Alternating Sign Matrices With Reflective Symmetry and Plane Partitions: $n + 3$ Pairs of Equivalent Statistics

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Outline

1. Alternating Sign Matrices and Descending Plane Partitions
2. Reflective Symmetry
3. $(n+3)$-Parameter Refinement of Alternating Sign Matrices
4. $(n+3)$-Parameter Refinement of Pairs of Plane Partitions
5. Sketch of the Proof

Alternating sign matrices and descending plane partitions

Definition (Mills, Robbins, Rumsey 1980s)

An alternating sign matrix (ASM) of order n is an $n \times n$-matrix with entries -1, 0 or $+1$ such that

- the entries in each row and each column sum to 1, and
- the nonzero entries alternate in sign along each row and each column.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Alternating sign matrices and descending plane partitions

Definition (Andrews 1979)

A descending plane partition (DPP) of order n is the filling of a shifted Young diagram with positive integers less than or equal to n such that

- the entries weakly decrease along rows
- and strictly decrease down columns, and
- the first part in each row is strictly larger than the length of the row
- but less than or equal to the length of the previous row.

11	10	10	10	7	5	4	4	3
	7	7	6	5	3	1		
		5	5	4	2			
			4	3	1			
				2				

ASMs and DPPs are equinumerous

Theorem (Andrews 1979, Zeilberger 1996)
ASMs and DPPs of the same order are equinumerous.

Example for $n = 3$:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} \quad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \quad \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} \quad \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

$$\emptyset \quad \boxed{2} \quad \boxed{3} \quad \boxed{3}\boxed{1} \quad \boxed{3}\boxed{2} \quad \boxed{3}\boxed{3} \quad \begin{array}{cc}\boxed{3}&\boxed{3}\\ &\boxed{2}\end{array}$$

Given $n+3$ statistics on generalised vertically symmetric ASMs of order $2n+1$, what do corresponding plane partition objects look like?

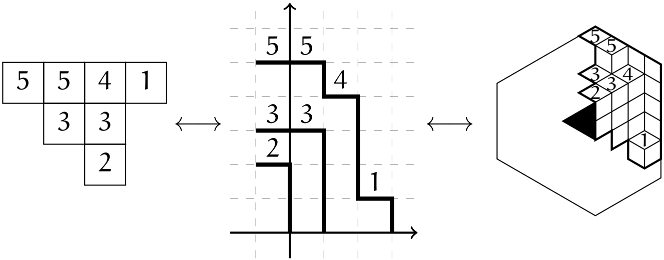
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \longleftrightarrow \quad ?$$

$$vX_2 \prod_{i=1}^{2} (uX_i + w + vX_i^{-1})$$

Plane partitions in disguise

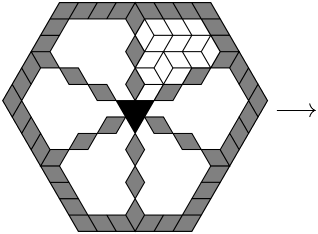
Theorem (Krattenthaler 2006)

There is a bijective correspondence between DPPs of order n and cyclically symmetric lozenge tilings of a hexagon with alternating side lengths $n-1$ and $n+1$ and a central triangular hole of size 2.

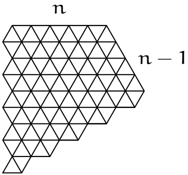


Adding a reflective symmetry I

Cyclically and vertically symmetric lozenge tilings of a hexagon with alternating side lengths $2n$ and $2n+2$ and a central triangular hole of size 2:



Fundamental domain:



Monotone triangles

Theorem
ASMs of order n are in bijective correspondence with monotone triangles with bottom row $1, 2, \ldots, n$.

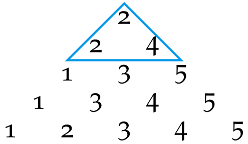
$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \longleftrightarrow \begin{array}{ccccccc} & & & 3 & & & \\ & & 1 & & 3 & & \\ & 1 & & 2 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

Definition
A monotone triangle (MT) of order n is a triangular array of integers with n rows such that the entries

- strictly increase along rows,
- weakly increase along $\nearrow$-diagonals, and
- weakly increase along $\searrow$-diagonals.

Adding a reflective symmetry II

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \overset{\circlearrowright}{\longleftrightarrow} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow$$



Theorem
Vertically symmetric ASMs (VSASMs) of order $2n + 1$ are in bijective correspondence with MTs of order n with bottom row $0, 2, \ldots, 2n - 2$.

Arrowed monotone triangles

Definition (Aigner, Fischer)

An arrowed monotone triangle (AMT) of order n is a MT of order n where each entry e carries a decoration from $\{\nwarrow, \nearrow, \nwarrow\!\!\!\nearrow\}$ such that the following two conditions are satisfied:

- If e has a $\nwarrow$-neighbor and is equal to it, then e must carry $\nearrow$.
- If e has a $\nearrow$-neighbor and is equal to it, then e must carry $\nwarrow$.

$$
\begin{array}{ccccccccccc}
 & & & & & {}^{\nwarrow}4^{\nearrow} & & & & & \\
 & & & & {}^{\nwarrow}4 & & {}^{\nwarrow}5 & & & & \\
 & & & {}^{\nwarrow}3^{\nearrow} & & 4^{\nearrow} & & 6^{\nearrow} & & & \\
 & & 2^{\nearrow} & & 3^{\nearrow} & & {}^{\nwarrow}6 & & {}^{\nwarrow}8^{\nearrow} & & \\
 & {}^{\nwarrow}1 & & 2^{\nearrow} & & {}^{\nwarrow}5^{\nearrow} & & 6^{\nearrow} & & {}^{\nwarrow}9 & \\
{}^{\nwarrow}0 & & {}^{\nwarrow}2 & & {}^{\nwarrow}4^{\nearrow} & & {}^{\nwarrow}6 & & {}^{\nwarrow}8^{\nearrow} & & 10^{\nearrow}
\end{array}
$$

$n + 3$ statistics

We assign the following weight to an AMT of order n:

$$u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\nearrow}$$

$$\times \prod_{i=1}^{n} X_i^{(\sum \text{ entries in row } i) - (\sum \text{ entries in row } i-1) + (\#\nearrow \text{ in row } i) - (\#\nwarrow \text{ in row } i)}.$$

The weight of

$$\begin{array}{ccccccccccc}
 & & & & & {}^{\nwarrow}4^{\nearrow} & & & & & \\
 & & & & {}^{\nwarrow}4 & & {}^{\nwarrow}5 & & & & \\
 & & & {}^{\nwarrow}3^{\nearrow} & & 4^{\nearrow} & & 6^{\nearrow} & & & \\
 & & 2^{\nearrow} & & 3^{\nearrow} & & {}^{\nwarrow}6 & & {}^{\nwarrow}8^{\nearrow} & & \\
 & {}^{\nwarrow}1 & & 2^{\nearrow} & & {}^{\nwarrow}5^{\nearrow} & & 6^{\nearrow} & & {}^{\nwarrow}9 & \\
{}^{\nwarrow}0 & & {}^{\nwarrow}2 & & {}^{\nwarrow}4^{\nearrow} & & {}^{\nwarrow}6 & & {}^{\nwarrow}8^{\nearrow} & & 10^{\nearrow}
\end{array}$$

is equal to

$$u^7 v^8 w^6 X_1^4 X_2^3 X_3^6 X_4^7 X_5^4 X_6^5.$$

Which families of (nonintersecting) lattice paths or plane partition objects have the same generating function as AMTs of order n with bottom row $0, 2, \ldots, 2n-2$?

- three signed combinatorial models in terms of lattice paths
- one signless combinatorial model in terms of pairs of plane partitions

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Pairs of plane partitions

$\geqslant$ (rows), $\wedge$ (columns)

$12 \geqslant$	9	7	7	5
$10 \geqslant$	8	6	6	4
$8 \geqslant$	6	5	3	
$6 \geqslant$	4	4		
$4 \geqslant$	3	2		
$2 \geqslant$	2			

$>$ (rows), $\not\wedge$ (columns)

$6 \geqslant$	6	5	4	2
$5 \geqslant$	5	4	3	1
$4 \geqslant$	4	2	1	
$3 \geqslant$	3	1		
$2 \geqslant$	2	1		
$1 \geqslant$	1			

Pairs (P, Q) of plane partitions of the same shape:

- n rows allowing rows of length 0
- P is column-strict
- Q is row-strict
- Row restrictions on P and Q

Weight:

$$w^{\binom{n+1}{2} - \#\text{entries in } Q} \prod_{i=1}^{n} X_i^{n-1} (uX_i)^{\#2i-1 \text{ in } P} (vX_i^{-1})^{\#2i \text{ in } P}$$

Magog triangles × symplectic tableaux

6	5	4	2
5	4	3	1
4	2	1	
3	1		
2	1		
1			

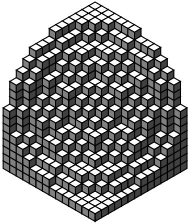
$\longleftrightarrow$

Magog triangle

1
1 2
1 2 3
1 2 3 4
1 1 3 3 5
1 1 2 3 4 6
1 1 1 3 4 6 7

$\longleftrightarrow$

TSSCPP



9	7	7	5
8	6	6	4
6	5	3	
4	4		
3	2		
2			

$\longleftrightarrow$

Symplectic tableau

5	4	4	3
$\bar{4}$	$\bar{3}$	$\bar{3}$	$\bar{2}$
$\bar{3}$	3	2	
$\bar{2}$	$\bar{2}$		
2	$\bar{1}$		
$\bar{1}$			

- Expansion into symplectic symmetric functions
- independently conjectured by Aigner

Sketch of the proof

- Starting point: operator formula for AMTs
- Transforming into antisymmetriser formula
- Specialising the bottom row: bialternant-type formula
- Several ways to transform into Jacobi-Trudi-type formula
- Signed enumeration of families of lattice paths via Lindström-Gessel-Viennot
- Eliminating the sign yields (two proofs of) plane partition interpretation

Generating function of AMTs

Schur polynomials $s_{(k_n,k_{n-1},\dots,k_1)}(X_1,\dots,X_n)$ are the generating functions of Gelfand-Tsetlin patterns with bottom row $k_1,k_2,\dots,k_n$ with respect to the weight

$$\prod_{i=1}^{n} X_i^{(\sum \text{entries in row } i)-(\sum \text{entries in row } i-1)}.$$

We have the following analogy:

Theorem (Aigner, Fischer)

The generating function of AMTs with bottom row $k_1,k_2,\dots,k_n$ is

$$\prod_{i=1}^{n}(uX_i+vX_i^{-1}+w)$$
$$\times \prod_{1\leqslant i<j\leqslant n}\left(uE_{k_i}+vE_{k_j}^{-1}+wE_{k_i}E_{k_j}^{-1}\right)s_{(k_n,k_{n-1},\dots,k_1)}(X_1,\dots,X_n),$$

where E_x denotes the *shift operator*, defined as $E_x p(x) := p(x+1)$.

From operator formula to antisymmetriser

Using the *antisymmetriser*

$$\mathbf{ASym}_{X_1,\dots,X_n}\left[f(X_1,\dots,X_n)\right] \coloneqq \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma\cdot f(X_{\sigma(1)},\dots,X_{\sigma(n)}),$$

we rewrite the generating function

$$\prod_{i=1}^{n}(uX_i+vX_i^{-1}+w)$$
$$\times\prod_{1\leqslant i<j\leqslant n}\left(uE_{k_i}+vE_{k_j}^{-1}+wE_{k_i}E_{k_j}^{-1}\right)s_{(k_n,k_{n-1},\dots,k_1)}(X_1,\dots,X_n),$$

as

$$\frac{\mathbf{ASym}_{X_1,\dots,X_n}\left[\prod_{1\leqslant i\leqslant j\leqslant n}\left(uX_j+vX_i^{-1}+w\right)\prod_{i=1}^{n}X_i^{k_i+n-i}\right]}{\prod_{1\leqslant i<j\leqslant n}(X_j-X_i)}.$$

The special case of VSASMs

For the case of VSASMs, we set $k_i = 2i - 2$. We also multiply the generating function with the symmetric expression

$$\prod_{1 \leqslant i < j \leqslant n} (u - v X_i^{-1} X_j^{-1}).$$

Thus, the generating function

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\leqslant i\leqslant j\leqslant n}\ (uX_j + vX_i^{-1} + w)\prod_{i=1}^{n} X_i^{k_i+n-i}\right]}{\prod_{1\leqslant i<j\leqslant n}(X_j - X_i)}.$$

becomes

$$\prod_{i=1}^{n} X_i^{n-2} \frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\leqslant i\leqslant j\leqslant n}\ (uX_j + vX_i^{-1} + w)\left(uX_j - vX_i^{-1}\right)\right]}{\prod_{1\leqslant i<j\leqslant n}(X_j - X_i)}.$$

Antisymmetriser lemma

Lemma (Aigner, Fischer, Konvalinka, Nadeau, Tewari 2020)

Let $\mathbf{Y} = (Y_1, \dots, Y_n), \mathbf{Z} = (Z_1, \dots, Z_n)$ be two sets of indeterminants. Then

$$\det_{1 \leqslant i,j \leqslant n} \left(Y_i^j - Z_i^j \right) = \overline{\mathbf{ASym}} \left[\prod_{1 \leqslant i \leqslant j \leqslant n} (Y_j - Z_i) \right]$$

with

$$\overline{\mathbf{ASym}} \left[f(\mathbf{Y}; \mathbf{Z}) \right] \coloneqq \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn} \sigma \cdot f(Y_{\sigma(1)}, \dots, Y_{\sigma(n)}; Z_{\sigma(1)}, \dots, Z_{\sigma(n)}).$$

Now set $Y_j = u^2 X_j^2 + uwX_j$ and $Z_i = v^2 X_i^{-2} + vwX_i^{-1}$,....

From antisymmetriser to bialternant

... apply the antisymmetriser lemma to

$$\prod_{i=1}^{n} X_i^{n-2} \frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\leqslant i\leqslant j\leqslant n}\ (uX_j + vX_i^{-1} + w)\left(uX_j - vX_i^{-1}\right)\right]}{\prod_{1\leqslant i<j\leqslant n}(X_j - X_i)}$$

and divide again by

$$\prod_{1\leqslant i<j\leqslant n} (u - vX_i^{-1}X_j^{-1})$$

to obtain the following bialternant-type formula

$$\prod_{i=1}^{n} X_i^{n-1} \frac{\det_{1\leqslant i,j\leqslant n}\left(\frac{\left(u^2X_i^2+uwX_i\right)^j-\left(v^2X_i^{-2}+vwX_i^{-1}\right)^j}{uX_i-vX_i^{-1}}\right)}{\prod_{1\leqslant i<j\leqslant n}\left(\left(uX_j - vX_j^{-1}\right) - \left(uX_i - vX_i^{-1}\right)\right)}.$$

From bialternant to Jacobi-Trudi I

Lemma (Aigner, Fischer)
Let $f_j(Y)$ be a formal Laurent series for $1 \leqslant j \leqslant n$, and define

$$f_j[Y_1, \ldots, Y_i] = \sum_{k \in \mathbb{Z}} \langle Y^k \rangle f_j(Y) \cdot h_{k-i+1}(Y_1, \ldots, Y_i),$$

where $\langle Y^k \rangle f_j(Y)$ denotes the coefficient of Y^k in $f_j(Y)$ and h_{k-i+1} denotes the complete homogeneous symmetric polynomial of degree $k - i + 1$. Then

$$\frac{\det_{1 \leqslant i,j \leqslant n} \left(f_j(Y_i) \right)}{\prod_{1 \leqslant i < j \leqslant n} (Y_j - Y_i)} = \det_{1 \leqslant i,j \leqslant n} \left(f_j[Y_1, \ldots, Y_i] \right).$$

This lemma can be applied to our bialternant-type formula in several ways. By using the Lindström-Gessel-Viennot lemma, we finally get the different combinatorial interpretations in terms of (non-intersecting) lattice paths.

From bialternant to Jacobi-Trudi II

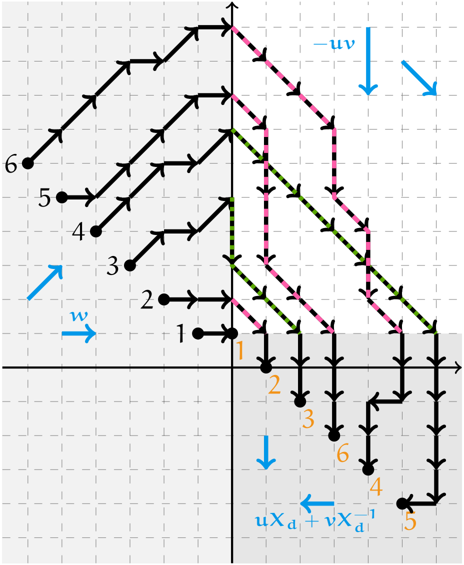
By applying the previous lemma with $Y_i = uX_i + vX_i^{-1}$, we obtain

$$\prod_{i=1}^{n} X_i^{n-1} \det_{1\leqslant i,j\leqslant n} (a_{i,j})$$

with $a_{i,j}$ equal to

$$\sum_{p\geqslant 1} w^{2i-p} \binom{i}{2i-p} \times \sum_{q\geqslant 1, 2|(p-q)} (-uv)^{(p-q)/2} \binom{(p+q)/2-1}{(p-q)/2} \times h_{q-j}(uX_1 + vX_1^{-1}, \ldots, uX_j + vX_j^{-1}).$$

Lattice paths interpretation



Family of n lattice paths:

- Starting points $(-1, 1)$, $(-2, 2)$,...,$(-n, n)$
- Endpoints $(0, 1)$,$(1, 0)$,...,$(n-1, -n+2)$
- Three regions with different step sets
  - $\{(x, y) | x \leqslant 0\}$
  - $\{(x, y) | x \geqslant 0, y \geqslant 1\}$
  - $\{(x, y) | x \geqslant 0, y \leqslant 1\}$
- Odd and even paths may intersect in the second region
- Signed generating function

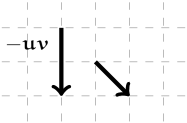
Eliminating the sign: Sign-reversing involutions!

Lemma
Let p, j be positive integers, then

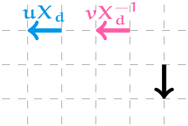
$$\sum_{q \geqslant 1, 2 | (p-q)} (-uv)^{(p-q)/2} \binom{(p+q)/2 - 1}{(p-q)/2} h_{q-j}(uX_1 + vX_1^{-1}, \ldots, uX_j + vX_j^{-1})$$

is the generating function of lattice paths from $(0, p)$ to $(j-1, -j+2)$ with step sets as given below, but without steps of type $(0, -2)$ and without consecutive pairs of horizontal steps with the first step being blue and the second step being red.

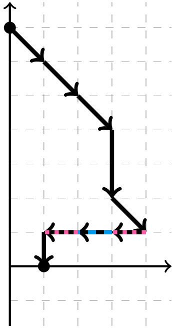
$\{(x, y) | x \geqslant 0, y \geqslant 1\}:$



$\{(x, y) | x \geqslant 0, y \leqslant 1\}:$

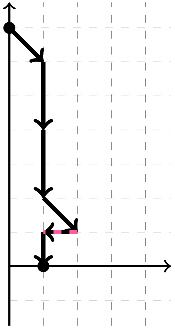


First sign-reversing involution



$(-uv)(vX_1^{-1})(uX_1)(vX_1^{-1})$
$= -u^2v^3X_1^{-1}$

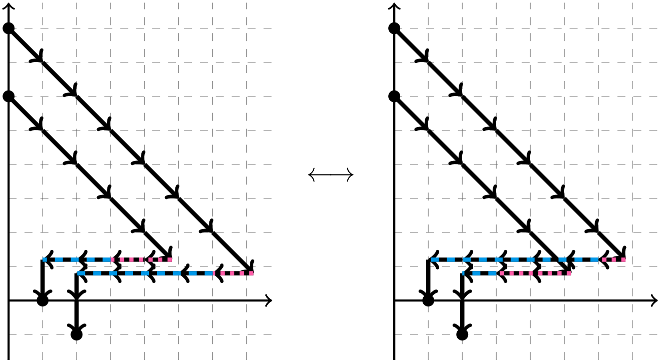
$\longleftrightarrow$



$(-uv)(-uv)(vX_1^{-1})$
$= u^2v^3X_1^{-1}$

Second sign-reversing involution

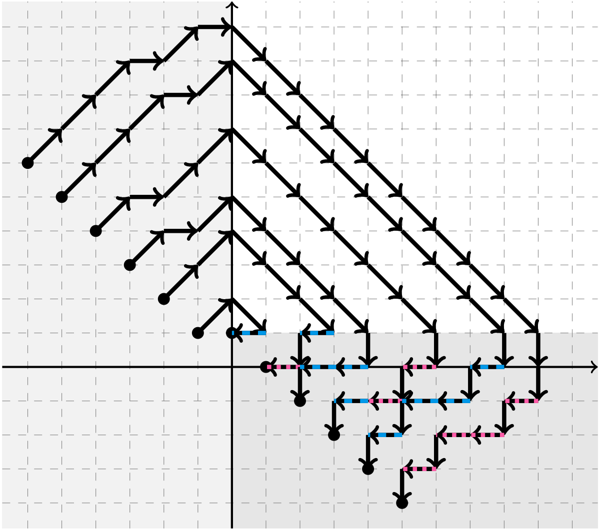
Center: 



$\longrightarrow$ changes the sign of the family of paths

Remainder: Paths that *touch* (intersections do not contain centers) but don't intersect in any other sense

Remaining lattice paths: read off plane partitions



Thank you for your attention!