Rogers - Ramanujan identities

& Cylindric partitions

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AEC - Vienna - 8 Juillet 2022















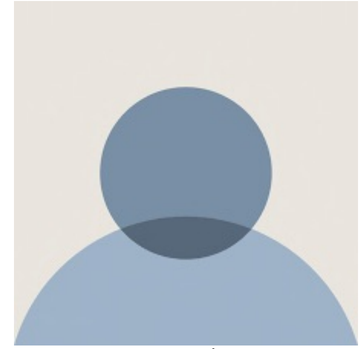
GRAPH THEORY






Yoo

S.C.



T. Welsh

Ann Comb (2019)

Proc of the AMS (2017)

S.C.

J. Dousse

A. Uncu

Proc of the AMS (2021)

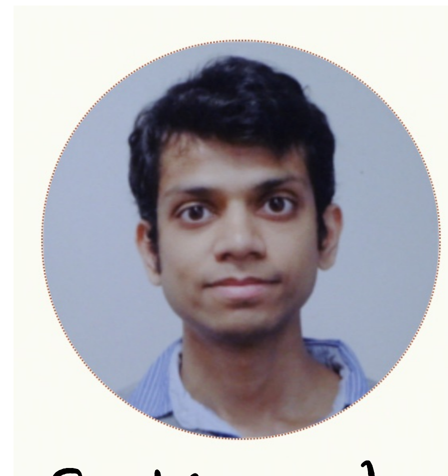
S.C.

J. Dousse

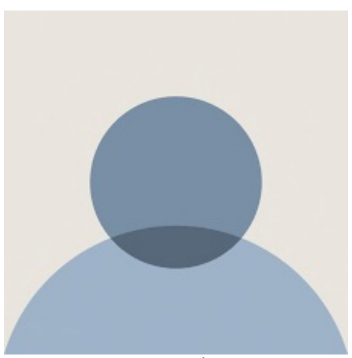
O. Foda

A. Uncu

O. Warnaar



S. Kanade



T. Welsh

S. Tsuchioka

M. Russell

Integer partitions

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$$

such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$

Partition of N if $\lambda_1 + \lambda_2 + \lambda_3 + \dots = N$

$(4,3,1)$ is a partition of 8.

Integer partitions

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$$

such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$

Partition of N if $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \dots = N$

$$\sum_{\lambda \mid \lambda_i \in S} q^{|\lambda|} = \prod_{i \in S} \frac{1}{1-q^i}$$

$$S = \{ i \mid i \equiv 1 \text{ or } 4 \bmod 5 \}$$

Integer partitions

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$

such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$

Interlacing partitions $\quad \lambda \succcurlyeq \mu$

if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \cdots$$

ex $\quad (4,3,1) \succcurlyeq (4,1,1)$

$$4 \geq 4 \geq 3 \geq 1 \geq 1 \geq 1$$

Integer partitions

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$

such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$

Interlacing partitions $\lambda \succcurlyeq \mu$

if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \ldots$$

$$\sum_{\lambda^{(n)} \succcurlyeq \cdots \succcurlyeq \lambda^{(1)} \succcurlyeq \emptyset} \prod_{i=1}^{n} x_i^{|\lambda_i| - |\lambda_{i-1}|} = S_{\lambda^{(n)}}(x_1, \ldots, x_n)$$

Rogers – Ramanujan identities (1910s)

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

$$(a;q)_n = \prod_{i=0}^{n-1} (1-aq^i) \qquad (a;q)_\infty = \prod_{i=0}^{\infty} (1-aq^i)$$

$$(a_1,\dots,a_k;q)_\infty = \prod_{j=1}^{k} \prod_{i=0}^{\infty} (1-a_j q^i)$$

Rogers – Ramanujan identities

$$\underbrace{\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n}}_{\text{Sum}} = \underbrace{\frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}}_{\text{Product}}$$

Sum = Product

Ramanujan identities

$$\underbrace{\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n}}_{\text{"Difference condition"}} = \underbrace{\frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}}_{\text{"Congruence classes"}}$$

Ramanujan identities

$$\underbrace{\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n}}_{} = \underbrace{\frac{1}{(q^1;q^5)_\infty (q^4;q^5)_\infty}}_{}$$

"Difference condition"

λ partition of N

$\lambda_i - \lambda_{i+1} \geq 2$

"Congruence classes"

μ partition of N

$\mu_i \equiv 1$ or $4 \bmod 5$

Ramanujan identities

$$\underbrace{\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n}} = \underbrace{\frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}}$$

"Difference condition" $\overset{?}{\longleftrightarrow}$ "Congruence classes"

λ partition of N | μ partition of N
$\lambda_i - \lambda_{i+1} \geq 2$ | $\mu_i \equiv 1$ or $4 \bmod 5$

$N = 10$ (10) (8,2) (7,3) (6,4) (6,3,1) | (9,1) (6,4) (6,1,1,1,1) (4,4,1,1) $(4,1^6)$ (1^{10})

Ramanujan identities (TODAY)

$$\frac{1}{(q;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q;q)_\infty}\cdot\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$$

Ramanujan identities (TODAY)

$$\frac{1}{(q;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n} = \underbrace{\frac{1}{(q;q)_\infty}\cdot\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}}_{\substack{\text{Cylindric}\\ \text{partitions of}\\ \text{profile } (3,0)}}$$

Ramanujan identities (TODAY)

$$\underbrace{\frac{1}{(q;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n}} = \frac{1}{(q;q)_\infty}\cdot\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$$

Cylindric partitions of profile (3, 0)

(Foda & Welsh 2016)

Ramanujan identities (TODAY)

$$\frac{1}{(q;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q;q)_\infty}\cdot\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$$

Cylindric partitions of profile (3,0)

Difference conditions

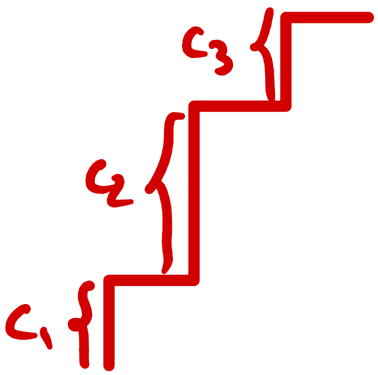
Congruence conditions

(Bijection, c. 2017)

Cylindric partitions

(Gessel & Krattenthaler) $d, r > 0$

c



c a composition $(c_1, c_2, \ldots, c_r)$

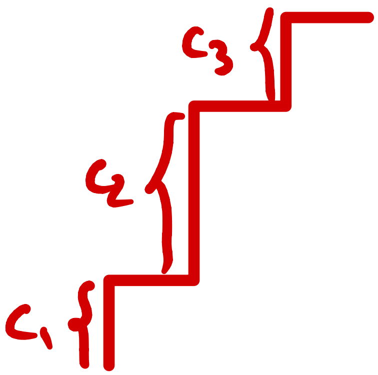
$c_1 + c_2 + \ldots + c_r = d$

ex $r = 3$ $d = 4$

$c = (1, 2, 1)$

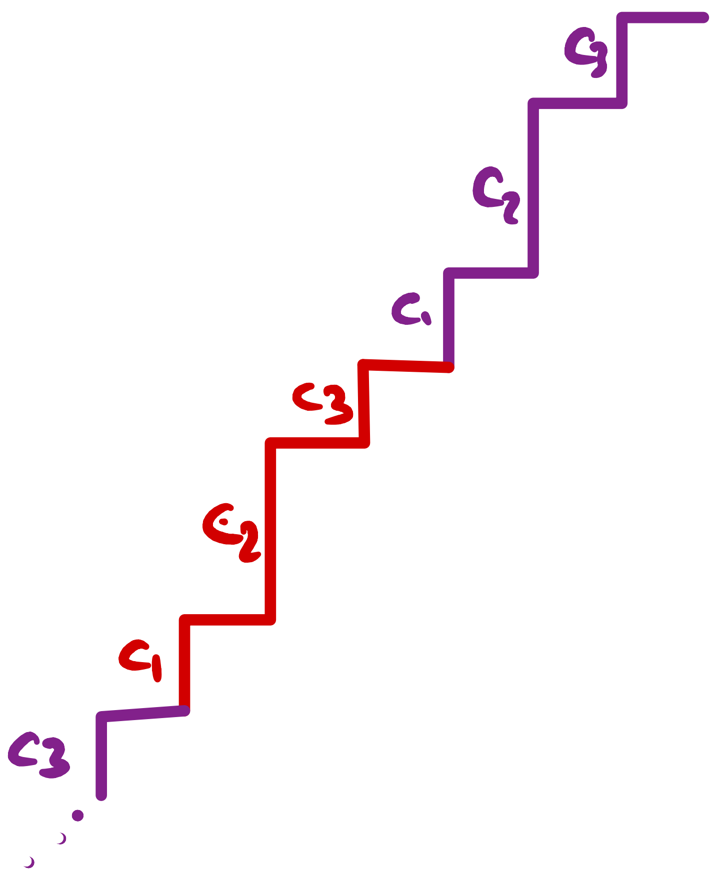
Cylindric partitions (Gessel & Krattenthaler) $d, r > 0$

c



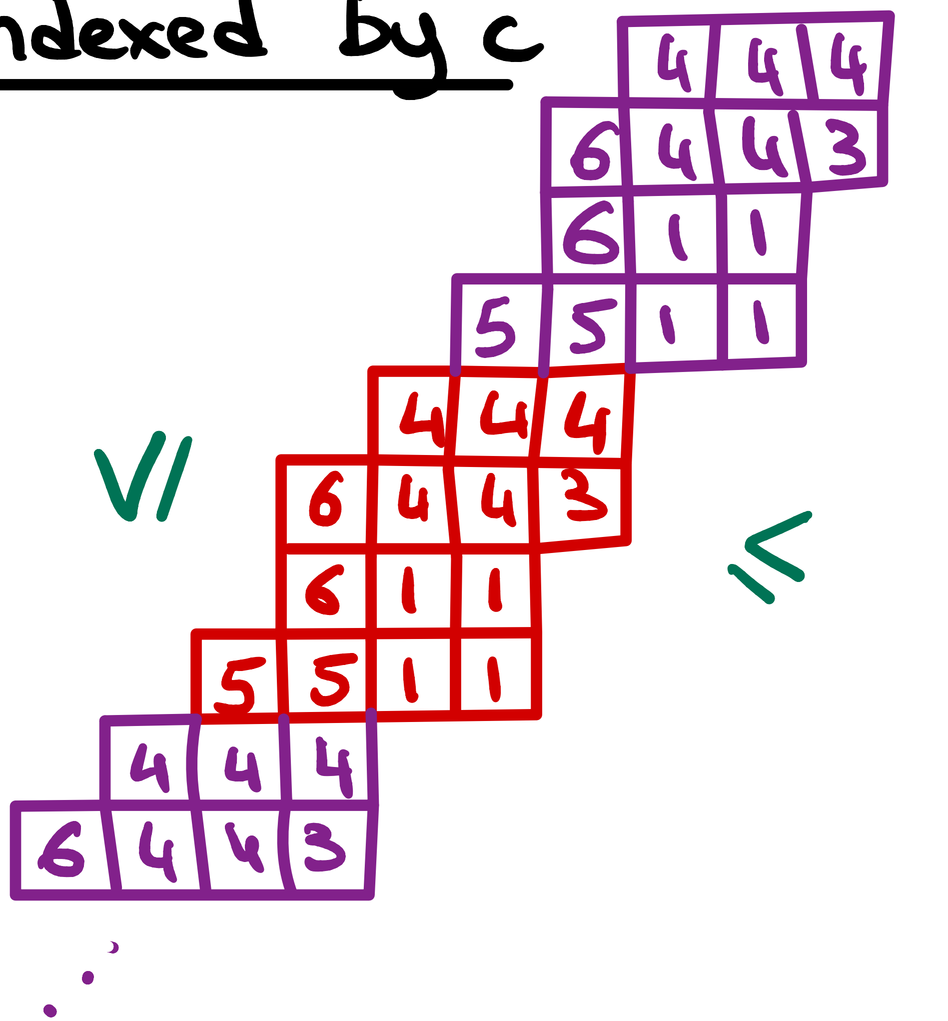
c a composition $(c_1, c_2, \ldots, c_r)$

$c_1 + c_2 + \ldots + c_r = d$

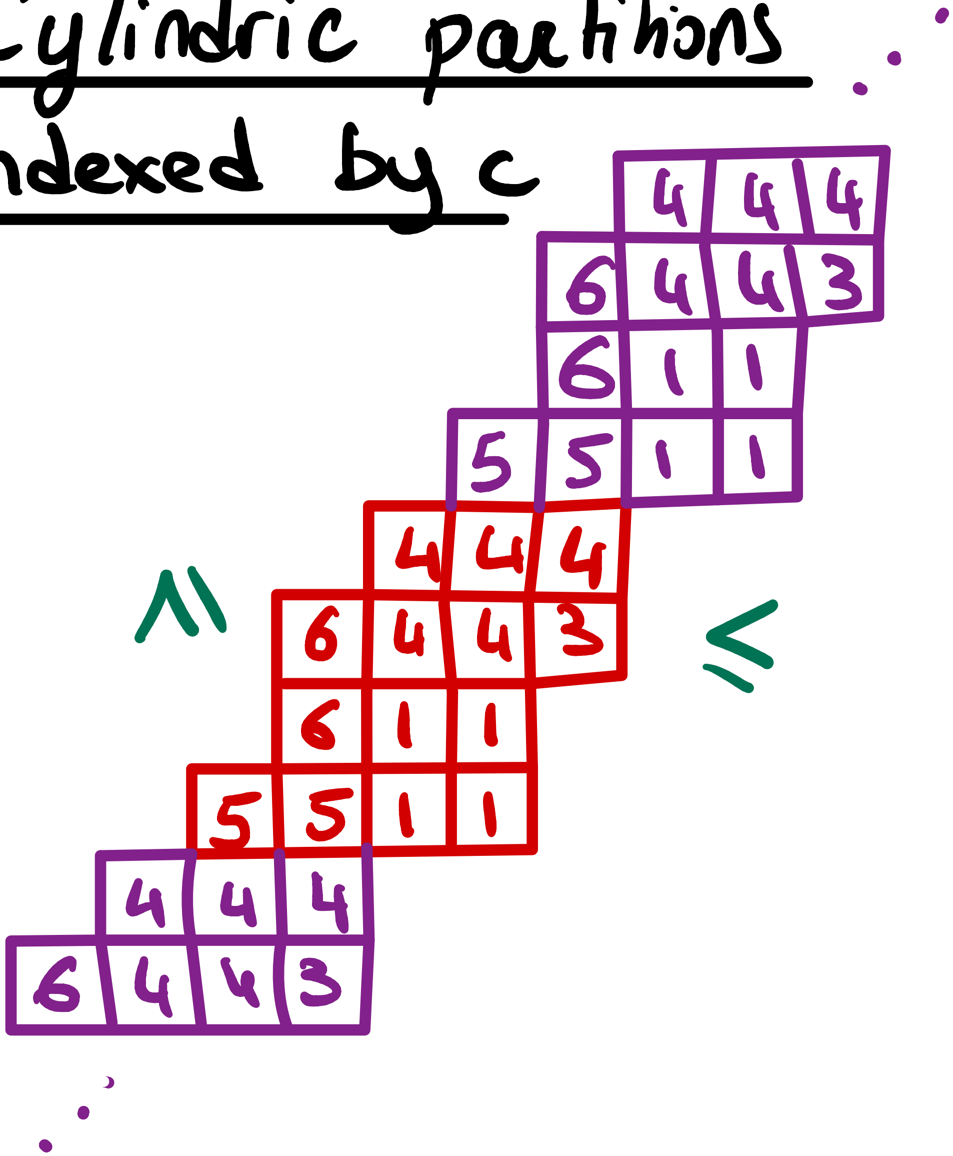


on the cylinder

Cylindric partitions indexed by c



Cylindric partitions indexed by c



Weight

$|\Lambda| = 49$

__Theorem__ (Borodin 2007) $d+r=t$

$$\sum_{\substack{a \text{ cylindric} \\ \text{of profile} \\ c}} q^{|A|} =$$ Beautiful product involving hooks

$$= \frac{1}{(q^t;q^t)_\infty} \prod_{n \geq 0} \prod_{\square \in c} \frac{1}{1-q^{nt+\text{cylhook}(\square)}}$$

Theorem (Borodin 2007) $d+r=t$

$$\sum_{\substack{\text{a cylindric} \\ \text{of profile} \\ c}} q^{|\Lambda|} = \text{Beautiful product involving hooks}$$

$$= \frac{1}{(q^t;q^t)_\infty} \prod_{n\geq 0} \prod_{ij\in c} \frac{1}{1-q^{nt+\mathrm{cylhook}(\square)}}$$

Proof: Vertex operators

Schur polynomials

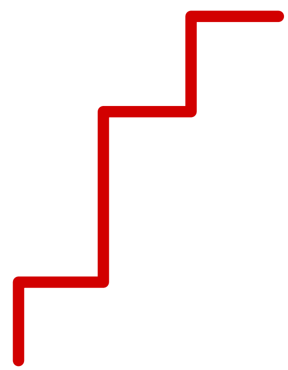
Other proofs: Hypergeometric series (Krattenthaler)

Bijective (Langer)

Cylindric hooks

$c = (1,2,1)$

$d = 4 \quad r = 3$



Cylindric hooks

$c = (1,2,1)$

$d = 4 \quad r = 3$

		1
	1	3
	2	4
1	4	6

Cylindric hooks

$c = (1, 2, 1)$

$d = 4 \quad r = 3$

3	6	1
5	1	3
6	2	4
1	4	6

Cylindric hooks

$c = (1,2,1)$

$d = 4 \quad r = 3$

3	6	1
5	1	3
6	2	4
1	4	6

$$\sum_{\lambda \text{ profile } (1,2,1)} q^{|\lambda|} = \frac{1}{(q^1,q^1,q^1,q^2,q^3,q^4,q^4,q^6;q^7)_\infty} \frac{1}{(q^3,q^5,q^6,q^6;q^7)_\infty} \frac{1}{(q^7;q^7)_\infty}$$

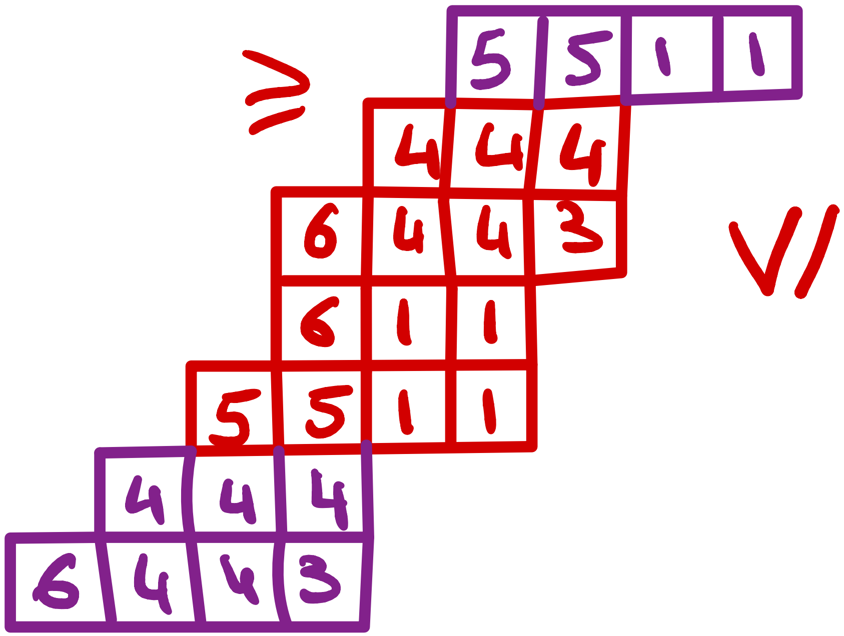
Theorem (Borodin)

$$\sum_{\substack{\lambda \text{ cylindric} \\ \text{of profile } c}} q^{|\lambda|} = \text{Beautiful product involving hooks}$$

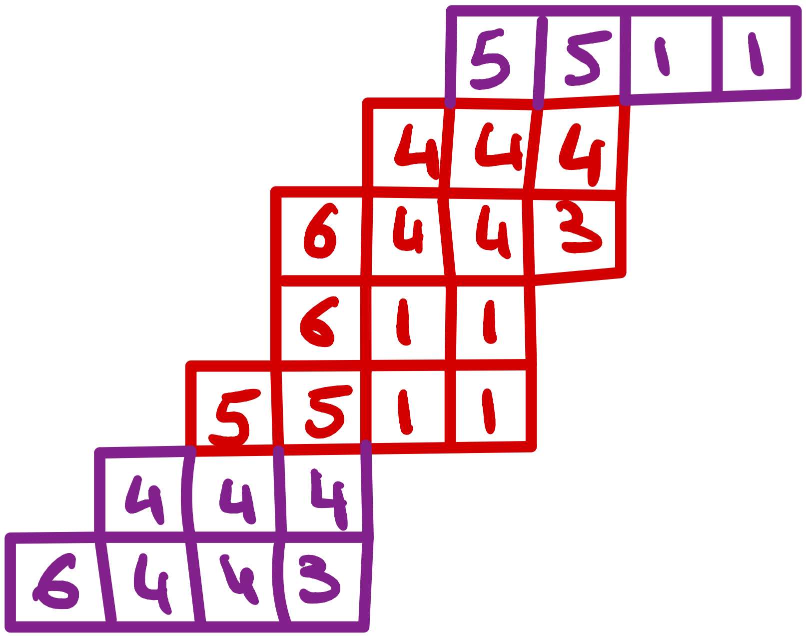
ex $\quad c = (2,1)$

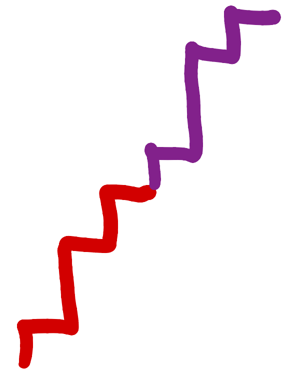
$$\text{Product} = \frac{1}{(q;q)_\infty (q,q^5)_\infty (q^4;q^5)_\infty}$$

Cylindric partitions



Cylindric partitions



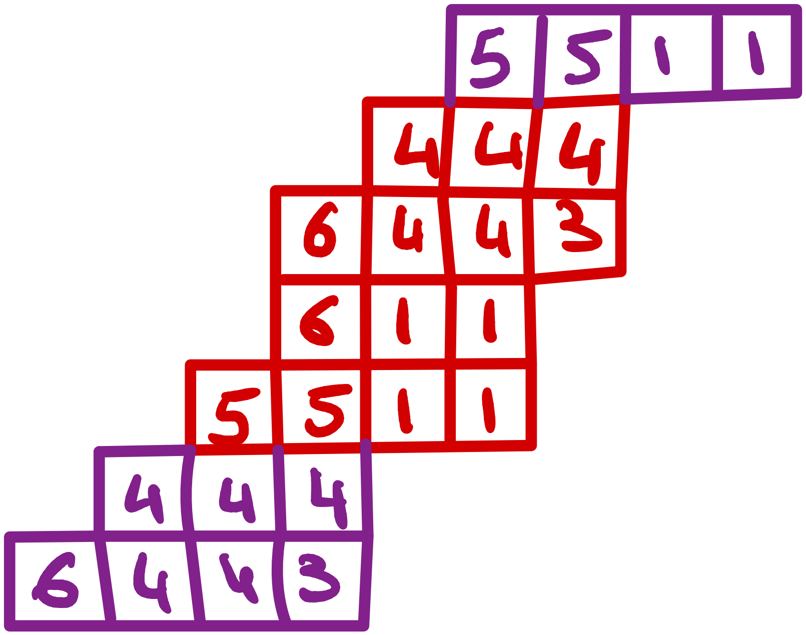


Cylindric partitions of profile (1, 2, 1)

$$\lambda^{(0)} \preccurlyeq \lambda^{(1)} \succcurlyeq \lambda^{(2)} \preccurlyeq \lambda^{(3)} \preccurlyeq \lambda^{(4)} \succcurlyeq \lambda^{(5)} \preccurlyeq \lambda^{(6)} \succcurlyeq \lambda^{(7)}$$

$$\lambda^{(0)} = \lambda^{(7)}$$

Cylindric partitions



c a composition $(c_1, c_2, \ldots, c_r)$

$\Big\downarrow$ interlacing word

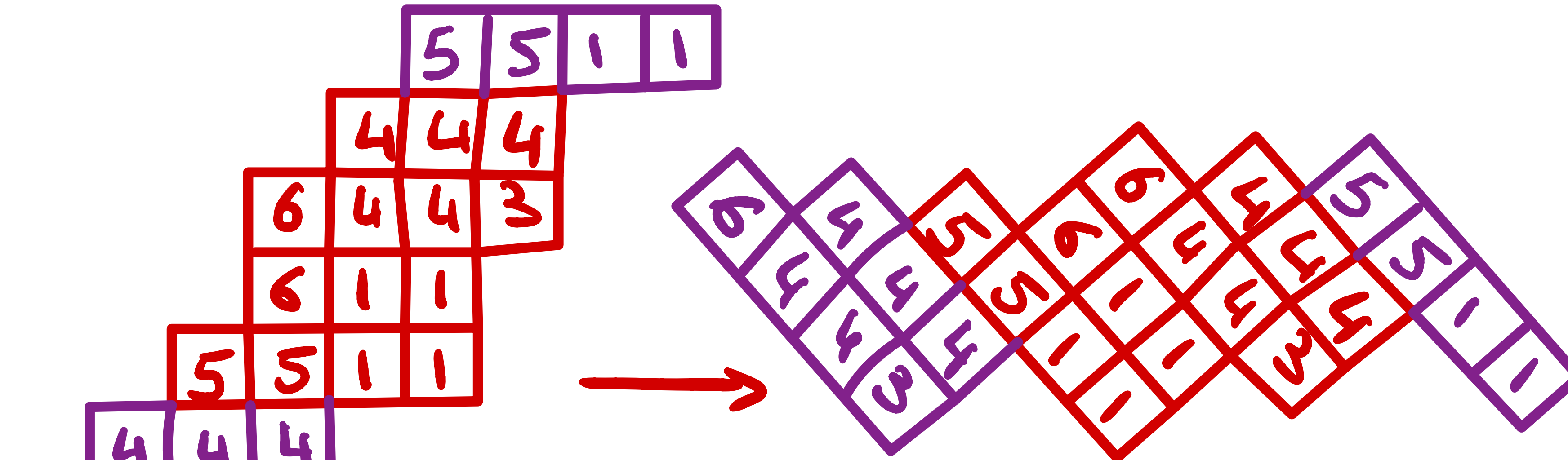
$$(\preccurlyeq)^{c_1} \succcurlyeq (\preccurlyeq)^{c_2} \succcurlyeq \cdots$$

Cylindric partitions of profile $(1,2,1)$

$$\lambda^{(0)} \preccurlyeq \lambda^{(1)} \succcurlyeq \lambda^{(2)} \preccurlyeq \lambda^{(3)} \preccurlyeq \lambda^{(4)} \succcurlyeq \lambda^{(5)} \preccurlyeq \lambda^{(6)} \succcurlyeq \lambda^{(7)}$$

$$\lambda^{(0)} = \lambda^{(7)}$$

Cylindric partitions

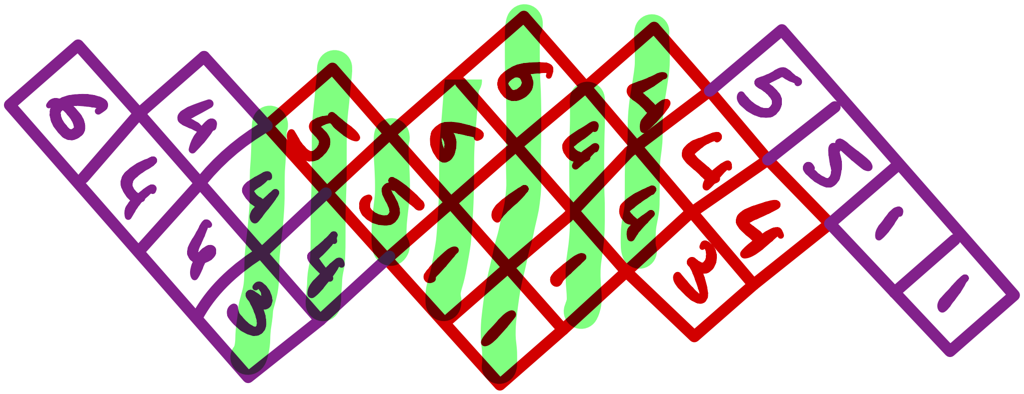


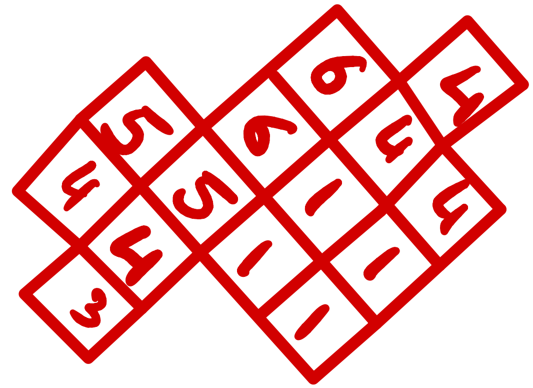
Cylindric partitions of profile $(1,2,1)$

$$\lambda^{(0)} \preccurlyeq \lambda^{(1)} \succcurlyeq \lambda^{(2)} \preccurlyeq \lambda^{(3)} \preccurlyeq \lambda^{(4)} \succcurlyeq \lambda^{(5)} \preccurlyeq \lambda^{(6)} \succcurlyeq \lambda^{(7)}$$

$$\lambda^{(0)} = \lambda^{(7)}$$

Cylindric partitions





Cylindric partitions of profile $(1,2,1)$

$$\lambda^{(0)} \preccurlyeq \lambda^{(1)} \succcurlyeq \lambda^{(2)} \preccurlyeq \lambda^{(3)} \preccurlyeq \lambda^{(4)} \succcurlyeq \lambda^{(5)} \preccurlyeq \lambda^{(6)} \succcurlyeq \lambda^{(7)} = \lambda^{(0)}$$

$$(4,3) \preccurlyeq (5,4) \succcurlyeq (5) \preccurlyeq (6,1) \preccurlyeq (6,1,1)$$
$$\succcurlyeq (4,1) \preccurlyeq (4,4) \succcurlyeq (4,3)$$

Theorem (Borodin)

$$\sum_{\substack{\text{a cylindric} \\ \text{of profile } c}} q^{|\Lambda|} = \text{Beautiful product involving hooks}$$

ex $c = (2,1)$

Rogers - Ramanujan identity I

$$\frac{1}{(q;q)_\infty} \sum_{n \geqslant 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)(q^4;q^5)} \cdot \frac{1}{(q;q)_\infty}$$

Theorem (Borodin)

$$\sum_{\substack{\lambda \text{ cylindric} \\ \text{of profile} \\ w_1 \ldots w_T}} q^{|\lambda|} = \text{Beautiful product involving hooks}$$

ex $c = (3,0)$

Rogers - Ramanujan identity II

$$\frac{1}{(q;q)_\infty} \sum_{n \geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \cdot \frac{1}{(q;q)_\infty}$$

Foda & Welsh (2016) $r = 2$ $c = (c_1, c_2)$ $c_1+c_2=d$

Andrews - Gordon - Bressoud identities

$c = (2k-i, i-1)$

$$\frac{1}{(q;q)_\infty} \sum_{n_1,\dots,n_{k-1}} \frac{q^{n_1^2+\dots+n_{k-1}^2 + n_i + \dots + n_{k-1}}}{(q)_{n_1-n_2}(q)_{n_2-n_3}\cdots(q)_{n_{k-2}-n_{k-1}}} \frac{1}{(q)_{n_{k-1}}}$$

$$= \frac{(q^{2k+1}, q^i, q^{2k+1-i}; q^{2k+1})_\infty}{(q)_\infty} \cdot \frac{1}{(q;q)_\infty}$$

$c = (2k+1-i, i-1)$

$$\frac{1}{(q;q)_\infty} \sum_{n_1,\dots,n_{k-1}} \frac{q^{n_1^2+\dots+n_{k-1}^2 + n_i + \dots + n_{k-1}}}{(q)_{n_1-n_2}(q)_{n_2-n_3}\cdots(q)_{n_{k-2}-n_{k-1}}} \frac{1}{(q^2;q^2)_{n_k}}$$

$$= \frac{(q^{2k}, q^i, q^{2k-i}; q^{2k})_\infty}{(q)_\infty} \cdot \frac{1}{(q;q)_\infty}$$

Again the cylindric partitions give

$$\frac{1}{(q;q)_\infty} \cdot \text{Product of the A-G-B identities.}$$

<u>Q</u>: How about $r \geq 3$?

$r = 3 \quad d = 4$

A_2 - Rogers Ramanujan identities

(Andrews, Schilling & Warnaar 2002

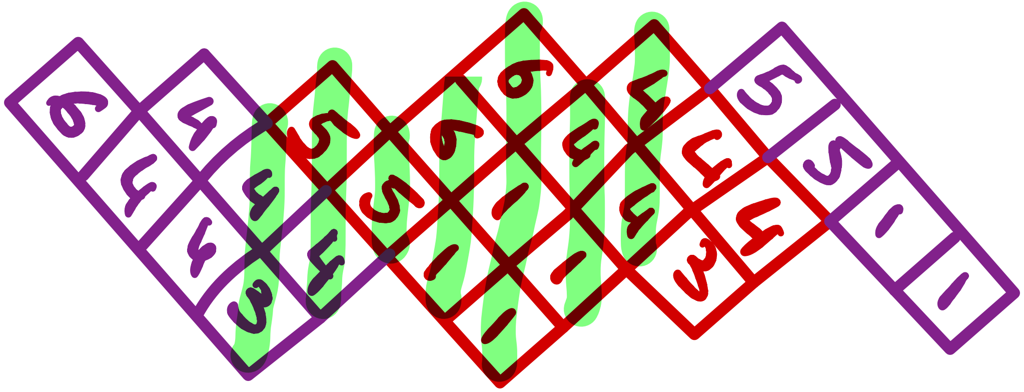
C., Welsh 2019)

Conjecture

There exists a Rogers–Ramanujan identity for each composition $(c_1, \ldots, c_r)$ of d

for all r & d

"Up to rotation and conjugation"



Rotation

$c = (1, 2, 1)$

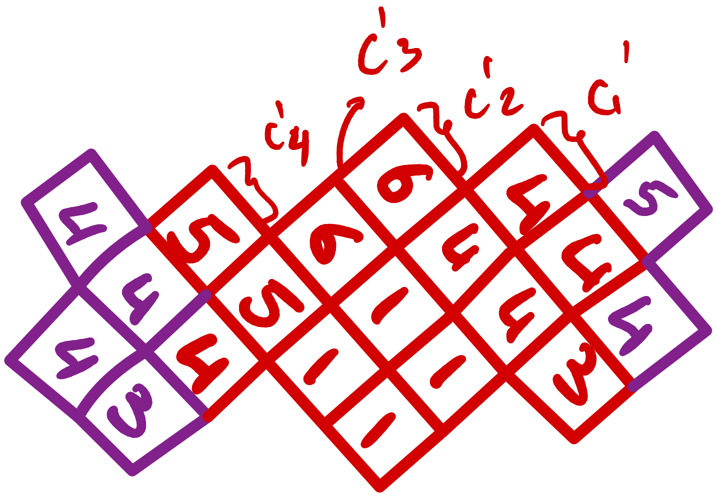
$\updownarrow$

$\tilde{c} = (2, 1, 1)$

$\updownarrow$

$\tilde{\tilde{c}} = (1, 1, 2)$

"Up to rotation and conjugation"



Rotation

$c = (1,2,1)$

$\updownarrow$

$c^r = (2,1,1)$

$\updownarrow$

$c^{rr} = (1,1,2)$

Conjugation

$c = (1,2,1) \longleftrightarrow c' = (1,1,0,1)$

Main Tool (Corteel & Welsh 2019)

$$c = (c_1, \ldots, c_r)$$

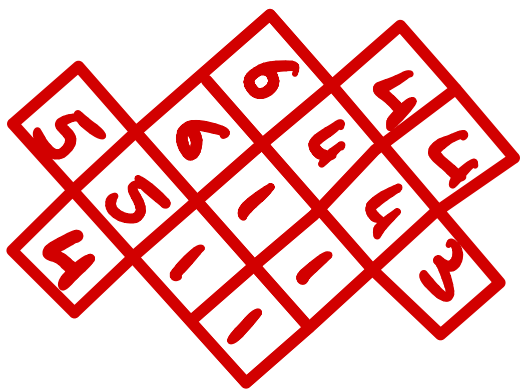
$$F_c(z;q) = \sum_{\lambda \text{ profile } c} z^{\max(\lambda)} q^{|\lambda|}$$

$$G_c(z,q) = (zq;q)_\infty F_c(z;q)$$

Main Tool (Corteel & Welsh)

$$c = (c_1, \ldots, c_r)$$

$$G_c(z;q) = (zq;q)_\infty \sum_{\Lambda \text{ profile } c} z^{\max(\Lambda)} q^{|\Lambda|}$$



$$|\Lambda| = 49$$

$$\max(\Lambda) = 6$$

Main Tool (Corteel & Welsh 2019)

$$c = (c_1, \dots, c_n)$$

$$G_c(z;q) = (zq;q)_\infty \sum_{\Lambda \text{ profile } c} z^{\max(\Lambda)} \, q^{|\Lambda|}$$

Theorem $$G_c(z,q) = \sum_{S \text{ subset of the corners}} (-1)^{|S|} \, G_{c(S)}(zq^{|S|};q)\,(zq;q)_{|S|-1}$$

Example $\quad r = 2 \quad d = 3$

$$G_{30}(z,q) = G_{2,1}(zq,q)$$

$$G_{21}(z,q) = G_{30}(zq,q) + G_{21}(zq,q) - G_{21}(zq^2,q)(1-zq)$$

$$= G_{21}(zq,q) + zq\, G_{21}(zq^2,q)$$

Example

$r = 2 \qquad d = 3$

$$G_{30}(z,q) = G_{2,1}(zq,q)$$

$$G_{21}(z,q) = G_{21}(zq,q) + zq\, G_{21}(zq^2,q)$$

Lemma

$$G_{30}(z,q) = \sum_{n \geq 0} \frac{z^n q^{n^2+n}}{(q;q)_n}$$

$$G_{21}(z,q) = \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q;q)_n}$$

Conjecture (Corteel & Welsh 2019, Warnaar 2021)

$c = (c_1, \ldots, c_r)$ $\quad c_1 + \ldots + c_r = d \quad k = \gcd(d, r)$

$$G_c(z, q) = \sum_n \frac{z^n}{(q^k; q^k)_n} P_{c,n}(q)$$

Conjecture (Corteel & Welsh, Warnaar)

$c = (c_1, \ldots, c_n) \qquad c_1 + \ldots + c_n = r \qquad d = \gcd(n, r)$

$$G_c(z,q) = \frac{1}{(zq;q)_\infty} \sum_{n_1} \frac{z^n}{(q^k;q^k)_n} P_{c,n}(q)$$

$P_{c,n}(q)$ is a polynomial into non negative coefficients

$$P_{c,n}(1) = k^n \left(\frac{1}{d+r} \binom{d+r}{r} - 1 \right)^n$$

What is known so far?

- $r = 1$ ✓
- $r = 2$ ✓
- $r = 3$ $\quad d = 2, 4, 5$

(C. 2016, C & Welsh 2019
C. Dousse & Uncu 2021
Warnaar 2021)

$d = 3$ (Tsuchioka 2022)

What is known so far?

- $n = 1$
- $n = 2$
- $n = 3$ $\quad r = 2, 4, 5$

  (C. 2016, C & Welsh 2019
  C. Dousse & Uncu 2021
  Warnaar 2021)

  $r = 3$ (Tsuchioka 2022)

<u>Conjecture</u> (Warnaar 2021) $\quad k > 0$

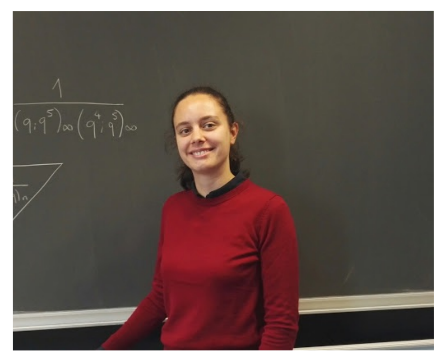
$c = (k, k-1, k-1)$ $\quad c = (3k-s, s-1, 0)$

$c = (k, k, k-1)$ $\quad c = (3k-s-1, s-1, 0)$

$1 \leq s \leq k+1$

Case $d = 5$, $r = 3$ (C., Dousse & Uncu 2021)

$$\begin{aligned}
G_{(5,0,0)}(z,q) &= G_{(4,1,0)}(zq,q),\\
G_{(4,1,0)}(z,q) &= G_{(4,0,1)}(zq,q) + G_{(3,2,0)}(zq,q) - (1-zq)G_{(3,1,1)}(zq^2,q),\\
G_{(4,0,1)}(z,q) &= G_{(5,0,0)}(zq,q) + G_{(3,1,1)}(zq,q) - (1-zq)G_{(4,1,0)}(zq^2,q),\\
G_{(3,2,0)}(z,q) &= G_{(3,1,1)}(zq,q) + G_{(3,0,2)}(zq,q) - (1-zq)G_{(2,2,1)}(zq^2,q),\\
G_{(3,1,1)}(z,q) &= G_{(4,1,0)}(zq,q) + G_{(3,0,2)}(zq,q) + G_{(2,2,1)}(zq,q)\\
&\quad - (1-zq)\big(G_{(4,0,1)}(zq^2,q) + G_{(3,2,0)}(zq^2,q) + G_{(2,2,1)}(zq^2,q)\big)\\
&\quad + (1-zq)(1-zq^2)G_{(3,1,1)}(zq^3,q),\\
G_{(3,0,2)}(z,q) &= G_{(4,0,1)}(zq,q) + G_{(2,2,1)}(zq,q) - (1-zq)G_{(3,1,1)}(zq^2,q),\\
G_{(2,2,1)}(z,q) &= G_{(3,2,0)}(zq,q) + G_{(3,1,1)}(zq,q) + G_{(2,2,1)}(zq,q)\\
&\quad - (1-zq)\big(G_{(3,1,1)}(zq^2,q) + G_{(3,0,2)}(zq^2,q) + G_{(2,2,1)}(zq^2,q)\big)\\
&\quad + (1-zq)(1-zq^2)G_{(2,2,1)}(zq^3,q).
\end{aligned}$$



C. Dousse & Uncu Automatic proofs

$$
\begin{aligned}
G_{(5,0,0)}(z,q) &= G_{(4,1,0)}(zq,q),\\
G_{(4,1,0)}(z,q) &= G_{(4,0,1)}(zq,q) + G_{(3,2,0)}(zq,q) - (1-zq)G_{(3,1,1)}(zq^2,q),\\
G_{(4,0,1)}(z,q) &= G_{(5,0,0)}(zq,q) + G_{(3,1,1)}(zq,q) - (1-zq)G_{(4,1,0)}(zq^2,q),\\
G_{(3,2,0)}(z,q) &= G_{(3,1,1)}(zq,q) + G_{(3,0,2)}(zq,q) - (1-zq)G_{(2,2,1)}(zq^2,q),\\
G_{(3,1,1)}(z,q) &= G_{(4,1,0)}(zq,q) + G_{(3,0,2)}(zq,q) + G_{(2,2,1)}(zq,q)\\
&\quad - (1-zq)\big(G_{(4,0,1)}(zq^2,q) + G_{(3,2,0)}(zq^2,q) + G_{(2,2,1)}(zq^2,q)\big)\\
&\quad + (1-zq)(1-zq^2)G_{(3,1,1)}(zq^3,q),\\
G_{(3,0,2)}(z,q) &= G_{(4,0,1)}(zq,q) + G_{(2,2,1)}(zq,q) - (1-zq)G_{(3,1,1)}(zq^2,q),\\
G_{(2,2,1)}(z,q) &= G_{(3,2,0)}(zq,q) + G_{(3,1,1)}(zq,q) + G_{(2,2,1)}(zq,q)\\
&\quad - (1-zq)\big(G_{(3,1,1)}(zq^2,q) + G_{(3,0,2)}(zq^2,q) + G_{(2,2,1)}(zq^2,q)\big)\\
&\quad + (1-zq)(1-zq^2)G_{(2,2,1)}(zq^3,q).
\end{aligned}
$$

Theorem (C., Dousse, Uncu 2021)

$$
\sum_{n_1,n_2,n_3,n_4\geq 0} \frac{q^{n_1^2+n_2^2+n_3^2+n_4^2-n_1n_2+n_2n_4}}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q = \frac{1}{(q,q,q^2,q^4,q^4,q^6,q^7,q^7;q^8)_\infty}.
$$

Automatic proof

C., Welsh

2019

$$G_c(z,q) = (zq;q)_\infty F_c(z,q)$$

New approach

Kanade & Russell (2022)

$$H_c(z,q) = \frac{(q;q)_\infty}{(zq,q)_\infty} F_c(z,q)$$

C. Welsh
2019

$$G_c(z,q) = (zq;q)_\infty F_c(z,q)$$

New approach

Kanade & Russell (2022)

$$H_c(z,q) = \frac{(q;q)_\infty}{(zq,q)_\infty} F_c(z,q)$$

$r=3$ $\quad c=(k-s, s, 0)$ $\quad 0 \leq s < \frac{k}{3}$

Andrews, Schilling, Warnaar (2002)

$r=3$ all compositions of $d \leq 7$, $d \neq 6$ (Kanade & Russell 2021)

all compositions of 8 (Uncu 2022)

Proof techniques

- Combinatorics
- q-series
- Computer algebra
- Lie theory

Proof techniques

- Combinatorics
- q - series
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Q: Can we combine all those to prove identities for $c = (c_1, \ldots, c_r)$ compositions of d?

Q: How to guess the sum side?

Combinatorics: could we find more statistics on cylindric partitions

1^{st} sum $\longleftrightarrow$ max(Λ)

2^{nd} sum $\longleftrightarrow$??

$\vdots$

Linz Fall 2022

A lovely special case $\quad d = r+1$

Conjecture $(c_1, \dots, c_r)$

There are C_r RR identities and they are of the form

$$\frac{1}{(q)_\infty} \sum_n \frac{P_{c,n}(q)}{(q;q)_{n_1}} = \prod_{\square \in c} \frac{1}{1 - q^{\mathrm{cychook}(\square)}}$$

A lovely special case $d = r+1$

Conjecture $(c_1, \ldots, c_r)$ Catalan

There are C_r RR identities and they are of the form

$$\frac{1}{(q)_\infty} \sum_n \frac{P_{c,n}(q)}{(q;q)_n} = \prod_{\square \in c} \frac{1}{1 - q^{\mathrm{cychook}(\square)}}$$

$P_{c,n}(q) \in \mathbb{N}[q]$ $\quad$ $P_{c,n}(1) = (C_r - 1)^n$

Open for $r \geq 4$


















Merci !