Phase transitions of composition schemes: Mittag-Leffler and mixed Poisson distributions

AEC Conference (TU Wien)

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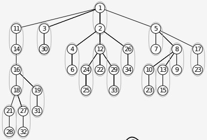
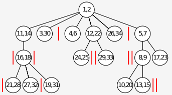
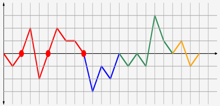
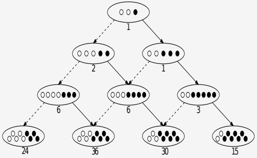
Phase transitions of composition schemes: Mittag-Leffler and mixed Poisson distributions, arXiv:2103.03751, submitted.

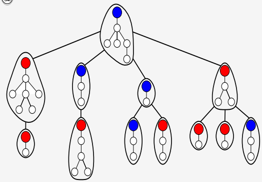
Combinatorial structures

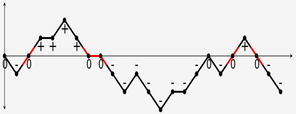


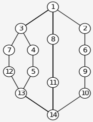


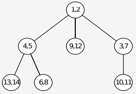


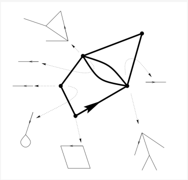


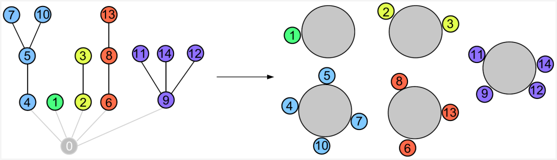












Frequent observation

Combinatorial structure = assemblage of basic building blocks

- random walks
- Pólya urns
- Galton–Watson processes
- trees
- permutations
- random mappings
- set partitions
- integer partitions
- tilings
- graphs
- maps
- ...

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A composition scheme for generating functions

$$F(z) = G\big(H(z)\big)M(z)$$

Let ρ_G and ρ_H be the radii of convergence of $G(z)$ and $H(z)$, resp. Then, the composition scheme is *critical* if $H(\rho_H) = \rho_G$ and $\rho_M \geq \rho_H$.

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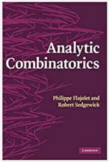
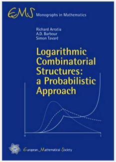
Examples:

- Bicoloured supertrees: $F(z) = C(2zC(z))$
- Factorization of walks: $W(z) = \frac{1}{1-A(z)}M(z)$





Goal 1: Analyse $F(z, u) = G(uH(z))M(z)$

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Number of $\mathcal{H}$-components: Define the discrete r.v. X_n of the *core size*:

$$\mathbb{P}\{X_n = k\} = \frac{[z^n u^k]F(z,u)}{[z^n]F(z,1)}$$

Note that $H(z)$ has typically the following singular expansion

$$H(z) = \tau_H + c_H\left(1 - \frac{z}{\rho_H}\right)^{\lambda_H} + \dots.$$

$\Rightarrow$ the asymptotic behaviour of $\mathbb{P}\{X_n = k\}$ depends on the *singular exponent* λ_H!

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Limit law of X_n related to certain distributions:

- $\lambda_H < 0$: scheme *not* critical as $H(z)$ diverges at $z = \rho_H$
  (called supercritical, typically Gaussian)
- $0 < \lambda_H < 1$: generalized Mittag-Leffler distribution (this talk!)
  ($\lambda_H = 1/2$, $M(z) = 1$: Rayleigh distribution)
- $1 < \lambda_H < 2$: related to stable laws of parameter λ_H
  ($\lambda_H = 3/2$, $M(z) = 1$: map-Airy distribution
  [Banderier, Flajolet, Schaeffer, Soria 01], [Stufler 22])
- $\lambda_H > 2$: related to Gaussian

Goal 2: Analyse $F_j(z,v) = G(H(z) - (1-v)h_j z^j)M(z)$

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Profile: Number of $\mathcal{H}$-components of given size j

Let $H(z) = \sum_{n\geq 0} h_j z^n$ and define the discrete random variable $X_{n,j}$:

$$\mathbb{P}\{X_{n,j} = k\} = \frac{[z^n v^k]F_j(z,v)}{[z^n]F_j(z,1)}$$

- $X_{n,j}$ naturally refines X_n:

$$\sum_{j\in\mathbb{N}} X_{n,j} = X_n.$$

- leads to *mixed Poisson distributions (also in this talk!)*

Main results

Three different regimes

Our model:

$$F(z,u) = G\big(uH(z)\big) \cdot M(z),$$

for $F/G/H/M$ analytic at the origin, with nonnegative coefficients, and singular exponents $\lambda_F/\lambda_G/\lambda_H/\lambda_M$, such that $0 < \lambda_H < 1$, $H(\rho_H) = \rho_G$, and $\rho_M = \rho_H$.
For example:

$$F(z) = \tau_F + c_F(1 - z/\rho_F)^{\lambda_F} + \dots$$

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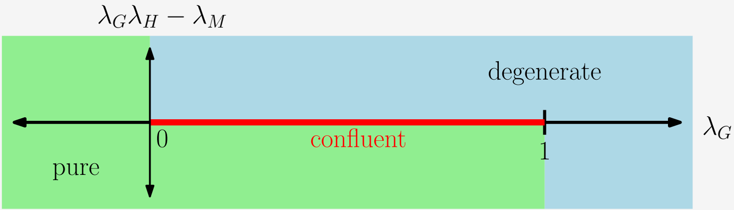

Composition scheme: pure case

The beta-Mittag-Leffler distribution $\mathrm{BML}(\alpha, \theta, \beta)$ has the density:

$$f(x) = \frac{\Gamma(\theta+\beta)}{\Gamma(\theta/\alpha)} \sum_{j\geq 0} \frac{(-1)^j}{j!\Gamma(\beta - j\alpha)} x^{\theta/\alpha + j - 1}.$$

Remark: $\mathrm{BML}(\alpha, \theta, \beta) \stackrel{d}{=} \mathrm{ML}(\alpha, \theta) \cdot \mathrm{Beta}(\theta, \beta)^{\alpha}$.

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Theorem

In a pure critical composition scheme

$$F(z,u) = G\big(uH(z)\big)M(z),$$

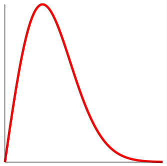
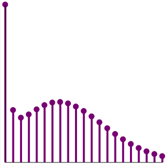
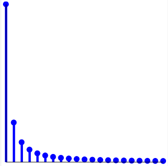
the core size X_n converges in distribution and moments to a beta-Mittag-Leffler:

$$\frac{X_n}{\kappa \cdot n^{\lambda_H}} \xrightarrow[m]{d} \mathrm{BML}(\alpha,\theta,\beta),$$

where $\alpha = \lambda_H, \quad \theta = -\lambda_G \lambda_H, \quad \beta = -\min(0,\lambda_M), \quad \kappa = \frac{\tau_H}{-c_H}$.

Moreover, we have a local limit theorem $\mathbb{P}\{X_n = x \cdot \kappa n^{\lambda_H}\} \sim \frac{1}{\kappa n^{\lambda_H}} \cdot f(x)$.

The three different regimes

Singular exponent	$\lambda_M > \lambda_G \lambda_H$ (pure scheme)	$\lambda_M = \lambda_G \lambda_H$ (confluent scheme)	$\lambda_M < \lambda_G \lambda_H$ (degenerate scheme)
Limit law	continuous (BML)	linear combination $(\mathrm{ML} + \mathcal{B})$	discrete (Boltzmann $\mathcal{B}$)
Example	 $X_n \sim C n^{\lambda_H} \mathrm{BML}$	 $X_n \sim \mathrm{LinComb}(n^{\lambda_H}\,\mathrm{ML}, \mathcal{B})$	 $\mathbb{P}\{X_n = k\} \sim \frac{g_k \rho_G^k}{G(\rho_G)}$

Bimodal case for confluent scheme:

1. first mode: small k (discrete Boltzmann)
2. second mode: larger $k \approx n^{\lambda_H}$ (continuous Mittag-Leffler)

Refined scheme

Theorem

Consider a size-refined pure critical composition scheme

$$F_j(z,v) = G\big(H(z) - (1-v)h_j z^j\big)M(z),$$

with $j \in \mathbb{N}$. *Let* $\xi_{n,j} = \frac{\rho_H^j}{-c_H} h_j n^{\lambda_H}$.

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1. $j \ll n^{\frac{\lambda_H}{1+\lambda_H}}$: *we have* $\xi_{n,j} \to +\infty$ *and*

$$\frac{X_{n,j}}{\xi_{n,j}} \xrightarrow[m]{d} X, \qquad \text{with} \qquad X \stackrel{d}{=} \mathrm{BML}(\alpha, \theta, \beta)$$

is beta-Mittag-Leffler with $\alpha = \lambda_H$, $\theta = -\lambda_G\lambda_H$, *and* $\beta = -\min(0, \lambda_M)$.

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2. $j \sim r \cdot n^{\frac{\lambda_H}{1+\lambda_H}}$, $r \in (0,\infty)$: *we have* $\xi_{n,j} \to \xi$ *with* $\xi = r^{-\frac{\lambda_H}{1+\lambda_H}} \cdot \frac{1}{-\Gamma(-\lambda_H)}$ *and*

$$X_{n,j} \xrightarrow[m]{d} MPo(\xi X),$$

   is a mixed Poisson distribution with mixing distribution X.
   $\left(\mathbb{P}\{X_{n,j} = \ell\} = \frac{\xi^\ell}{\ell!} \int_{\mathbb{R}^+} X^\ell e^{-\xi X} dU \right)$

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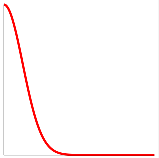
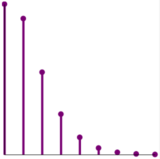
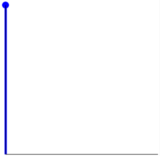
3. $j \gg n^{\frac{\lambda_H}{1+\lambda_H}}$: *we have* $\xi_{n,j} \to 0$ *and* $X_{n,j}$ *converges to a Dirac distr. at* 0.

Universal phase transition for the profile

Scale	$j \ll n^{\frac{\lambda_H}{1+\lambda_H}}$	$j = \Theta\left(n^{\frac{\lambda_H}{1+\lambda_H}}\right)$	$j \gg n^{\frac{\lambda_H}{1+\lambda_H}}$
Limit law	continuous (BML)	discrete (MPo(ξ BML))	discrete (Dirac)
Example			
	$X_{n,j} \sim C\, h_j \rho_H^j n^{\lambda_H}$ BML	$X_{n,j} \sim$ MPo(ξ BML)	$\mathbb{P}\{X_{n,j} \geq 1\} \sim 0$

1. For large n there are many small ($j \ll n^{\frac{\lambda_H}{1+\lambda_H}}$), some giant ($j \sim r n^{\frac{\lambda_H}{1+\lambda_H}}$), and no super-giant ($j \gg n^{\frac{\lambda_H}{1+\lambda_H}}$) $\mathcal{H}$-components of size j.
2. Universality of the window $\Theta(n^{1/3})$: ubiquitous square-root behaviour ($\lambda_H = \frac{1}{2}$) $\Rightarrow$ universality of the window $j = \Theta\left(n^{\frac{\lambda_H}{1+\lambda_H}}\right) = \Theta(n^{1/3})$.

Applications

Applications

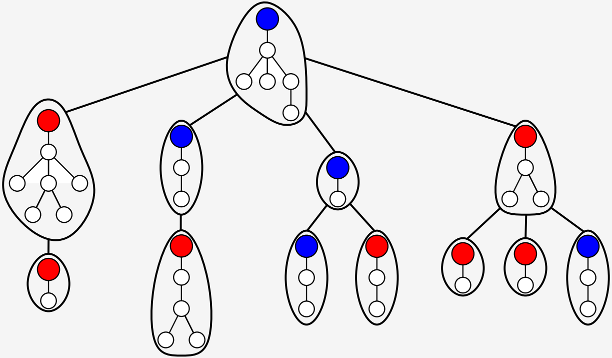
1. Core size of **supertrees**
2. Returns to zero in **walks and bridges** with drift zero
3. Initial returns in **coloured bridges**
4. Sign changes in **Motzkin walks**
5. Table sizes in the **Chinese restaurant process**
6. Compositions in **balanced triangular urn models**
7. Root degree and branching structure in **bilabelled increasing trees**

See our paper for full details extending/unifying works of
[Drmota, Soria 97], [Banderier, Flajolet, Schaeffer, Soria 01], [Janson 06 and 10], [Pitman 06], [Flajolet, Dumas, Puyhaubert 06], [Kuba, Panholzer 06], [James 15], [Goldschmidt, Haas, Sénizergues 20], . . .

Ex. 1: Bicoloured supertrees

Composition scheme: $F(z,u) = C\big(u \cdot 2zC(z)\big)$

where $C(z) = \frac{1-\sqrt{1-4z}}{2}$ is the generating function of plane trees.



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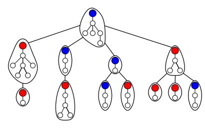


Theorem

The core size X_n in supertrees of size n has factorial moments

$$\mathbb{E}(X_{\underline{n}}^{\underline{s}}) \sim n^{s/2} \cdot \mu_s, \qquad \mu_s = \frac{\Gamma(s-\frac{1}{2})\Gamma(-\frac{1}{4})}{\Gamma(-\frac{1}{2})\Gamma(\frac{s}{2}-\frac{1}{4})}.$$

Convergence in distribution and all moments to a generalized Mittag-Leffler:

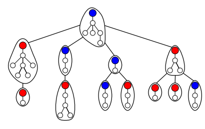
$$\frac{X_n}{n^{1/2}} \xrightarrow[m]{d} \mathrm{ML}\left(\frac{1}{2}, -\frac{1}{4}\right).$$

Moreover, we have the local limit theorem $\mathbb{P}\{X_n = x \cdot n^{1/2}\} \sim n^{-1/2} f(x)$.

Ex. 1: Bicoloured supertrees refined

Refined scheme: $F_j(z,v) = C\big(2zC(z) + (v-1)2c_{j-1}z^j\big)$

where $C(z) = \frac{1-\sqrt{1-4z}}{2}$ is the generating function of plane trees.



Theorem (Size-refined)

The number of coloured trees of size j in supertrees of size n has factorial moments of mixed Poisson type given by

$$\mathbb{E}(X_{n,j}^{\underline{s}}) = \xi_{n,j}^s \cdot \mu_s \cdot (1 + o(1)),$$

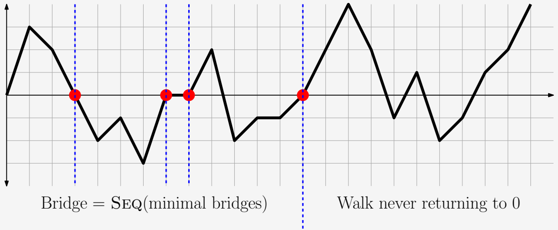
with $\xi_{n,j} = 2(\frac{1}{4})^{j-1}c_{j-1}n^{1/2}$ and mixing distribution $X = \mathrm{ML}\left(\frac{1}{2}, -\frac{1}{4}\right)$.

Furthermore, the random variable $X_{n,j}$ possesses the three distinct asymptotic régimes (ML, MPo, Dirac), with a phase transition at $j = \Theta(n^{1/3})$.

Ex. 2: Returns to zero in walks

Walk: Sequence of vectors $(v_1, \ldots, v_n) \in \mathcal{S}^n$

- Step set: $\mathcal{S} = \{s_1, \ldots, s_m\} \subset \mathbb{Z}$ with weights $\{p_1, \ldots, p_m\}$
- Step polynomial $P(u) = \sum_{i=1}^{m} p_i u^{s_i}$ $\Rightarrow$ drift 0: $P'(1) = 0$



- Walk "=" initial bridge $B(z)$ + final walk $M(z) = \frac{W(z)}{B(z)}$ (not returning to 0)
- Bridge contains all returns to zero
- Decompose bridge into a sequence of "minimal bridges" $B(z) = \frac{1}{1-A(z)}$

$$\Rightarrow \qquad W(z,u) = \frac{1}{1-uA(z)} \frac{W(z)}{B(z)}$$

Ex. 2: Profile of returns to zero

Corollary (Size-refined counting)

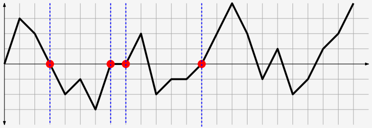
Let $X_{n,j}$ be the number of distance-j-zeroes in walks (bridges) with zero drift of length n. Then, $X_{n,j}$ has factorial moments of mixed Poisson type

$$\mathbb{E}(X_{n,j}^{\underline{s}}) = \xi_{n,j}^{s} \cdot \mathbb{E}(X^s)\,(1 + o(1))\,,$$

with $\xi_{n,j} = \sqrt{\frac{P(1)}{2P''(1)}}\frac{h_j}{P(1)^j} \cdot n^{1/2}$, *where X is given by*

$$X = \begin{cases} \mathit{Halfnormal}(\sigma) & \text{for walks,} \\ \mathit{Rayleigh}(\sigma) & \text{for bridges,} \end{cases} \qquad \sigma = \sqrt{\frac{P(1)}{P''(1)}}.$$

Furthermore, the random variable $X_{n,j}$ possesses our three distinct asymptotic régimes (BML, *MPo, Dirac*), *with a phase transition at* $j = \Theta(n^{1/3})$.



Conclusion: automatic limit laws for schemes!

Composition scheme	**Symbolic form**	**Limit law**
Ordinary	$F(z,u) = G\big(uH(z)\big)$	generalized Mittag-Leffler
Extended	$F(z,u) = M(z)G\big(uH(z)\big)$	beta-Mittag-Leffler and Boltzmann distribution
Cyclic	$F(z,u) = -\log\Big(1 - uH(z)\Big)$	Mittag-Leffler
Multivariate extended	$F(z,\mathbf{u}) = M(z)\prod_{\ell=1}^{m} G_\ell\big(u_\ell H_\ell(z)\big)$	multivariate product distribution
Refined	$F(z,v) = M(z)G\big(H(z) - z^j h_j(1-v)\big)$	mixed Poisson type phase transition
Refined cyclic	$F(z,v) = -\log\Big(1-\big(H(z)-(1-v)h_j z^j/j!\big)\Big)$	mixed Poisson type phase transition
Multivariate size-refined	$F(z,\mathbf{v}) = M(z)\prod_{\ell=1}^{m} G_\ell\big(H_\ell(z) - z^{j_\ell} h_{\ell,j_\ell}(1-v_\ell)\big)$	mv. mixed Poisson type phase transition

Conclusion: automatic limit laws for schemes!

Composition scheme	**Symbolic form**	**Limit law**
Ordinary	$F(z,u) = G\big(uH(z)\big)$	generalized Mittag-Leffler
Extended	$F(z,u) = M(z)G\big(uH(z)\big)$	beta-Mittag-Leffler and Boltzmann distribution
Cyclic	$F(z,u) = -\log\Big(1 - uH(z)\Big)$	Mittag-Leffler
Multivariate extended	$F(z,\mathbf{u}) = M(z)\prod_{\ell=1}^{m} G_\ell\big(u_\ell H_\ell(z)\big)$	multivariate product distribution
Refined	$F(z,v) = M(z)G\big(H(z) - z^j h_j(1-v)\big)$	mixed Poisson type phase transition
Refined cyclic	$F(z,v) = -\log\Big(1-\big(H(z)-(1-v)h_j z^j/j!\big)\Big)$	mixed Poisson type phase transition
Multivariate size-refined	$F(z,\mathbf{v}) = M(z)\prod_{\ell=1}^{m} G_\ell\big(H_\ell(z) - z^{j_\ell} h_{\ell,j_\ell}(1-v_\ell)\big)$	mv. mixed Poisson type phase transition

Dankeschön!

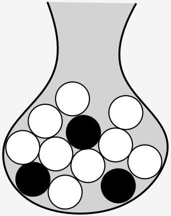
Bonus

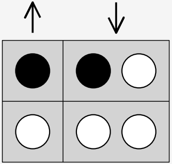
Ex. 3: Balanced triangular Pólya urns

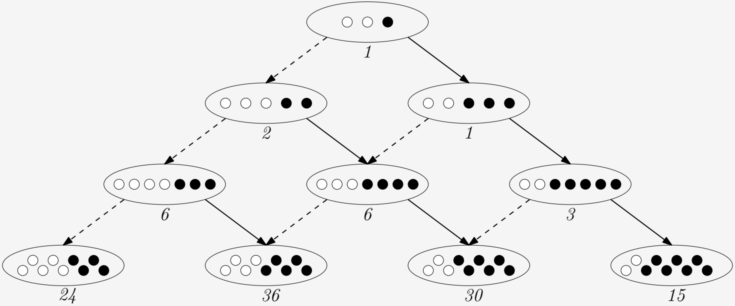
Replacement matrix

Let $\alpha, \beta > 0$, $\sigma = \delta = \alpha + \beta$.

$$\begin{array}{c} \\ \circ \\ \bullet \end{array} \begin{array}{c} \begin{array}{cc} \circ & \bullet \end{array} \\ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ \circ \\ \bullet \end{array} \begin{array}{c} \begin{array}{cc} \circ & \bullet \end{array} \\ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \end{array}$$







Limit law for balanced triangular Pólya urns

Problem 1.15. [Janson 06]

Find better descriptions of the limits of triangular Pólya urns.

- Closed form of the moments known [Theorem 1.7, Janson 06]
- For $b_0 > 0$ and $w_0 = 0$ (or β) Janson observed a moment-tilted stable law

History generating function [Flajolet, Dumas, Puyhaubert 06]:

$$F(z,u) = u^{w_0}(1-\sigma z)^{-b_0/\sigma}\left(1 - u^{\alpha}\left(1-(1-\sigma z)^{\alpha/\sigma}\right)\right)^{-w_0/\alpha}.$$

Corollary

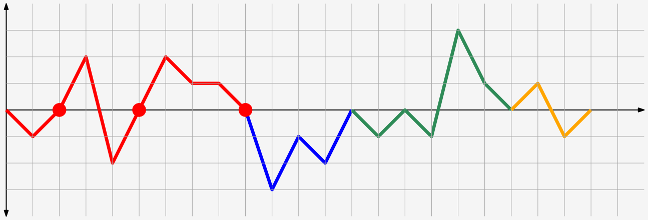
Let $\mathcal{W}_n$ be the rv for the number of white balls in a balanced triangular urn with initially $w_0 > 0$ white and $b_0 \geq 0$ black balls. Then, we have a convergence in distr., with convergence of all moments, to a beta-Mittag-Leffler distr.

$$\frac{\mathcal{W}_n}{\alpha n^{\alpha/\sigma}} \xrightarrow[m]{d} \mathrm{BML}\left(\frac{\alpha}{\sigma}, \frac{w_0}{\alpha}, \frac{b_0}{\alpha}\right).$$

Same limit for urns with noninteger weights [Goldschmidt, Haas, Sénizergues 20]

Ex. 4: Initial returns in coloured walks with zero drift

A 4-coloured bridge, with all its initial returns to zero marked by red dots:



Generating functions for m-colored bridges and walks:

$$B_m(z,u) = \left(\frac{1}{1-uA(z)} - 1\right)(B(z)-1)^{m-1}$$

$$W_m(z,u) = (1 + B_m(z,u))\frac{W(z)}{B(z)}$$

⇒ apply our blackbox theorems!

Corollary

The random variable X_n counting the number of initial returns in a m-coloured walk (resp. bridge) of length n satisfies

$$\mathbb{E}(X_n^{\underline{s}}) \sim n^{s/2} \left(\frac{\sigma}{\sqrt{2}}\right)^n \mu_s, \quad \sigma = \sqrt{\frac{P(1)}{P''(1)}}, \quad \mu_s = \begin{cases} \frac{\Gamma(s+1)\Gamma((m+1)/2)}{\Gamma((m+s+1)/2)}, & \text{for walks,} \\ \frac{\Gamma(s+1)\Gamma(m/2)}{\Gamma((m+s)/2)}, & \text{for bridges.} \end{cases}$$

The random variable $X_n/n^{1/2}$ converges in distribution with convergence of all moments to the product of a Rayleigh and a scaled beta distribution:

$$\frac{X_n}{n^{1/2}} \xrightarrow[m]{d} X, \qquad X \stackrel{d}{=} \text{Rayleigh}(\sigma) \cdot B^{1/2},$$

with independent random variables

$$\text{Rayleigh}(\sigma) \quad \text{and} \quad B = \begin{cases} \text{Beta}\left(\frac{1}{2}, \frac{m}{2}\right), & \text{for walks,} \\ \text{Beta}\left(\frac{1}{2}, \frac{m-1}{2}\right), & \text{for bridges.} \end{cases}$$

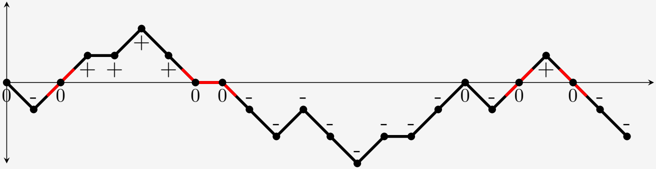
We have the local limit theorem $\mathbb{P}\{X_n = x \cdot n^{1/2}\} \sim n^{-1/2} \cdot f_X(x)$, where, for bridges

$$f_X(x) = \sqrt{\frac{2}{\pi\sigma^2}}\, \Gamma\left(\frac{m}{2}\right) e^{-\frac{x^2}{2\sigma^2}}\, U\left(\frac{m}{2} - 1, \frac{1}{2}, \frac{x^2}{2\sigma^2}\right),$$

where $U(a, b, x)$ is the confluent hypergeometric function of the second kind. For walks, one replaces m by $m+1$.

Ex. 5: Sign changes in Motzkin walks with zero drift

A Motzkin walk (i.e., step set $\mathcal{S} = \{-1, 0, 1\}$) with 4 sign changes marked in red.



Corollary (Size-refined counting)

Let $X_{n,j}$ be the number of distance-j-sign changes in Motzkin walks/bridges of length n with zero drift. Then, $X_{n,j}$ has factorial moments of mixed Poisson type

$$\mathbb{E}(X_{n,j}^{\underline{s}}) = \xi_{n,j}^{s} \cdot \mathbb{E}(X^{s})\,(1 + o(1))\,,$$

with $\xi_{n,j} = \frac{1}{2}\sqrt{\frac{P''(1)}{2P(1)}}\frac{h_j}{P(1)^j} \cdot n^{1/2}$ *and mixing distributions*

$$X \stackrel{d}{=} \begin{cases} \text{Halfnormal}(\sigma) & \text{for walks,} \\ \text{Rayleigh}(\sigma) & \text{for bridges,} \end{cases} \qquad \sigma = \frac{1}{2}\sqrt{\frac{P''(1)}{P(1)}}.$$

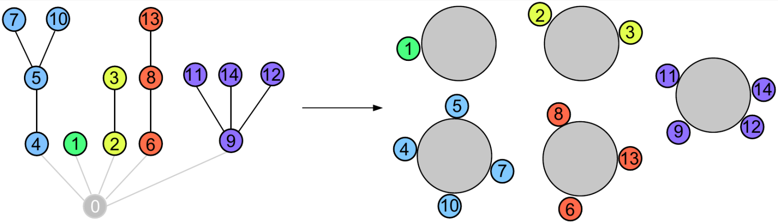
Furthermore, the r.v. $X_{n,j}$ (for walks and for bridges) possesses our three distinct asymptotic régimes (BML, *MPo, Dirac*), *with a phase transition at $j = \Theta(n^{1/3})$.*

Ex. 6: Tables in the Chinese restaurant process

- Studied by Aldous, Pitman, Yor, ...
- Discrete-time stochastic process: at time n a set partition of $\{1, \dots, n\}$
  - Start at time $n = 1$ with the partition $\{\{1\}\}$
  - Given partition $T = \{t_1, \dots, t_k\}$ of $[n]$ either add $n+1$ to $t_i \in T$ with prob.

$$\mathbb{P}\{n+1 \hookrightarrow t_i\} = \frac{|t_i| - \alpha}{n + \theta}, \quad 1 \le i \le k,$$

  - or as a new singleton block with remaining probability.



Embedding into plane-oriented recursive trees [Kuba, Panholzer 16]

$\Rightarrow$ Number of tables with j customers $\stackrel{d}{=}$ branches of size j

Theorem (Size-refined counting)

Let $a > 0$, $b > -1$. The random variable $X_{n,j}$ counting the number of tables with j customers in a Chinese restaurant process of parameter

$$\alpha = \frac{1}{1+a} \qquad \theta = \frac{b}{1+a},$$

with a total of $n-1$ customers possesses our three distinct asymptotic régimes, with a phase transition at $j = \Theta(n^{1/(a+2)})$:

1. *For $j \ll n^{\frac{1}{a+2}}$ we have $\xi_{n,j} = \frac{\alpha n^{\alpha}}{j}\binom{j-1-\alpha}{j-1} \to \infty$ and $\frac{X_{n,j}}{\xi_{n,j}}$ converges in distr. with convergence of all moments, to a generalized Mittag-Leffler distr.:*

$$\frac{X_{n,j}}{\xi_{n,j}} \xrightarrow[m]{d} X \qquad \text{with} \qquad X \stackrel{d}{=} \mathrm{ML}(\alpha, \theta).$$

2. *For $j \sim r \cdot n^{\frac{1}{a+2}}$, $r \in (0, \infty)$, we have $\xi_{n,j} \to \xi$, and the $X_{n,j}$ converges in distr. with convergence of all moments, to a mixed Poisson distr.:*

$$X_{n,j} \xrightarrow[m]{d} MPo(\xi X).$$

3. *For $j \gg n^{\frac{1}{a+2}}$, $\xi_{n,j} \to 0$, and $X_{n,j}$ converges to a Dirac distribution at 0.*

Pure case: simplifications

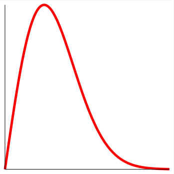
1. $\lambda_M \geq 0$ (which includes $F(z,u) = G(uH(z))$):

   $$X_1 \stackrel{d}{=} \mathrm{BML}(\lambda_H, -\lambda_G\lambda_H, 0) \stackrel{d}{=} \mathrm{ML}(\lambda_H, -\lambda_G\lambda_H)$$

   In particular, for $\lambda_G = -1$ and $\lambda_H = \frac{1}{2}$:

   $$X_1 \stackrel{d}{=} \text{Rayleigh}$$

   Sequence scheme [Drmota, Soria 97]



$$f_{X_1}(x) = \frac{x}{2} \exp\left(-\frac{x^2}{4}\right)$$

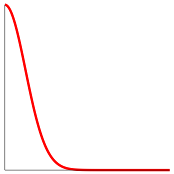
2. $\lambda_M < 0$, $\lambda_G = -1$, and $\lambda_H - \lambda_M = 1$:

   $$X_2 \stackrel{d}{=} \mathrm{BML}(\lambda_H, \lambda_H, 1-\lambda_H) \stackrel{d}{=} \mathrm{ML}(\lambda_H).$$

   In particular, for $\lambda_H = \frac{1}{2}$:

   $$X_2 \stackrel{d}{=} \text{Halfnormal}$$

   Sequence scheme [Wallner 20]



$$f_{X_2}(x) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{4}\right)$$

Note that $\lambda_G = -1$ "=" *sequences* of $\mathcal{H}$-components

Composition scheme: degenerate case

Theorem

In a degenerate critical composition scheme

$$F(z,u) = G\big(uH(z)\big)M(z)$$

the core size X_n converges for $0 < \lambda_G < 1$ and $\lambda_M < \lambda_G \lambda_H$ to a Boltzmann distribution:

$$\mathbb{P}\{X_n = k\} \to \mathbb{P}\{\mathcal{B}_G(\rho_G) = k\} = \frac{g_k \rho_G^k}{G(\rho_G)}.$$

The case $\lambda_G > 1$ is similar.

Definition (Boltzmann distribution $\mathcal{B}_G(x)$)

Let $G(z) = \sum_{n\geq 0} g_n z^n$ be a generating function and $x > 0$ inside the radius of convergence. Then, the *Boltzmann distribution* $\mathcal{B}_G(x)$ is defined by

$$\mathbb{P}\{X = n\} = \frac{g_n x^n}{G(x)}, \quad n \geq 0.$$

Composition scheme: confluent case

Theorem

In a confluent (i.e., $0 < \lambda_G < 1$ and $\lambda_M = \lambda_G \lambda_H$) ext. crit. comp. scheme

$$F(z,u) = G\big(uH(z)\big)M(z)$$

the core size X_n is a convex combination of a Boltzmann distribution $\mathcal{B}_G(\rho_G)$ and an asymptotically continuous random variable Z_n:

$$X_n \sim \mathrm{Be}(p)\cdot \mathcal{B}_G(\rho_G) + (1-\mathrm{Be}(p))\cdot Z_n, \qquad \frac{Z_n}{\kappa \cdot n^{\lambda_H}} \xrightarrow{d} \mathrm{ML}(\lambda_H, -\lambda_G\lambda_H),$$

where $p = \frac{c_M G(\rho_G)}{c_M G(\rho_G) + \tau_M c_G (-c_H/\rho_G)^{\lambda_G}}$, and indep. rv's $\mathrm{Be}(p)$, $\mathcal{B}_G(\rho_G)$, Z_n, and ML.

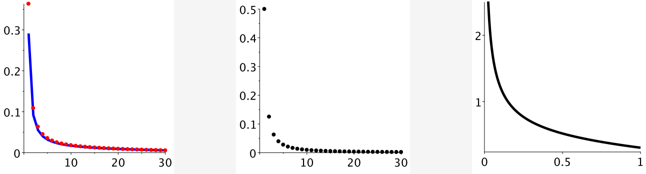


Figure: Core size in first part of pairs of supertrees: $\frac{1}{2}\,\mathcal{B}_C\left(\frac{1}{4}\right) + \frac{1}{2}\sqrt{n}\,\mathrm{ML}\left(\frac{1}{2}, -\frac{1}{4}\right)$.

(Generalized) Mittag-Leffler distribution

- A positive random var. S_α follows a **stable law of parameter** $\alpha \in (0,1)$ if

$$\mathbb{E}(e^{-tS_\alpha}) = e^{-t^\alpha}.$$

- A random variable M_α follows a **Mittag-Leffler distribution** ML(α) if

$$M_\alpha \stackrel{d}{=} (S_\alpha)^{-\alpha}.$$

$\Rightarrow$ Its MGF $\mathbb{E}(e^{xM_\alpha})$ is the Mittag-Leffler function $E_\alpha(x) = \sum_{k\geq 0} \frac{x^k}{\Gamma(1+\alpha k)}$.

Definition ([Pitman 06, James 15])

Let $\alpha \in (0,1)$ and $\theta > -\alpha$. Then, the **generalized Mittag-Leffler distribution** ML(α,θ) is uniquely defined by its moments

$$\mathbb{E}(X^s) = \frac{\Gamma\left(s+\frac{\theta}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma(\alpha s+\theta+1)\Gamma\left(\frac{\theta}{\alpha}+1\right)} = \frac{\Gamma\left(s+\frac{\theta}{\alpha}\right)\Gamma(\theta)}{\Gamma(\alpha s+\theta)\Gamma\left(\frac{\theta}{\alpha}\right)}.$$

- ML$(\alpha, 0) = M_\alpha$
- ML$(1/2, 0)$: *half-normal distribution* $|\mathcal{N}(0,\sigma^2)|$ of parameter $\sigma = \sqrt{2}$
- ML$(1/2, 1/2)$: *Rayleigh distribution* of parameter $\sqrt{2}$

Beta-Mittag-Leffler distribution

The distributions of *critical composition schemes* will be the **beta-Mittag-Leffler distributions** $\mathrm{BML}(\alpha, \theta, \beta)$ defined as

$$Z \stackrel{d}{=} Y \cdot B^{\alpha}$$

where $Y \stackrel{d}{=} \mathrm{ML}(\alpha, \theta)$ and $B \stackrel{d}{=} \mathrm{Beta}(\theta, \beta)$ are independent, such that $0 < \alpha < 1$, $\theta > 0$, and $\beta \geq 0$.

Lemma

The beta-Mittag-Leffler distribution $\mathrm{BML}(\alpha, \theta, \beta)$ *has the following moments*

$$\mathbb{E}(Z^s) = \frac{\Gamma\left(s + \frac{\theta}{\alpha}\right)\Gamma\left(\theta + \beta\right)}{\Gamma\left(\alpha s + \theta + \beta\right)\Gamma\left(\frac{\theta}{\alpha}\right)}.$$

One has the following identity

$$Z \stackrel{d}{=} \mathrm{ML}(\alpha, \theta)\,\mathrm{Beta}(\theta, \beta)^{\alpha} \stackrel{d}{=} \mathrm{ML}(\alpha, \theta + \beta)\,\mathrm{Beta}(\frac{\theta}{\alpha}, \frac{\beta}{\alpha}).$$

- distribution with moments of Gamma type [Janson 10]
- explicit representation of its density by integrals or hypergeometric functions

Mixed Poisson distribution

- First introduced for actuarial math./insurance modelling [Dubourdieu 39]
- studied by Lundberg under the name "compound Poisson processes"
- used in bacteriology [Neyman 39]
- unimodality properties [Masse, Theodorescu 05]
- tail asymptotics [Willmot, Lin 01]

Definition

Let X be a nonneg. random variable with cumulative distribution function U. Then, Y has a **mixed Poisson distribution with mixing distribution U** and scale parameter $\xi \geq 0$, if its probability mass function is given for $\ell \geq 0$ by

$$\mathbb{P}\{Y = \ell\} = \frac{\xi^\ell}{\ell!} \int_{\mathbb{R}^+} X^\ell e^{-\xi X} dU = \frac{\xi^\ell}{\ell!} \mathbb{E}(X^\ell e^{-\xi X}).$$

Notation: $Y \stackrel{d}{=} \mathrm{MPo}(\xi U)$ or $Y \stackrel{d}{=} \mathrm{MPo}(\xi X)$.

Important: $\mathbb{E}(Y^{\underline{s}}) = \xi^s \mathbb{E}(X^s), \quad s \geq 1.$