More Recent Progress on Cylindric Partitions

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The Algorithmic and Enumerative Combinatorics Conference

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Thanks

The talk is not over yet

Thanks

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There is a lot more to Cylindric Partitions

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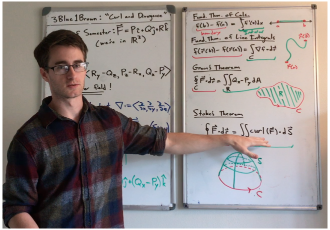
Corteel–Welsh recurrence and the idea behind is so fundamental that we can apply it to a large class of objects.
That's exactly what I did with Walter Bridges.



W. Bridges and A. K. Uncu, *Weighted Cylindric Partitions*, https://arxiv.org/abs/2201.03047.

Cylindric Partitions

A vector of k partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ for a *profile* $c = (c_1, c_2, \ldots, c_k)$ $(c_i \in \mathbb{Z}_{\geq 0})$ is called a *cylindric partition* if for every $i \in \{1, 2, \ldots, k-1\}$

$$\lambda_{i,j} \geq \lambda_{i+1, j+c_{i+1}}, \text{ and } \lambda_{k,j} \geq \lambda_{1, j+c_1},$$

where $\lambda_{i,j}$ is the j-th element of the i-th partition π_i.

$\pi = (\ (3,2,2,1), (4,3,3,1,1), (4,1,1,1)\ )$ is a cylindric partition of profile $(2,1,2)$ with largest part size 4 and total size 27.

							4	1	1	1
					3	2	2	1		
			4	3	3	1	1			
		4	1	1	1					
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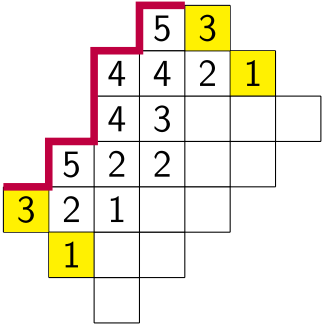
$$\begin{array}{cccccccccc}
 & & & & & & & 4 & 1 & 1 & 1 \\
 & & & & & 3 & 2 & 2 & 1 & & \\
 & & & 4 & 3 & 3 & 1 & 1 & & & \\
 & & 4 & 1 & 1 & 1 & & & & & \\
3 & 2 & 2 & 1 & & & & & & &
\end{array}$$

Cylindric Partitions - Equivalent Definition

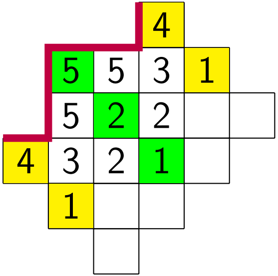
A *cylindric partition* (CP) of *width* h with *profile* $\delta = (\delta_1, \ldots, \delta_h) \in \{\pm 1\}^h$ is an $(h+1)$-tuple of integer partitions $(\lambda^0, \ldots, \lambda^h)$ such that $\lambda^0 = \lambda^h$ and $\lambda^{j-1} \succeq \lambda^j$ (resp. $\lambda^{j-1} \preceq \lambda^j$) if $\delta_j = -1$ (resp. if $\delta_j = 1$).



A cylindric partition of width 8, profile $(-1, 1, -1, 1, 1, -1, 1, -1)$, size 42.

Symmetric Cylindric Partitions

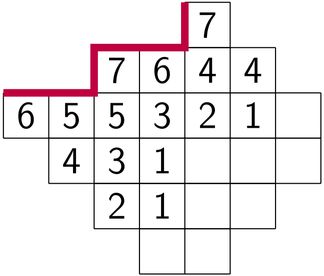
A *symmetric cylindric partition* λ is a cylindric partition of the form $(\lambda^h, \dots, \lambda^1, \lambda^0, \lambda^1, \dots, \lambda^h)$ with profile $\delta = (-\delta_h, \dots, -\delta_1, \delta_1, \dots, \delta_h)$.



A symmetric cylindric partition of width 6, profile $(-1, 1, 1, -1, -1, 1)$ and size 38.

Skew Double Shifted Plane Partitions

A *skew double shifted plane partition* (DSPP) of *width h* with *profile* $\delta = (\delta_1, \ldots, \delta_h) \in \{\pm 1\}^h$ is an $(h+1)$-tuple of integer partitions $(\lambda^0, \ldots, \lambda^h)$ such that $\lambda^{j-1} \succeq \lambda^j$ (resp. $\lambda^{j-1} \preceq \lambda^j$) if $\delta_j = -1$ (resp. if $\delta_j = 1$).



A DSPP of width 6, profile $(-1, -1, 1, -1, -1, 1)$ and size 61.

Generating Functions - Product Representations

Theorem (Borodin, 2007)

Let $\delta = (\delta_1, \dots, \delta_h)$ *be a profile and*

$$W_3(\delta) := \{h\} \cup \{j-i : 1 \le i < j \le h, \delta_i > \delta_j\} \cup \{h-(j-i) : 1 \le i < j \le h, \delta_i < \delta_j\}.$$

Then

$$\mathrm{CP}_\delta(q) := \sum_{\lambda \in \mathcal{CP}_\delta} q^{|\lambda|} = \prod_{k \in W_3(\delta)} \frac{1}{(q^k; q^h)_\infty}.$$

Weighted Generating Functions - Product Representations

Let

$$A_k := \sum_{j=0}^{k-1} a_j.$$

Theorem (Bridges, U 2022 - Slight Generalization of Han-Xiang's products)

Let $\delta = (\delta_1, \dots, \delta_h)$ be a profile, let $\mathbf{a} = (a_0, \dots, a_{h-1}) \in \mathbb{R}^h_{\geq 0}$ and

$$W_3^{\mathbf{a}}(\delta) := \{A_h\} \cup \{A_j - A_i : 1 \leq i < j \leq h, \delta_i < \delta_j\} \cup \{A_h - (A_j - A_i) : 1 \leq i < j \leq h, \delta_i > \delta_j\}.$$

If $0 \notin W_3^{\mathbf{a}}(\delta)$, then

$$\mathrm{CP}_\delta^{\mathbf{a}}(q) := \sum_{\lambda \in \mathcal{CP}_\delta} q^{|\lambda|_{\mathbf{a}}} = \prod_{k \in W_3^{\mathbf{a}}(\delta)} \frac{1}{(q^k; q^{A_h})_\infty}.$$

Weighted Generating Functions - Product Representations

Theorem (Bridges, U 2022 - Slight Generalization of Han-Xiang's products)

Let $\delta = (\delta_1, \dots, \delta_h)$ *be a profile, let* $\mathbf{a} = (a_0, \dots, a_h) \in \mathbb{R}^{h+1}_{\geq 0}$, *and*

$$\begin{aligned} W_1^{\mathbf{a}}(\delta) :=& \{A_{h+1}\} \cup \{A_i : \delta_i = -1\} \cup \{A_{h+1} - A_i : \delta_i = 1\}, \\ W_2^{\mathbf{a}}(\delta) :=& \{A_i + A_j : 1 \leq i < j \leq h,\ \delta_i = \delta_j = -1\} \\ & \cup \{2A_{h+1} - A_i - A_j : 1 \leq i < j \leq h,\ \delta_i = \delta_j = 1\} \\ & \cup \{2A_{h+1} - (A_j - A_i) : 1 \leq i < j \leq h,\ \delta_i < \delta_j\} \\ & \cup \{A_j - A_i : 1 \leq i < j \leq h,\ \delta_i > \delta_j\}. \end{aligned}$$

If $0 \notin W_1^{\mathbf{a}}(\delta) \cup W_2^{\mathbf{a}}(\delta)$, *then*

$$\mathrm{DSPP}^{\mathbf{a}}_{\delta}(q) := \sum_{\lambda \in \mathcal{DSPP}_\delta} q^{|\lambda|_{\mathbf{a}}} = \prod_{\substack{k \in W_1^{\mathbf{a}}(\delta) \\ \ell \in W_2^{\mathbf{a}}(\delta)}} \frac{1}{(q^k; q^{A_{h+1}})_\infty (q^\ell; q^{2A_{h+1}})_\infty}.$$

Weighted Generating Functions - Product Representations

How did we observe it?

Symmetric cylindric partitions can be viewed as weighted DSPPs—that is, for $\delta = (\delta_1, \dots, \delta_h)$ we have

$$\mathrm{SCP}_{(-\mathrm{rev}(\delta),\delta)}(q) = \mathrm{DSPP}_{\delta}^{(1,2,2,\dots,2,1)}(q),$$

where $rev(\delta)$ the profile delta written in reverse order.

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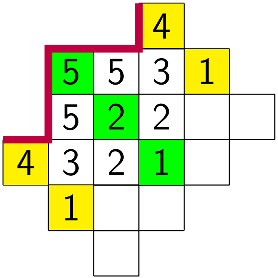

Finding Combinatorial Connections Using the Products

These weighted counts can be used to generate direct combinatorial connections.

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$$\mathrm{CP}^{(b_1,b_2-b_1,\dots,b_{r+1}-b_r)}_{(-1,-1,\dots,-1,1)}(q) = \frac{1}{(q^{b_1},q^{b_2},\dots,q^{b_r},q^{b_{r+1}};q^{b_{r+1}})_\infty},$$

where $0 < b_1 \leq b_2 \leq \cdots \leq b_{r+1}$.

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where $0<b_1\leq b_2\leq\dots\leq b_{r+1}$.

Or we can use more complicated profiles:

$$\mathrm{CP}^{(1,3,1)}_{(-1,-1,1)}(q)=\mathrm{CP}^{(1,3,1,5)}_{(-1,-1,1,1)}(q)=\mathrm{CP}^{(1,4,4,1,5)}_{(-1,1,-1,1,1)}(q)=\frac{1}{(q,q^4,q^5;q^5)_\infty},$$

$$\mathrm{CP}^{(2,2,1)}_{(-1,1,1)}(q)=\mathrm{CP}^{(2,2,3,3)}_{(-1,1,-1,1)}(q)=\mathrm{CP}^{(2,3,3,2,5)}_{(-1,1,-1,1,1)}(q)=\frac{1}{(q^2,q^3,q^5;q^5)_\infty}.$$

Finding Connections Using the Products

Or we can have mix match of families:

$$\mathrm{CP}^{(1,0,1,0,0)}_{(-1,-1,-1,1,1)}(q) = \mathrm{DSPP}^{(0,1,0)}_{(1,-1)}(q) = \frac{1}{(q;q)_\infty^3 (q;q^2)_\infty}$$

Finding Connections Using the Products

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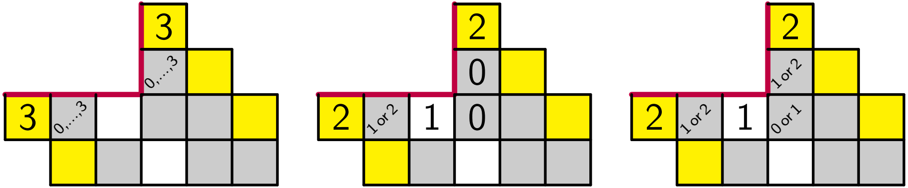
$$\mathrm{CP}^{(1,0,1,0,0)}_{(-1,-1,-1,1,1)}(q) = \mathrm{DSPP}^{(0,1,0)}_{(1,-1)}(q) = \frac{1}{(q;q)^3_\infty (q;q^2)_\infty}$$

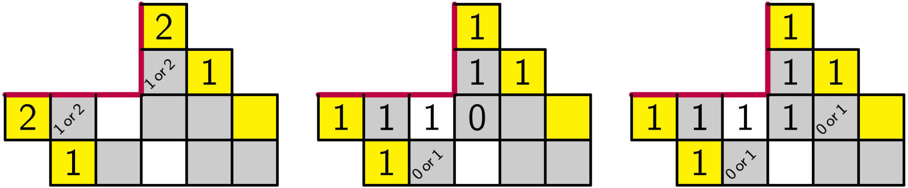
$$= 1 + 4q + 13q^2 + 36q^3 + 90q^4 + 208q^5 \dots$$

CPs:



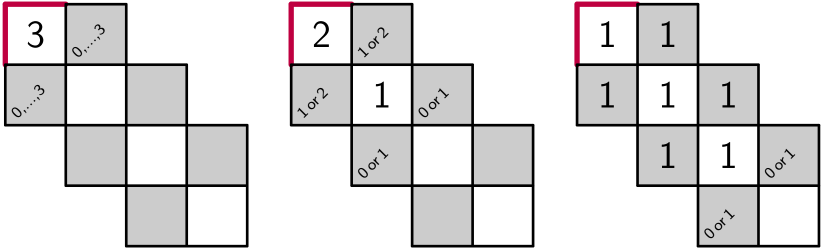
16 | 2 | 8

CPs:



4 | 2 | 4

DSPPs:



16 16 4

Corteel-Welsh style Functional Equations

q-Difference equations similar to the ones Corteel-Welsh defined on cylindric partitions directly carries over to weighted CPs, SCPs, and DSPPs.

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Proposition (Corteel–Welsh q-Difference Equations Reformulated)

Let $\delta = (\delta_1, \ldots, \delta_h)$ be a profile, and for convenience define $\delta_0 := \delta_h$. Define

$$I_\delta := \{0 \leq j \leq h-1 : (\delta_j, \delta_{j+1}) = (1, -1)\}.$$

For a subset $\emptyset \subsetneq J \subseteq I_\delta$, define a new profile $\sigma_J(\delta)$ by swapping the signs of (δ_j, δ_{j+1}) for $j \in J$. Then

$$\mathrm{CP}_\delta(z;q) = \sum_{\emptyset \subsetneq J \subseteq I_\delta} (-1)^{|J|-1} \frac{\mathrm{CP}_{\sigma_J(\delta)}\left(zq^{|J|};q\right)}{1 - zq^{|J|}}.$$

Corteel-Welsh style Functional Equations

Proposition

Let $\delta = (\delta_1, \ldots, \delta_h)$ *be a profile and let* $\mathbf{a} \in \mathbb{R}^h_{\geq 0}$. *We have*

$$\mathrm{CP}^{\mathbf{a}}_{\delta}(z;q) = \sum_{\emptyset \subsetneq J \subseteq I_\delta} (-1)^{|J|-1} \frac{\mathrm{CP}^{\mathbf{a}}_{\sigma_J(\delta)}\left(zq^{\sum_{j\in J} a_j}; q\right)}{1 - zq^{\sum_{j\in J} a_j}}.$$

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$$G^{\mathbf{a}}_{\delta}(z) := (zq;q)_\infty \mathrm{CP}^{\mathbf{a}}_{\delta}(z;q).$$

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The above recurrence becomes

$$G^{\mathbf{a}}_{\delta}(z) = \sum_{\emptyset \subsetneq J \subseteq I_\delta} (-1)^{|J|-1} (zq;q)_{\sum_{j\in J} a_j - 1}\, G^{\mathbf{a}}_{\sigma_J(\delta)}\left(zq^{\sum_{j\in J} a_j}\right).$$

Sum-Product Identities

With conventional weights

Theorem (Bridges, U 2022)

$$\mathrm{SCP}_{(-1,-1,1,1)}(z;q) = \frac{(-zq^4, zq^2; q^4)_\infty}{(zq;q)_\infty},$$

$$\mathrm{SCP}_{(1,-1,1,-1)}(z;q) = \frac{(-zq^3, zq; q^4)_\infty}{(zq;q)_\infty},$$

$$\mathrm{SCP}_{(-1,1,-1,1)}(z;q) = \frac{(-zq, zq^3; q^4)_\infty}{(zq;q)_\infty}.$$

Corollary (Bridges, U 2022)

$$\sum_{n\geq 0} (-1)^n q^{4n^2} \frac{(q^2,-q^4;q^4)_n}{(q^4;q^4)_{2n}} \left(1 - \frac{q^{4n+1}z}{1+q^{4n+2}}\right) z^{2n} = (zq, -zq^3; q^4)_\infty.$$

SCP width 6 identities:

Theorem (Bridges, U 2022)

$$\sum_{n,m\geq 0} (-1)^m q^{3\binom{n+1}{2}-3m(m+1)} \frac{(-q,-q^5;q^6)_m}{(q^6;q^6)_m(q^3;q^3)_{n-2m}} = \frac{(q^4,q^8;q^{12})_\infty}{(q^6;q^{12})_\infty},$$

$$\sum_{n,m\geq 0} (-1)^{m+1} q^{3\binom{n-1}{2}+2n-3m(m+1)-1} \frac{(-q,-q^5;q^6)_m}{(q^6;q^6)_m(q^3;q^3)_{n-2m}}(1-q^{3n+1}+q^{3n-6m}) = \frac{(q;q^6)_\infty(q^{10};q^{12})_\infty}{(q^5;q^6)_\infty},$$

$$\sum_{n,m\geq 0} (-1)^{m+1} q^{3\binom{n+1}{2}-3m(m+1)-1} \frac{(-q,-q^5;q^6)_m}{(q^6;q^6)_m(q^3;q^3)_{n-2m}}(1-q^{3n+1}+q^{3n-6m}) = \frac{(q^2,q^{10};q^{12})_\infty}{(q^6;q^{12})_\infty},$$

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$$\times (1+q^{3n-12m-3}(1+q^{1+6m})(q^{3n}-q^{6m}(1+q^3)) = \frac{(q^2,q^5,q^{11};q^{12})_\infty}{(q;q^6)_\infty}.$$

What works for 5, works for 8

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Cylindric partitions with small rank proves Rogers–Ramanujan Identities.

What works for 5, works for 8

Cylindric partitions with small rank proves Rogers–Ramanujan Identities.

Where are the Little Göllnitz and Göllnitz-Gordon identities then?

We look at the width 3 equation system of DSPPs (with denominators cleared):

$$\begin{aligned}
H_{(1,1,1)}(z) &= H_{(1,1,-1)}(zq),\\
H_{(1,1,-1)}(z) &= H_{(1,-1,1)}(zq^2),\\
H_{(1,-1,1)}(z) &= H_{(1,1,-1)}(zq) + H_{(-1,1,1)}(zq) - (1-zq)H_{(1,-1,1)}(zq^2),\\
H_{(-1,1,1)}(z) &= H_{(1,1,1)}(zq) + H_{(1,-1,1)}(zq) - (1-zq)H_{(1,1,-1)}(zq^2).
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Little Göllnitz and Göllnitz-Gordon identities

Fall under DSPPs with conventional weights

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\end{aligned}$$

$$H_\gamma(z) = \sum_{n\geq 0} h_\gamma(n) z^n$$

Then

$$(1-q^{2n})h_{(1,-1,1)}(n) = q^{2n-1}(1+q^{2n-1})h_{(1,-1,1)}(n-1).$$

Little Göllnitz and Göllnitz-Gordon identities
Fall under DSPPs with conventional weights

Then

$$(1-q^{2n})h_{(1,-1,1)}(n) = q^{2n-1}(1+q^{2n-1})h_{(1,-1,1)}(n-1).$$

By iterating, we obtain

$$h_{(1,-1,1)}(n) = \frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n^2},$$

therefore,

$$\sum_{n\geq 0}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n^2} = H_{(1,-1,1)}(1) = (q;q)_\infty \mathrm{DSPP}_{(1,-1,1)}(q)$$

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$$= \frac{(q;q)_\infty}{(q,q^2,q^3,q^4;q^4)_\infty (q,q^4,q^7;q^8)_\infty}.$$

Andrews-Paule Diamond partitions

Theorem (Andrews-Paule, 2021)

Let

$$\lambda_1 \geq \begin{matrix} \lambda_2 \\ \lambda_3 \end{matrix} \geq \lambda_4 \geq \begin{matrix} \lambda_5 \\ \lambda_6 \end{matrix} \geq \lambda_7 \geq \begin{matrix} \lambda_8 \\ \lambda_9 \end{matrix} \geq \lambda_{10} \geq \begin{matrix} \lambda_{11} \\ \lambda_{12} \end{matrix} \geq \lambda_{13} \geq \dots .$$

then

$$\sum_{\lambda \in \diamondsuit} q^{\lambda_1+\lambda_4+\lambda_7+\dots} = \frac{(-q;q)_\infty}{(q;q)_\infty^3}.$$

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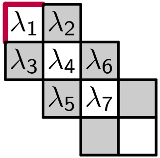

Andrews-Paule Diamond partitions

These diamond partitions are nothing but DSPPs with profile $(1,-1)$. For Andrews-Paule result, we need the weights $(0,1,0)$. Then we see,

$$\sum_{\lambda\in\diamondsuit} z^{\lambda_1} q^{\lambda_1+\lambda_4+\lambda_7+\dots} = \mathrm{DSPP}^{(0,1,0)}_{(1,-1)}(z;q).$$

and the product generating functions directly prove the above result when $z=1$. By solving the q-difference system related to DSPPs with width 2 and rank 1 profiles, we can actually prove

$$\sum_{\lambda\in\diamondsuit} z^{\lambda_1} q^{\lambda_1+\lambda_4+\lambda_7+\dots} = \frac{(-zq;q)_\infty}{(zq;q)^3_\infty}.$$

Schmidt's Problem

Theorem (Schmidt, 1999)

Let $\pi = (\lambda_1, \lambda_2, \dots)$ *be a partition into distinct parts, then*

$$\sum_{\pi \in \mathcal{D}} q^{\lambda_1+\lambda_3+\lambda_5+\dots} = \frac{1}{(q;q)_\infty}.$$

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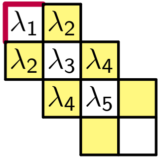

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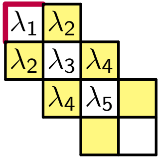

This time we need cylindric partitions into distinct parts.

Schmidt's Problem

Distinct Cylindric Partitions

We can form the recurrences for distinct cylindric partitions (DCPs) as well. However, the recurrences are inhomogeneous. Once we solve the recurrence for the Schmidt's theorem related DCP system we get

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Schmidt's Problem
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Corollary

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This implies Schmidt's theorem.

Thank you

More Recent Progress on Cylindric Partitions

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The Algorithmic and Enumerative Combinatorics Conference

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