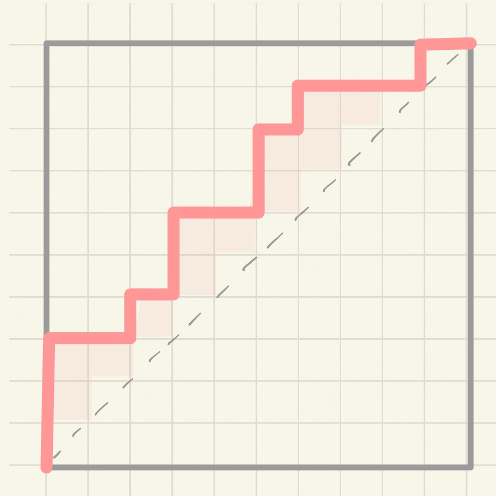
Outline

Dyck path



$$LLT_\gamma(x;q) = \sum \; \text{!} \; \tilde{H}_\mu(x;q)$$

$$X_\gamma(x;q) = \sum_\mu \; \text{!} \; P_\mu(x;q)$$

Natural unit interval order

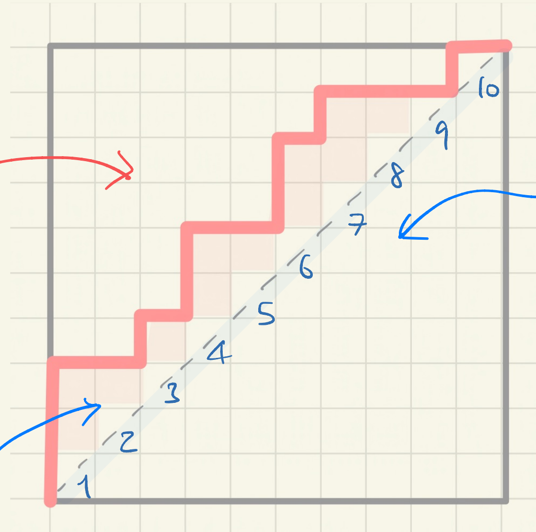
$\mathbf{m} = (m_1, m_2, \dots, m_{n-1})$; $m_1 \leq m_2 \leq \cdots \leq m_{n-1} \leq n$,

$m_i \geq i$, $\forall i$, positive integers

Def. The natural unit interval order $P(\mathbf{m})$ is the poset on $[n]$ with the order relation given by

$$i <_{P(\mathbf{m})} j \quad \text{if } i < n \text{ and } m_i + 1 \leq j \leq n.$$

Example. $m = (3, 3, 4, 6, 6, 8, 9, 9, 9)$



order relation in P

vertices of the incomparability graph $\mathrm{inc}(P)$

edges of the incomparability graph $\mathrm{inc}(P)$

Chromatic quasisymmetric functions

Def. $G = (V, E)$, a simple graph

The chromatic quasisymmetric function is

$$X_G(x;q) = \sum_{\kappa \text{ proper}} q^{\mathrm{asc}(\kappa)} x^{\kappa},$$

where

$$\mathrm{asc}(\kappa) = \left| \{ \{i,j\} \in E : i < j \text{ and } \kappa(i) < \kappa(j) \} \right|.$$

e-positivity conjecture

[Shareshian–Wachs, '2016]

If G is the incomparability graph of a natural unit interval order, then

$$X_G(x;q) \text{ is } \mathbf{e}\text{-positive}.$$

LLT Polynomials in a nutshell

- [Lascoux-Leclerc-Thibon, '1995] introduced a family of symmetric polynomials; ribbon symmetric ftns
- One motivation was to construct heighest weight vectors for the Fock space representation of $U_q(\widehat{sl}(n+1))$.
- [Bylund-Haiman] gave an alternative representation of LLT polynomials in which the functions are indexed by vectors of skew shapes.

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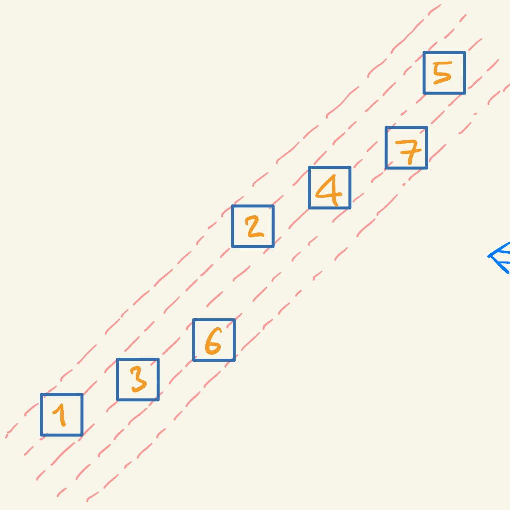
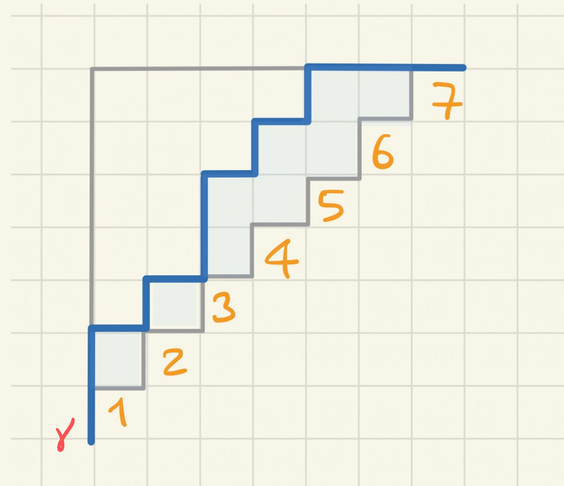
- One motivation was to construct heighest weight vectors for the Fock space representation of $U_q(\widehat{sl}(n+1))$.
- [Bylund-Haiman] gave an alternative representation of LLT polynomials in which the functions are indexed by vectors of skew shapes.

When these skew shapes are all single cells, then they are called "unicellular LLT"

Unicellular LLT polynomials



$\Longleftrightarrow$



Def. Given a Dyck path γ, the unicellular LLT polynomial is

$$LLT_\gamma(x;q) = \sum_{w \in \mathbb{Z}_{>0}^n} q^{\mathrm{inv}(w)} x^w ,$$

where $\mathrm{inv}(w) = \left| \{ (i,j) \in \pi_\gamma : w(i) > w(j) \} \right|$.

Curiouser and Curiouser

The chromatic quasisymmetric function;

$$X_G(x;q) = \sum_{\substack{w\in\mathbb{Z}^n_{>0}\\ \text{proper}}} q^{\mathrm{inv}(w)} x^w ,$$

where $\mathrm{inv}(\pi_G, w) = |\{(i,j) : (i,j)\in \mathrm{Area}(\pi_G),\ w_i > w_j\}|$.

VS

The unicellular LLT polynomial;

$$\mathrm{LLT}_\gamma(x;q) = \sum_{w\in\mathbb{Z}^n_{>0}} q^{\mathrm{inv}(w)} x^w .$$

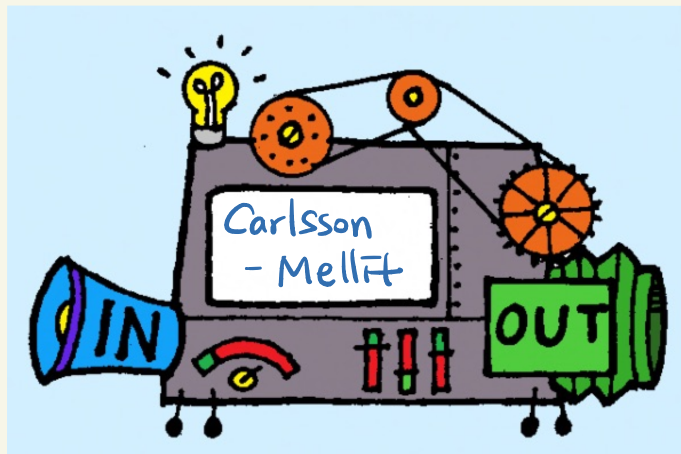
Voilà!

Prop. [Carlsson–Mellit, '2018]

$$X_\gamma(x;q) = (q-1)^{-n}\, \mathrm{LLT}_\gamma[(q-1)X;q]$$

Hall-Littlewood expansions

$$LLT_\gamma(x;q) = \sum_\mu q^{-|\lambda|} d_{\gamma\mu}(q) (q-1)^{n-\ell(\mu)} \tilde{H}_\mu(x;q)$$



Hall-Littlewood polynomials

$$X_\gamma(x;q) = \sum_\mu q^{\binom{n-1}{2}-1-n(\mu)+\ell(\mu)} d_{\gamma\mu}(q^{-1}) \prod_i [m_i(\mu)]_q! \, P_\mu(x;q)$$

Hall-Littlewood coefficients

Thm. [Lee-Y. '2022+]

"Rook polynomial"

$$X_\gamma(x;q) = \sum_\mu q^{\mathrm{area}(\gamma)-n(\mu)}\, r_{\lambda(\gamma),\mu}(q) \prod_i [m_i(\mu)]_q!\; P_\mu(x;q),$$

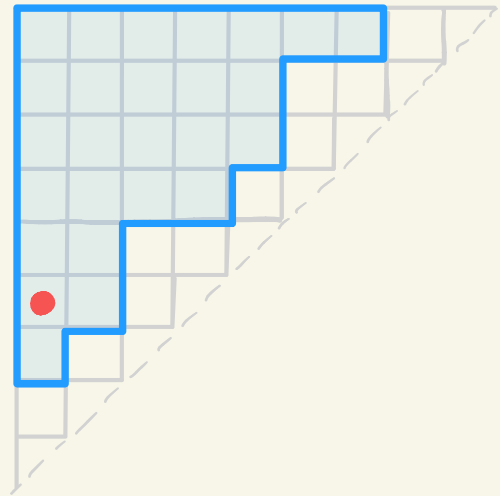
where

$n(\mu) = \sum_i (i-1)\mu_i$ and $m_i(\mu) = \#\, i\text{'s in } \mu$.

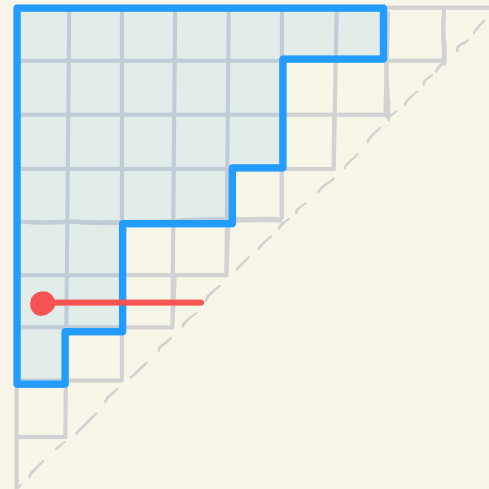
Also,

$$LLT_\gamma(x;q) = \sum_\mu q^{\ell(\mu)-n}\, r_{\lambda(\gamma),\mu}(q^{-1})\,(q-1)^{n-\ell(\mu)}\, \tilde{H}_\mu(x;q).$$

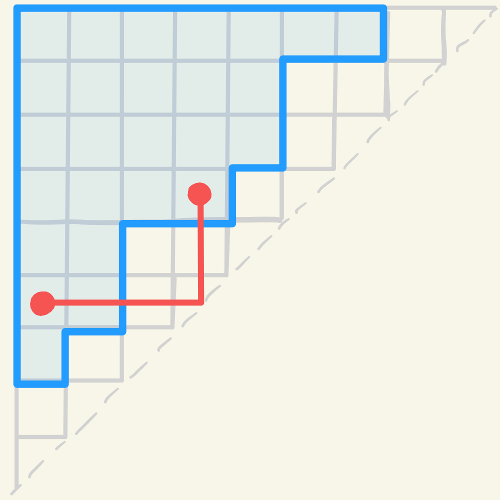
Linked rook placements



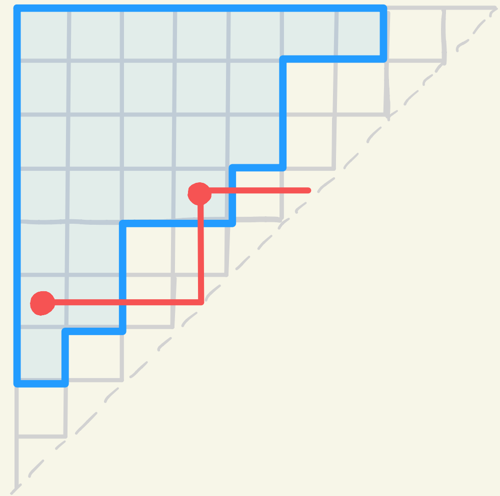
Linked rook placements



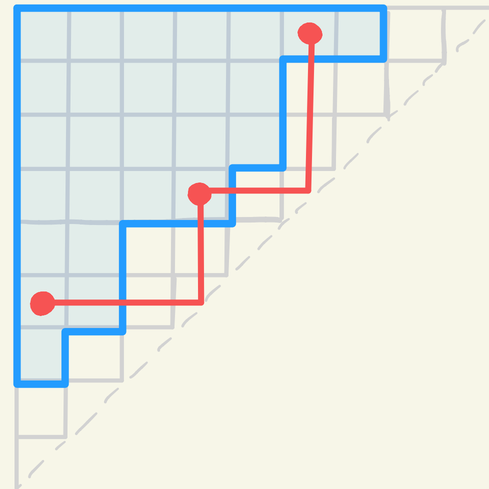
Linked rook placements



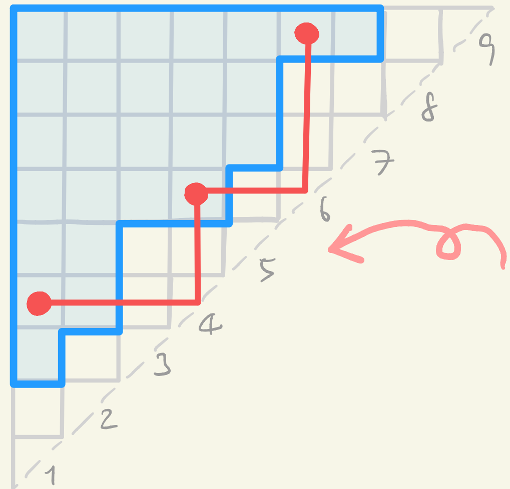
Linked rook placements



Linked rook placements



Linked rook placements



A linked rook on the cells

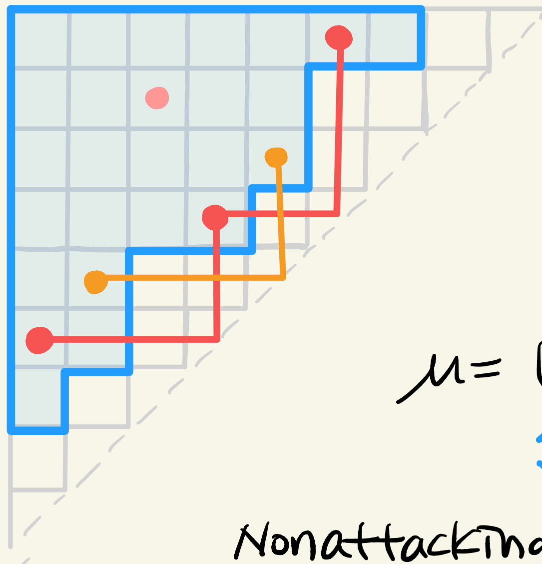
$(1,4) \to (4,6) \to (6,9)$

of length 3

$\longleftrightarrow$ A subset

$\{1, 4, 6, 9\}$

Linked rook placements of type μ

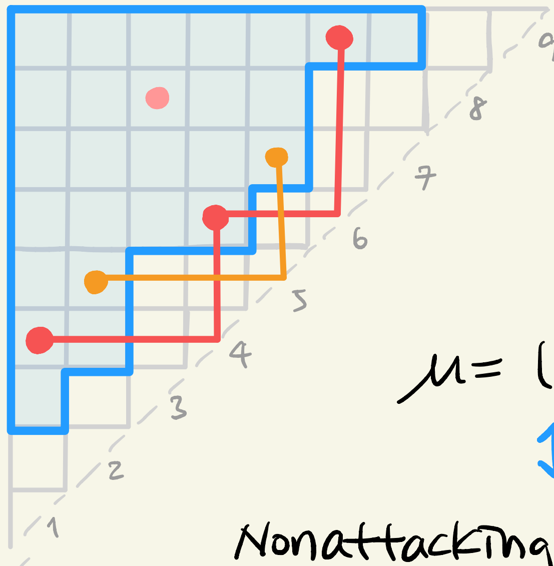


$\mu = (4, 3, 2)$

$\Updownarrow$

Nonattacking linked rook placements of length 3, 2, 1.

Linked rook placements of type μ



$\mu = (4, 3, 2)$

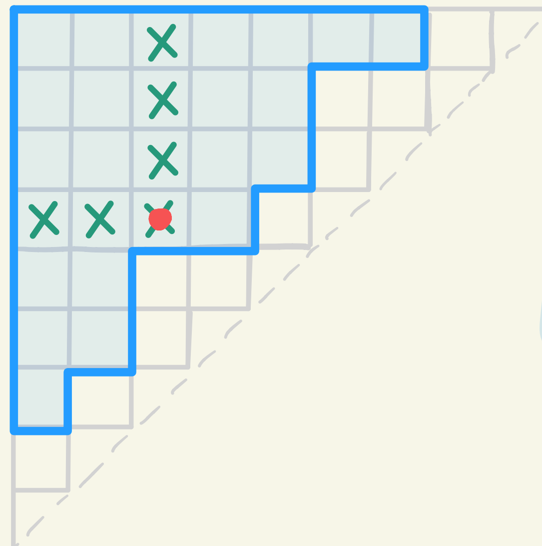
$\Updownarrow$

Nonattacking linked rook placements of length 3, 2, 1.

$\longleftrightarrow$ Set partition $\{1, 4, 6, 9\}, \{2, 5, 7\}, \{3, 8\}$

Rook cancellation

Rule 1 Every rook cancels the cells above.

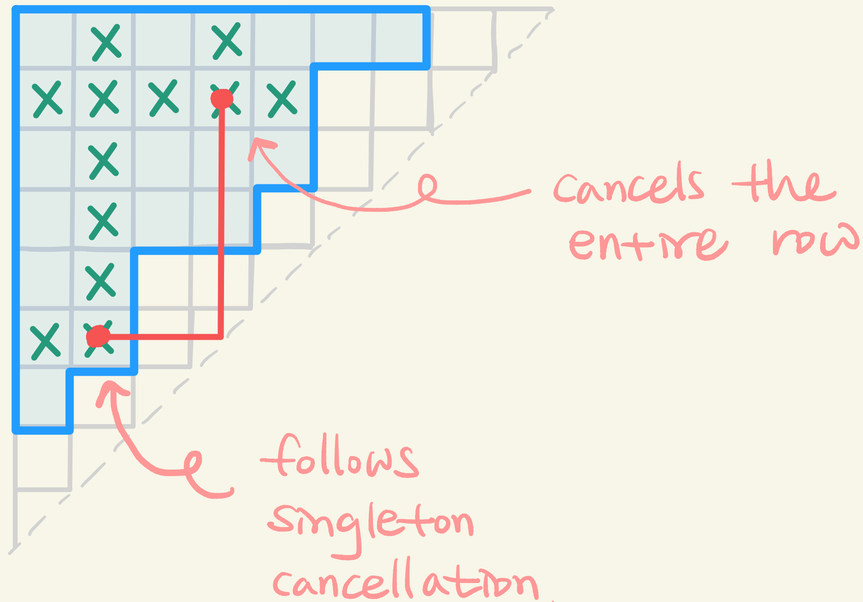


Rule 2

Singletons cancel the cells to the left.

Rook cancellation

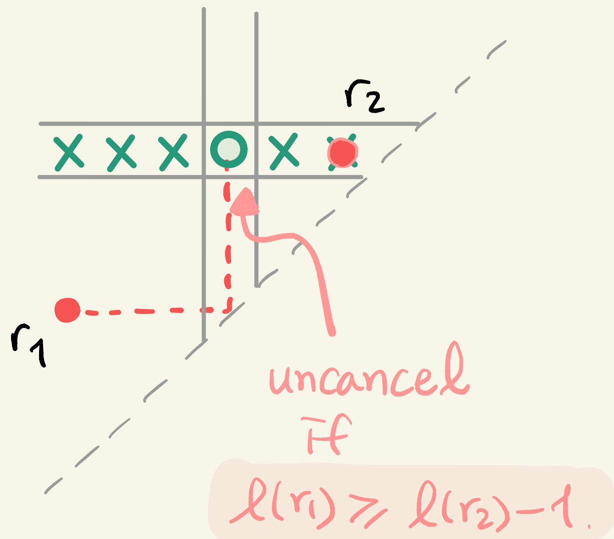
Rule 3



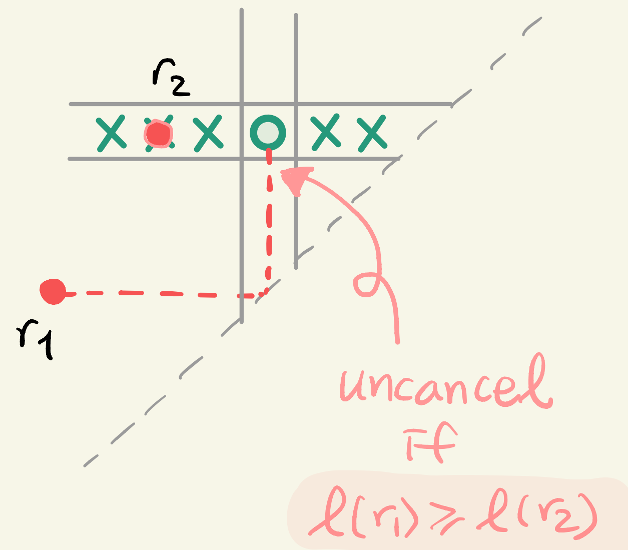
Rook cancellation

Rule 4 ; uncancellation

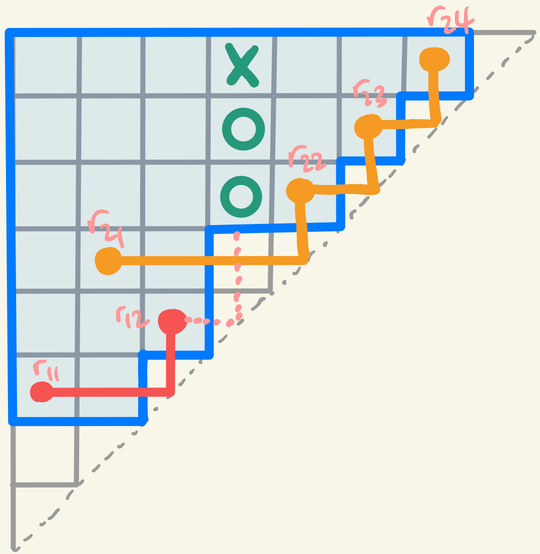
(a)

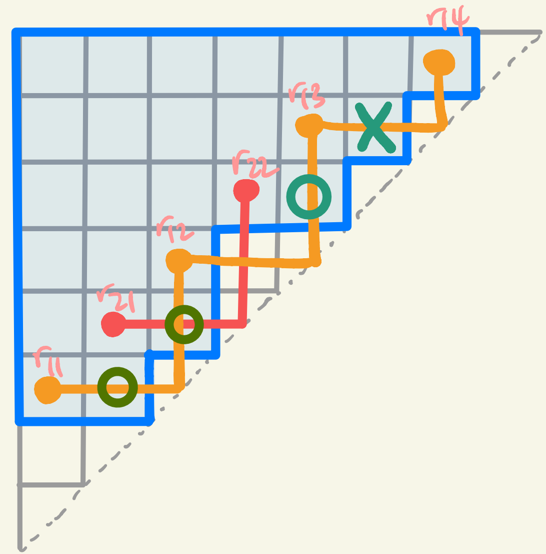


(b)



Example.





Rook polynomial of type μ

$$r_{\lambda,\mu}(q) = \sum_{P} q^{\mathrm{unc}(P)},$$

where the sum is over all the possible linked rook placements P of type μ and $\mathrm{unc}(P)$ is the number of uncancelled cells.

Hall-Littlewood coefficients

"Rook polynomial"

Thm. [Lee-Y. '2022$^+$]

$$X_\gamma(x;q) = \sum_\mu q^{\mathrm{area}(\gamma)-n(\mu)}\, r_{\lambda(\gamma),\mu}(q) \prod_i [m_i(\mu)]_q!\, P_\mu(x;q),$$

where

$n(\mu) = \sum_i (i-1)\mu_i$ and $m_i(\mu) = \#\, i$'s in μ.

Also,

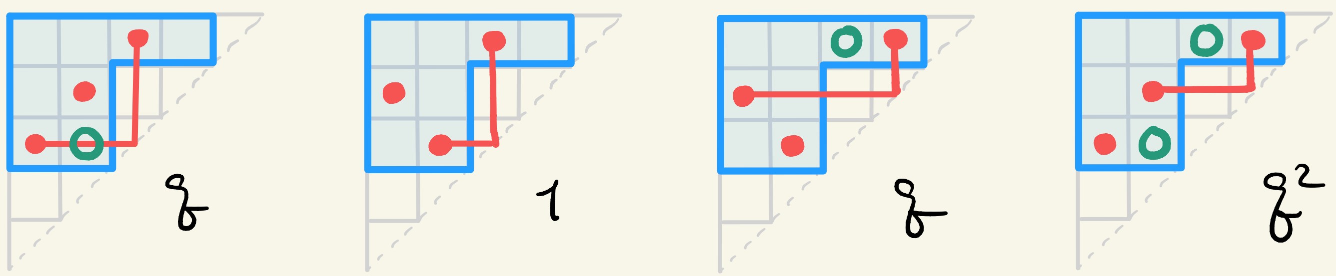
$$LLT_\gamma(x;q) = \sum_\mu q^{\ell(\mu)-n}\, r_{\lambda(\gamma),\mu}(q^{-1})\,(q-1)^{n-\ell(\mu)}\, \tilde{H}_\mu(x;q).$$

Example. $\gamma = (2,2,4,4,5)$ column heights

The $P_{(3,2)}(x;q)$ coefficient of $X_\gamma(x;q)$ is

$$q^2 + 2q + 1.$$

- $\mathrm{area}(\gamma) - n(\mu) = 2 - 2 = 0$
- $r_{\lambda(\gamma),\mu}(q)$;



Remark.

- The Schur expansion of the chromatic quasisymmetric function is well-known [Shareshian-Wachs, '2016]:

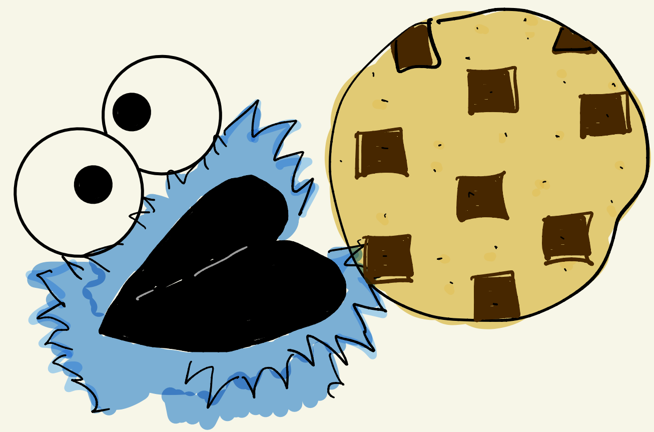
$$X_\gamma(x;q)=\sum_\lambda\Big(\sum_{\substack{T:\,P\text{-tab.}\\ sh(T)=\lambda}} q^{\mathrm{inv}(T)}\Big)\, s_\lambda(x).$$

- $$s_\lambda(x)=\sum_\mu K_{\lambda\mu}(q)\,P_\mu(x;q),$$

  where $$K_{\lambda\mu}(q)=\sum_{T\in SSYT(\lambda,\mu)} q^{ch(T)}.$$

- Linked rook placements

  $\overset{?}{\longleftrightarrow} \{(P\text{-tab.},\ SSYT)\}$



Thank you for your attention!