The birth of the strong components

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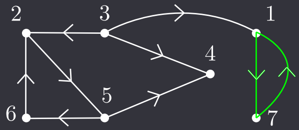
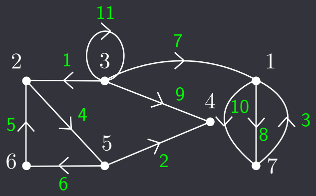
Simple digraph. Labeled vertices, unlabeled directed edges, loops and multiple edges forbidden



What about 2-cycles? Distinction between strict and simple digraphs.

$D(n,p)$. n vertices, each possible directed edge is present with probability p.

Multidigraph. Labeled vertices, labeled directed edges, loops and multiple edges allowed



Multigraph $D(n, p)$. The number of edges between any two vertices follows a Poisson law of parameter p.

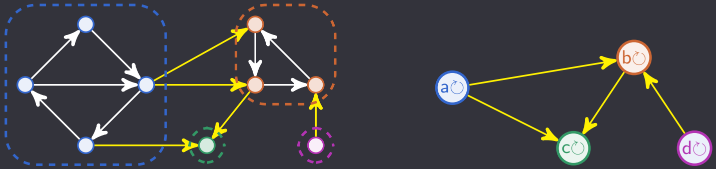
Simpler formulae with multidigraphs, see the arXiv article for the simple digraph versions.

Introduction. Digraph structure

Strong component. Maximal set of vertices, any oriented pair of them linked by a directed path.

Directed Acyclic Graph (DAG). No directed cycle.

Condensation. Each vertex of a digraph belongs to a unique strong component. Contracting each strong component to a vertex turns the digraph into a DAG



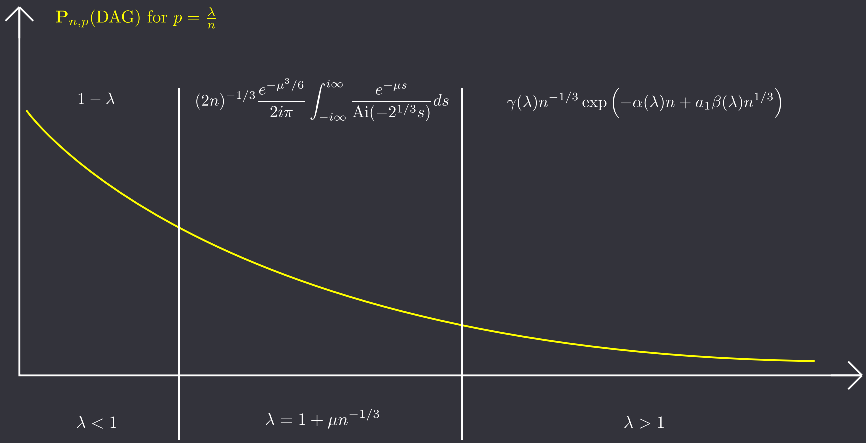
Introduction. Directed Acyclic Graphs (DAGs)

Exact enumeration of DAGs by Liskovets, Wright, Gessel, Robinson, between 1969 and 1977.

Asymptotic probability of DAGs in $D(n, p)$ for fixed p by Bender Richmond Robinson Wormald 1986. Quadratic expected number of edges.

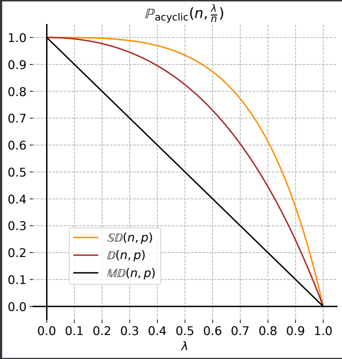
Our result $p = \lambda/n$. Linear expected number of edges.

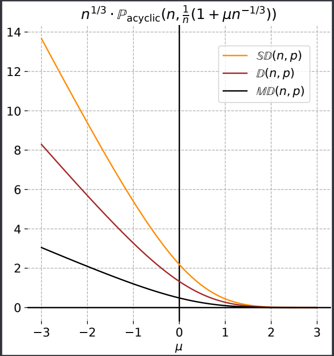
Introduction. Directed Acyclic Graphs (DAGs)



$a_1 \approx -2.338107$ is the smallest zero of $\mathrm{Ai}(z)$

$$\alpha(\lambda) = \frac{\lambda^2 - 1}{2\lambda} - \log(\lambda), \quad \beta(\lambda) = (2\lambda)^{-1/3}(\lambda - 1), \quad \gamma(\lambda) = \frac{2^{-2/3}}{\mathrm{Ai}'(a_1)}\lambda^{5/6}e^{(\lambda-1)/6}$$


$\mathbb{P}_{\text{acyclic}}(n, \frac{\lambda}{n})$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
λ
$\mathbb{SD}(n,p)$
$\mathbb{D}(n,p)$
$\mathbb{MD}(n,p)$


$n^{1/3} \cdot \mathbb{P}_{\text{acyclic}}(n, \frac{1}{n}(1 + \mu n^{-1/3}))$
14
12
10
8
6
4
2
0
−3 −2 −1 0 1 2 3
μ
$\mathbb{SD}(n,p)$
$\mathbb{D}(n,p)$
$\mathbb{MD}(n,p)$

Consider $D(n,p)$

Sub- and super-critical. Karp 1990 and Luczak 1990

$p < 1/n - \epsilon$. all strong components have bounded size, are either cycles or single vertices w.h.p.

$p > 1/n + \epsilon$. there exists a unique strong component of size $\Theta(n)$, while all the other strong components have logarithmic size w.h.p.

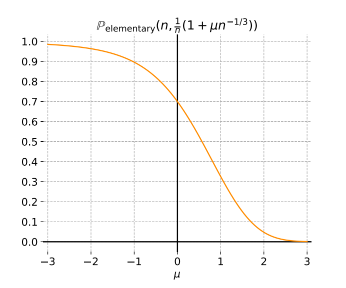
Critical.

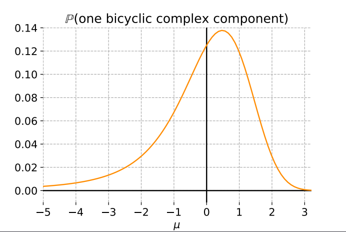
- Luczak Seierstad 2009 obtained the width of the transition window $p = n^{-1}(1 + \Theta(n^{-1/3}))$ and the size $\Theta(n^{1/3})$ of the largest component (see also Coulson 2019).
- Goldschmidt Stephenson 2021 gave the scaling limit.

Denser digraphs Cooper Frieze 2004. Sizes of the largest components in a random digraph with a given degree sequence.

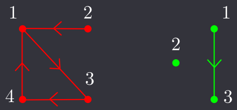
Elementary digraph. Each strong component is a single vertex or a cycle.

Our result. $D(n,p)$ with $p = n^{-1}(1 + \mu n^{-1/3})$, probability of elementary digraphs, or with one complex strong component.





combinatorial family	generating function
$\mathcal{A}$	$A(z) = \sum_{a\in\mathcal{A}} \frac{z^{\lvert a\rvert}}{\lvert a\rvert!} = \sum_{n\geq 0} \lvert\mathcal{A}_n\rvert \frac{z^n}{n!}$
disjoint union	sum
$\mathcal{C} = \mathcal{A} \uplus \mathcal{B}$	$C(z) = A(z) + B(z)$
relabeled Cartesian product	product
$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$C(z) = \sum_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}}} \binom{\lvert a\rvert + \lvert b\rvert}{\lvert a\rvert} \frac{z^{\lvert a\rvert+\lvert b\rvert}}{(\lvert a\rvert+\lvert b\rvert)!} = A(z)B(z)$



Set. If $\mathcal{B} = \text{Set}(\mathcal{A})$, then

$$B(z) = \sum_k \frac{A(z)^k}{k!} = e^{A(z)}.$$

Example. Generating function of graphs

$$G(z) = \sum_n 2^{\binom{n}{2}} \frac{z^n}{n!}.$$

A graph is a set of connected components

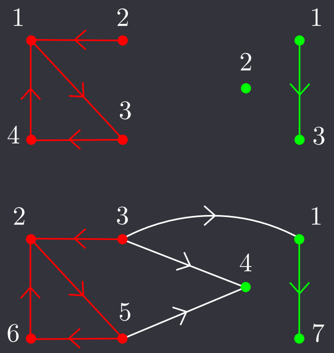
$$G(z) = e^{C(z)}$$

So the exponential gf of connected graphs is

$$C(z) = \log\left(\sum_n 2^{\binom{n}{2}} \frac{z^n}{n!}\right).$$

Arrow product. $\mathcal{C} = \mathcal{A} \ominus \mathcal{B}$.

- Relabel a pair of digraphs a from $\mathcal{A}$ and b from $\mathcal{B}$,
- write a on the left and b on the right,
- add arbitrary edges from left to right.



Symbolic method. Graphic generating functions

$A_{n,m}$ = number of digraph from $\mathcal{A}$ with n vertices and m edges.

Exponential gf. $$A(z,w) = \sum_{n,m} A_{n,m} \frac{w^m}{m!} \frac{z^n}{n!}$$

Graphic gf. $$\hat{A}(z,w) = \sum_{n,m} A_{n,m} e^{-n^2 w/2} \frac{w^m}{m!} \frac{z^n}{n!}.$$

Product. Corresponds to the arrow product $\mathcal{C} = \mathcal{A} \ominus \mathcal{B}$

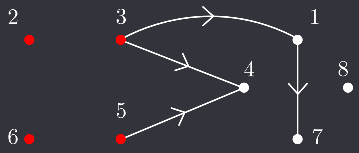
$$\hat{C}(z,w) = \sum_{k+\ell=n} e^{-n^2 w/2} \binom{n}{k} e^{k\ell w} \left(\sum_m A_{k,m} \frac{w^m}{m!} \right) \left(\sum_m B_{\ell,m} \frac{w^m}{m!} \right) \frac{z^n}{n!}$$

$$= \sum_{k+\ell=n} e^{-k^2 w/2} e^{-\ell^2 w/2} \left(\sum_m A_{k,m} \frac{w^m}{m!} \right) \left(\sum_m B_{\ell,m} \frac{w^m}{m!} \right) \frac{z^k}{k!} \frac{z^\ell}{\ell!} = \hat{A}(z,w)\hat{B}(z,w).$$

Arcless digraphs. $$\widehat{\mathrm{Set}}(z,w) = \sum_{n\geq 0} e^{-n^2 w/2} \frac{z^n}{n!}.$$

DAG (Directed Acyclic Graph) (Robinson, Gessel, Liskovets). Consider $\widehat{\mathrm{DAG}}(z,w,u)$ where u marks the sources (in-degree 0), and apply inclusion-exclusion

$$\widehat{\mathrm{DAG}}(z,w,u+1) = \widehat{\mathrm{Set}}(uz,w) \times \widehat{\mathrm{DAG}}(z,w)$$



The only DAG without source is the empty DAG, so for $u=-1$

$$1 = \widehat{\mathrm{DAG}}(z,w,0) = \widehat{\mathrm{Set}}(-z,w) \times \widehat{\mathrm{DAG}}(z,w),$$

$$\widehat{\mathrm{DAG}}(z,w) = \frac{1}{\widehat{\mathrm{Set}}(-z,w)}.$$

Exact enumeration. Generating function translation

Undirected multigraphs and (multi)digraphs.

$$\mathrm{MG}(z,w) = \sum_{n\ge 0} e^{n^2 w/2} \frac{z^n}{n!}, \qquad \hat{D}(z,w) = \sum_{n\ge 0} e^{n^2 w} e^{-n^2 w/2} \frac{z^n}{n!} = \mathrm{MG}(z,w).$$

Arcless digraphs.

$$\widehat{\mathsf{Set}}(z,w) = \sum_{n\ge 0} e^{-n^2 w/2} \frac{z^n}{n!}.$$

Exponential Hadamard product.

$$\sum_n a_n \frac{z^n}{n!} \odot_z \sum_n b_n \frac{z^n}{n!} = \sum_n a_n b_n \frac{z^n}{n!}.$$

Translation between exponential and graphic gfs.

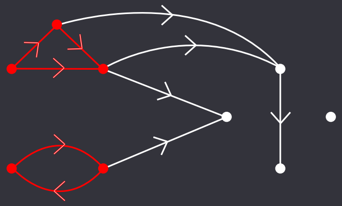
$$\hat{A}(z,w) = \sum_{n,m} A_{n,m} e^{-n^2 w/2} \frac{w^m}{m!} \frac{z^n}{n!} = A(z,w) \odot_z \widehat{\mathsf{Set}}(z,w),$$

$$A(z,w) = \hat{A}(z,w) \odot_z \mathrm{MG}(z,w).$$

Exact enumeration. Strongly connected digraphs

Strongly connected digraphs (Robinson, Gessel, Liskovets). Consider $\hat{D}(z,w,u)$ where u marks the source-like components and apply inclusion-exclusion

$$\hat{D}(z,w,u+1) = \left(e^{u\,\mathrm{Strong}(z,w)} \odot_z \widehat{\mathrm{Set}}(z,w)\right) \hat{D}(z,w)$$



The only digraph without source-like component is the empty digraph, so for $u=-1$

$$1 = \left(e^{-\,\mathrm{Strong}(z,w)} \odot_z \widehat{\mathrm{Set}}(z,w)\right) \mathrm{MG}(z,w),$$

$$\mathrm{Strong}(z,w) = -\log\left(\mathrm{MG}(z,w) \odot_z \frac{1}{\mathrm{MG}(z,w)}\right).$$

Digraphs where strong components must belong to a family S

$$\hat{D}_S(z, w, u+1) = \left(e^{uS(z,w)} \odot_z \widehat{\mathsf{Set}}(z,w) \right) \hat{D}_S(z,w)$$

$$\hat{D}_S(z,w) = \frac{1}{e^{-S(z,w)} \odot_z \widehat{\mathsf{Set}}(z,w)}$$

Elementary digraphs. Strong components are single points or cycles

$$\hat{D}_{\mathrm{elem}}(z,w) = \frac{1}{(1-wz)e^{-z} \odot_z \widehat{\mathsf{Set}}(z,w)}$$

Elementary digraph plus one strong component in S.

$$\hat{D}_{\mathrm{elem},S}(z,w) = \frac{(1-wz)S(z,w)e^{-z} \odot_z \widehat{\mathsf{Set}}(z,w)}{\left((1-wz)e^{-z} \odot_z \widehat{\mathsf{Set}}(z,w)\right)^2}$$

Asymptotics. Integral representation

Linearization.

$$\hat{A}(z,w) = \sum_n A_n(w) e^{-n^2 w/2} \frac{z^n}{n!}$$
$$= \sum_n A_n(w) \frac{1}{\sqrt{2\pi w}} \int_{-\infty}^{+\infty} e^{-nix} \exp\left(-\frac{x^2}{2w}\right) dx \frac{z^n}{n!}$$
$$= \frac{1}{\sqrt{2\pi w}} \int_{-\infty}^{+\infty} A\left(ze^{-ix}, w\right) \exp\left(-\frac{x^2}{2w}\right) dx$$

Generalized deformed exponential. Define

$$\phi_r(z,w) = \frac{1}{\sqrt{2\pi w}} \int_{-\infty}^{+\infty} (1 - wze^{-ix})^r \exp\left(-\frac{x^2}{2w} - ze^{-ix}\right) dx$$

then the gfs of DAGs and elementary digraphs are

$$\widehat{\mathrm{DAG}}(z,w) = \frac{1}{\phi_0(z,w)}, \qquad \hat{D}_{\mathrm{elem}}(z,w) = \frac{1}{\phi_1(z,w)}.$$

Multidigraphs $D(n,p)$. The number of edges between any two vertices follows a Poisson law of parameter p.

Probability for a random $D(n,p)$ (multi)digraph to belong to $\mathcal{F}$

$$\mathbb{P}_{n,p}(\mathcal{F}) = \sum_{G \in \mathcal{F}_n} \frac{(n^2 p)^{m(G)}}{m(G)!} e^{-n^2 p} \frac{1}{n^{2m(G)}} = e^{-n^2 p/2} n! [z^n] \hat{F}(z,p).$$

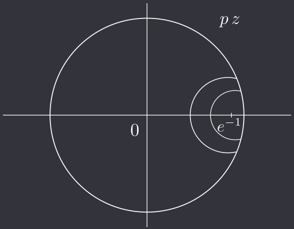
Thus

$$\mathbb{P}_{n,p}(\mathrm{DAG}) = e^{-n^2 p/2} n! [z^n] \frac{1}{\phi_0(z,p)},$$
$$\mathbb{P}_{n,p}(\text{elementary}) = e^{-n^2 p/2} n! [z^n] \frac{1}{\phi_1(z,p)}.$$

Asymptotics. Generalized deformed exponential

$$\phi_r(z,p) = \frac{1}{\sqrt{2\pi p}} \int_{-\infty}^{+\infty} (1 - pze^{-ix})^r \exp\left(-\frac{x^2}{2p} - ze^{-ix}\right) dx$$

Asymptotics estimates of $\phi(z,p)$ as $p \to 0$ in 3 zones, using the saddle-point method.



Isolated zeros of $\phi(z(p),p)$.

Singularity analysis of $[z^n]\dfrac{1}{\phi_r(z,p)}$ for $p = \lambda/n$ with $\lambda < 1$, $\lambda > 1$ or $\lambda = 1 + \mu n^{-1/3}$.

Conclusion

Numerical tests. We checked almost all assertions using computer algeba systems

https://gitlab.com/sergey-dovgal/strong-components-aux

Open problems and futur research.

- work on $D(n, m)$ instead of $D(n, p)$
- full description of the structure distribution in the critical window
- limit probability of satisfiability for 2-SAT formulae in the critical window.