Growing Connections Between Partition Crank, Mex, and Frobenius Symbols

Brian Hopkins, Saint Peter's University
Jersey City, New Jersey, USA
Joint work with James Sellers, Dennis Stanton, and Ae Ja Yee

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Outline

- partition rank and crank
- a combinatorial crank result
- minimal excluded part (mex)
- connecting mex & crank
- Frobenius symbols
- further connections and questions
- references

Partitions and rank

Write $p(n)$ for the number of partitions of n.

MacMahon provided Hardy and Ramanujan $p(n)$ values through $n = 200$. In 1919, Ramanujan proved (analytically) that

- $p(5n+4) \equiv 0 \bmod 5$,
- $p(7n+5) \equiv 0 \bmod 7$, and
- $p(11n+6) \equiv 0 \bmod 11$.

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In 1944, a young Freeman Dyson defined the rank of $\lambda = (\lambda_1, \dots, \lambda_\ell)$ as $\lambda_1 - \ell$ and conjectured that this simple partition statistic combinatorially verifies the modulo 5 and 7 results by grouping the appropriate partitions into 5 or 7 equally numerous classes. Proven correct by Atkin–Swinnerton-Dyer, 1954.

Partitions and rank

Dyson, Some guesses in the theory of partitions, *Eureka* 1944

After a few preliminaries I state certain properties of partitions which I am unable to prove; these guesses are then transformed into algebraic identities which are also unproved, although there is conclusive evidence in their support; finally, I indulge in some even vaguer guesses concerning the existence of identities which I am not only unable to prove but also unable to state.

The rank statistic shows the modulo 5 and 7 results, but not the modulo 11 identity. Dyson suggested that some "more recondite" partition statistic should. He gave it a name ("crank") and a purpose, but no definition!

Partitions and crank

Definition (Andrews–Garvan 1988)

Given a partition λ, let $\omega(\lambda)$ be the number of ones in λ and let $\mu(\lambda)$ be the number of parts of λ greater than $\omega(\lambda)$. Then

$$\text{crank}(\lambda) = \begin{cases} \lambda_1 & \text{if } \omega(\lambda) = 0, \\ \omega(\lambda) - \mu(\lambda) & \text{if } \omega(\lambda) > 0. \end{cases}$$

They showed that this definition of the "elusive crank" does all that Dyson hoped for and gives a combinatorial verification of the modulo 5 and 7 identities, too (with different groupings).

Some crank results

For integers m and $n > 1$, let $M(n, m)$ be the number of partitions of n with crank m. We use standard q-series notation.

Theorem (Garvan 1988)

$$\sum_{n \geq 0} M(m,n) q^n = \frac{1}{(q;q)_\infty} \sum_{n \geq 1} (-1)^{n-1} q^{n(n-1)/2 + n|m|} (1 - q^n),$$

$$M(m,n) = M(-m,n).$$

Compare the "not completely different" rank generating function

$$\sum_{n \geq 0} N(m,n) q^n = \frac{1}{(q;q)_\infty} \sum_{n \geq 1} (-1)^{n-1} q^{n(3n-1)/2 + n|m|} (1 - q^n).$$

Bounded crank

Given $j \geq 0$, we're interested in the number of partitions λ of n with $\text{crank}(\lambda) \geq j$.

$$\sum_{m \geq j} \sum_{n \geq 0} M(m,n) q^n = \frac{1}{(q;q)_\infty} \sum_{n \geq 0} (-1)^n q^{n(n+1)/2 + j(n+1)} \tag{G}$$

$$= \sum_{n \geq 0} \frac{q^{(n+1)(n+j)}}{(q;q)_n (q;q)_{n+j}} \tag{HSY}$$

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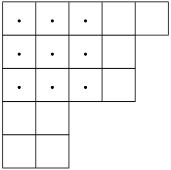
Note that there is no alternating sum in the Hopkins–Sellers–Yee expression, more conducive to combinatorial proofs.

Proof ingredient: j-Durfee rectangles

Definition

The j-Durfee rectangle of a partition λ is the largest rectangle of size $d \times (d + j)$ that fits inside the Ferrers diagram of λ.



$(5, 4, 4, 2, 2)$ has 0-Durfee rectangle (Durfee square) size 3×3,

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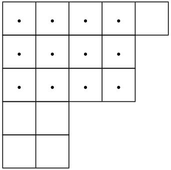

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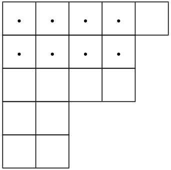

$(5, 4, 4, 2, 2)$ has 0-Durfee rectangle (Durfee square) size 3×3, 1-Durfee rectangle size 3×4, 2-Durfee rectangle size 2×4, etc.

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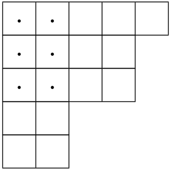

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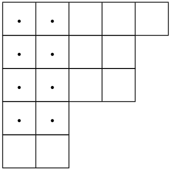

$(5, 4, 4, 2, 2)$ has 0-Durfee rectangle (Durfee square) size 3×3, 1-Durfee rectangle size 3×4, 2-Durfee rectangle size 2×4, etc. Also, -1-Durfee rectangle size 3×2, -2-Durfee rectangle size 4×2, etc.

Proof ingredient: symmetry insight

Use $\text{crank}(\lambda) \leq -j$ rather than $\text{crank}(\lambda) \geq j$.

Equal count since $M(m, n) = M(-m, n)$, but nonpositive cranks only come from the second part of the definition:

$$\text{crank}(\lambda) = \begin{cases} \lambda_1 & \text{if } \omega(\lambda) = 0, \quad \leftarrow \text{only positive crank} \\ \omega(\lambda) - \mu(\lambda) & \text{if } \omega(\lambda) > 0. \end{cases}$$

Combinatorial proof

H., Sellers, Yee 2022

$$\sum_{m\geq j}\sum_{n\geq 0} M(m,n)q^n = \sum_{m\leq -j}\sum_{n\geq 0} M(m,n)q^n = \sum_{d\geq 0}\frac{q^{(d+1)(d+j)}}{(q;q)_d(q;q)_{d+j}}$$

Nonpositive crank means $\omega(\lambda) > 0$. For crank $-j$, consider the j-Durfee rectangle, size $d \times (d+j)$.

Claim: $\omega(\lambda) \geq d + j$. If $\omega(\lambda) < d + j$, then $\mu(\lambda) \geq d$ since $\lambda_d \geq d + j$ (i.e., all parts in the j-Durfee rectangle) and

$$\text{crank}(\lambda) = \mu(\lambda) - \omega(\lambda) \ > \ d - (d+j) = -j.$$

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Nonpositive crank implies $\omega(\lambda) > 0$. Consider the j-Durfee rectangle, size $d \times (d+j)$. Since crank$(\lambda) \leq -j$, we know $\omega(\lambda) \geq d+j$.

The generating function for such λ: The j-Durfee rectangle contributes $d(d+j)$ towards the partition weight, $\omega(\lambda)$ gives at least $(d+j)$, together $(d+1)(d+j)$. Boxes to the right of the j-Durfee rectangle account for $(q;q)_d$, boxes below $(q;q)_{d+j}$.

Mex

The mex of a partition is the smallest missing (positive) part, e.g.,

$$\text{mex}(2,2,2) = 1, \quad \text{mex}(3,1,1,1) = 2, \quad \text{mex}(3,2,1) = 4.$$

Terminology from combinatorial game theory (at least by 1973, Grundy values), combination of minimal excluded number.

References in partitions:

- Grabner–Knopfmacher 2006 "least gap"
- Andrews 2011 "smallest number that is *not* a summand"
- Andrews–Newman 2019 "minimal excludant"/mex

Definition

Let $m_{a,b}(n)$ be the number of partitions of n with mex congruent to a modulo b.

Also, write superscript e for the number of partitions with an even number of parts, similarly for superscript o.

n	2	3	4	5	6	7	8	9	10	11	12
$m_{1,2}(n)$	1	2	3	4	6	8	12	16	23	30	42
$m_{1,4}(n)$	1	1	2	2	4	4	7	8	13	15	23
$m_{3,4}(n)$	0	1	1	2	2	4	5	8	10	15	19
$m^{o}_{1,2}(n)$	1	1	2	2	3	4	6	8	11	15	21
$m^{e}_{1,2}(n)$	0	1	1	2	3	4	6	8	12	15	21

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Splitting the mexes

H., Sellers, Yee 2022

$$m^o_{1,2}(n) = \begin{cases} m^e_{1,2}(n) + (-1)^{m+1} & \text{when } n = m(3m \pm 1), \\ m^e_{1,2}(n) & \text{otherwise.} \end{cases}$$

Combinatorial proof comes down to considering triples (π, μ, ν) where π is a partition into distinct even parts, μ is a partition into odd parts, and ν is a partition into distinct odd parts.

A sign-reversing involutions leaves just $(\pi, \emptyset, \emptyset)$, then apply Franklin's bijection to $(\pi_1/2, \pi_2/2, \ldots)$.

Splitting the mexes

Andrews, Newman 2019

$m_{1,2}(n)$ is almost always even and is odd exactly when $n = m(3m \pm 1)$ for some m.

HSY proof:

$$m_{1,2}(n) = m^o_{1,2}(n) + m^e_{1,2}(n)$$
$$= \begin{cases} 2m^e_{1,2}(n) + (-1)^{m+1} & \text{when } n = m(3m \pm 1), \\ 2m^e_{1,2}(n) & \text{otherwise.} \end{cases}$$

Connecting crank and mex

Andrews, Newman 2020; H., Sellers 2020

The number of partitions of n with nonnegative crank equals the number of partitions of n with odd mex. I.e., $M_{\geq 0}(n) = m_{1,2}(n)$.

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Generalization: For j a part in λ, let $\text{mex}_j(\lambda)$ to be the least integer greater than j that is not a part of λ.

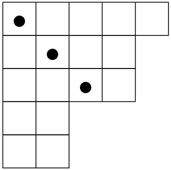
H., Sellers, Stanton 2022

The number of partitions λ of n with $\text{crank}(\lambda) \geq j$ equals the number of partitions of n with odd mex_j that include j as a part.

Recent combinatorial proof by Isaac Konan.

Frobenius symbols

$(5, 4, 4, 2, 2)$ Ferrers diagram and Frobenius symbol



$$\begin{pmatrix} 4 & 2 & 1 \\ 4 & 3 & 0 \end{pmatrix}$$

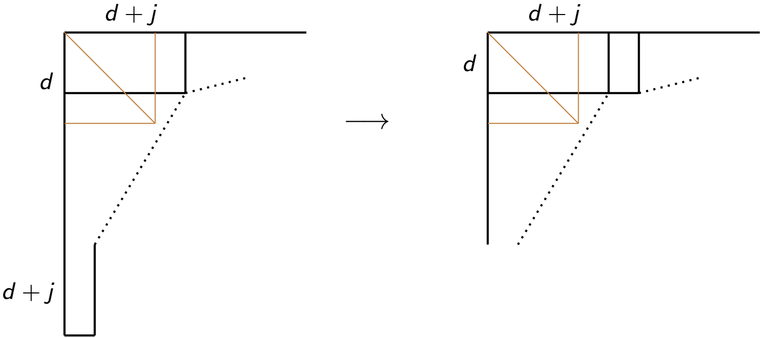
Andrews 2011

The number of partitions of n with no 0 in the top row of their Frobenius symbols equals $m_{1,2}(n)$ (and now $M_{\geq 0}(n)$.)

Crank and Frobenius symbols

H., Sellers, Stanton 2022

The number of partitions of $n - j$ with no j in the top row of their Frobenius symbols equals the number of partitions λ of n with $\text{crank}(\lambda) \geq j$.



Crank and Frobenius symbols

H., Sellers, Yee 2022

The number of partitions λ of n with $\text{crank}(\lambda) = 0$ equals the number of partitions of n whose Frobenius symbol has no 0 and the first two entries of the bottom row differ by 1.

Andrews, Dastidar, Morill 2021

The number of partitions λ of n with $\text{crank}(\lambda) > j$ equals one-half the number of j's in the Frobenius symbols of all partitions of n.

Frobenius symbols in disguise

Blecher–Knopfmacher 2022 consider partitions with "fixed points" where $\lambda_i = i$. A partition (in nonincreasing order) has 0 or 1 fixed points. They wonder whether there are always more partitions of n without fixed points than with fixed points.

E.g., $(5, 4, 4, 2, 2)$ does not have a fixed point, $(5, 4, 3, 3, 2)$ does.

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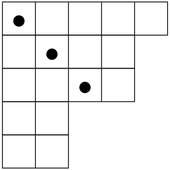
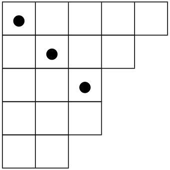


$$\begin{pmatrix} 4 & 2 & 1 \\ 4 & 3 & 0 \end{pmatrix},$$



$$\begin{pmatrix} 4 & 2 & 0 \\ 4 & 3 & 1 \end{pmatrix}$$

A partition without a fixed point has no 0 in the top row of its Frobenius symbol. With a fixed point, the top row does end in 0.

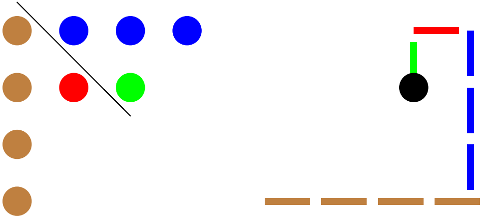
Frobenius symbols in disguise

The answer to Blecher and Knopfmacher's open question is yes.

no fixed point	?	fixed point
$\updownarrow$		$\updownarrow$
top Frob. no 0		top Frob. 0
$\updownarrow$		$\updownarrow$
$m_{1,2}(n)$		$m_{0,2}(n)$
$\updownarrow$		$\updownarrow$
$M_{\geq 0}(n)$		$M_{<0}(n)$
$\updownarrow$		$\updownarrow$
$M_{\geq 0}(n)$	$>$	$M_{>0}(n)$

Greater by the number of crank 0 partitions.

Concatenable spiral self-avoiding walks: Guttmann, Hirschhorn, Wormald 1984



More with an odd or even number of turns?
turns odd $\sim m_{1,2}(n)$, # turns even $\sim m_{0,2}(n)\ldots$

Where are the split odd mexes?

Recall $m_{1,2}(n) = m_{1,4}(n) + m_{3,4}(n)$. Where are these as subsets of the nonpositive crank partitions? Of Frobenius symbols with no 0 on the top? Of partitions without fixed points? Of CSSAWs with an odd number of turns?

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Huh, Kim 2021

$$m_{1,4}(n) = M^e_{\leq 0}(n), \quad m_{3,4}(n) = M^o_{\leq 0}(n).$$

Note that Konan's (current) bijection does not show this.

We don't yet know the $m_{1,4}(n)$, $m_{3,4}(n)$ subsets for the other equinumerous sets.

Also, how do the refinements such as $m^e_{1,2}(n)$ and $m^o_{3,4}(n)$ manifest in nonnegative crank partitions? Might help with bijective proofs relating those statistics.

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