THE DOMINATION NUMBER IN GALTON-WATSON TREES

V.R. MISANANTENAINA, F. RAKOTONIAINA, AND D. RALAIVAOSAONA

Abstract. We study the distribution of the domination number in a random rooted plane tree where a random tree is generated from the conditioned Galton-Watson model with offspring distribution ξ that satisfies $\mathbb{E}\xi = 1$ and $0 < \mathrm{Var}\xi < \infty$. We show that the domination number in such a random tree is asymptotically normal as the order of the tree tends to infinity. We also prove that the same result holds for the distribution of the total domination number. Our method is based on the analysis of the Cockayne-Goodman-Hedetniemi (CGH) algorithm and a recent extension of Janson's result on local additive functionals.

Keywords. Domination number, algorithm, Galton-Watson trees, central limit theorem.

1. Introduction and result statement

Let $G = (V, E)$ be a simple graph. A set $X \subseteq V$ is a dominating set of G if every vertex in $V \setminus X$ is adjacent to a vertex in X. The *domination number* of G is the cardinality of the smallest dominating set of G. If in the definition above $V \setminus X$ is replaced by V, then we say that X is a total dominating set, and the *total domination number* is the cardinality of the smallest total dominating set of G. Domination is one of the fastest growing areas within graph theory, with several applications in computer science and engineering. For literature on both parameters, we refer the reader to [4, 5]. In our context, we focus on the domination of conditioned Galton-Watson trees. The domination number has been studied for other models of random trees, see for example, the recent paper by Fuchs *et al.* [3] for the random recursive tree and the random binary search tree.

Given a discrete random variable ξ with support on $\{0, 1, 2, \ldots\}$, the Galton-Watson tree $\mathcal{T}$ with offspring distribution ξ is the random tree generated in the following way: we start with a root and generate a number of children according to ξ. For each of the children of the root, generate a number of children of their own according to ξ independently of the other vertices. Then, we repeat this process until it stops. The standard assumptions on ξ are $\mathbb{E}\xi = 1$ and $0 < \mathrm{Var}\xi < \infty$. Under these assumptions, the process described above stops at a finite time almost surely, and the resulting tree is the random tree $\mathcal{T}$. We are interested in the domination number and the total domination number of the random tree $\mathcal{T}_n$ which is defined to be the Galton-Watson tree conditioned to have exactly n vertices. To the best of our knowledge we are not aware of any results in this direction in the literature. Let us now state our main result.

Theorem 1.1. *Let $D(T)$ denote the domination number of a tree T, and let $\mathcal{T}$ be the Galton-Watson tree of offspring distribution ξ with mean $\mathbb{E}\xi = 1$ and finite non-zero variance. Let $\mathcal{T}_n$ denote the conditioned Galton-Watson tree of order n. Then there exist constants $\delta > 0$ and $\gamma^2 \geq 0$ such that $\mathbb{E}D(\mathcal{T}_n) = \delta n + o(\sqrt{n})$, and*

$$\frac{D(\mathcal{T}_n) - \delta n}{\sqrt{n}} \xrightarrow{\mathrm{d}} \mathcal{N}(0, \gamma^2), \quad \textit{as} \quad n \to \infty.$$

Similarly, if $\hat{D}(T)$ denotes the total domination number of a tree T, then there exist constants $\hat{\delta} > 0$ and $\hat{\gamma}^2 \geq 0$ such that $\mathbb{E}\hat{D}(\mathcal{T}_n) = \hat{\delta} n + o(\sqrt{n})$, and

$$\frac{\hat{D}(\mathcal{T}_n) - \hat{\delta} n}{\sqrt{n}} \xrightarrow{\mathrm{d}} \mathcal{N}(0, \hat{\gamma}^2), \quad \textit{as} \quad n \to \infty.$$

2. Ideas of proof

Many graph theoretic parameters of trees satisfy the so-called additive property. A tree functional $F : \mathbb{T} \to \mathbb{R}$ (where $\mathbb{T}$ denotes the class of finite rooted plane trees) is called additive if there exists a functional $f : \mathbb{T} \to \mathbb{R}$, known as the toll function, such that for every $T \in \mathbb{T}$, $F(T) = \sum_i F(T_i) + f(T)$, where $T_1, T_2, \dots$ are the branches of T. Janson showed in [6] that if the toll function is bounded and local, then the $F(\mathcal{T}_n)$ is asymptotically normal. Here, a toll function f is local if there exists a fixed integer $M > 0$ such $f(T^{(M)}) = f(T)$ for any tree T, where $T^{(M)}$ denotes the tree consisting of all vertices of T at a distance at most M from the root.

Using the CHG algorithm, see [1, 5], each of the parameters that we are interested in can be approximated by an additive functional $F(T)$ where the corresponding toll function f is bounded but not local. However, CHG algorithm allows us to partition the class of trees $\mathbb{T}$ into a finite number of subclasses $\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_m$ and we show that the toll function f is only a linear combination of the indicator functions of these subclasses. Using the properties of this partition, we are able to show that the toll function f is almost-local, according to [7], so the main result in that paper can be applied to prove Theorem 1.1.

3. Numerical results and concluding remarks

Many practical examples of Galton-Watson trees are equivalent to simply generated tree models, see [2]. In those instances, tree enumeration problems and some parameter studies can be done via generating functions and the methods of analytic combinatorics. Using this approach, the constants δ and $\hat{\delta}$ can be expressed in terms of the solutions of some systems of equations which, in turn, can be solved numerically. Below is the table of numerical values for δ and $\hat{\delta}$ for three of these random tree models.

	ξ	δ	$\hat{\delta}$
Binary trees	Binomial B(2, 1/2)	0.35411	0.37563
Plane trees	Geometric G(1/2)	0.34729	0.38316
Labelled plane trees	Poisson Pois(1)	0.36008	0.43695

Table 1. Numerical values of δ and $\hat{\delta}$.

It might also be possible to prove the central limit theorems in Theorem 1.1 using the system of equations method in [2], but for the condition Galton-Watson trees that are equivalent to simply generated tree models, the offspring distribution ξ needs to satisfy stronger moment conditions. Therefore, we did not consider this alternative approach. It appears that Theorem 1.1 can be rewritten more generally in terms of additive functionals with toll functions that are linear combinations of indicator functions. In this way, more graph theoretic parameters can be covered, such as the independence number, the size of kernel, etc. The full paper version of this extended abstract is currently under preparation and will be made available on ArXiv soon.

References

[1] E. Cockayne, S. Goodman, and S. Hedetniemi. A linear algorithm for the domination number of a tree. *Information Processing Letters*, 4(2):41–44, 1975.

[2] M. Drmota. *Random trees: an interplay between combinatorics and probability.* Springer, 2009.

[3] M. Fuchs, C. Holmgren, D. Mitsche, and R. Neininger. A note on the independence number, domination number and related parameters of random binary search trees and random recursive trees. *Discret. Appl. Math.*, 292:64–71, 2021.

[4] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning. *Topics in Domination in Graphs.* Developments in Mathematics. Springer, 2020.

[5] M. A. Henning and A. Yeo. *Total domination in Graphs.* Springer Monographs in Mathematics, New York, 2013.

[6] S. Janson. Asymptotic normality of fringe subtrees and additive functionals in conditioned galton watson trees. *Random Structures Algorithms*, 48(1):57–101, 2016.

[7] D. Ralaivaosaona, M. Šileikis, and S. Wagner. A central limit theorem for almost local additive tree functionals. *Algorithmica*, 82(3):642–679, 2020.