Counting linear intervals in the Tamari and Dyck lattices

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Abstract

Several classical lattice structures can be put on Dyck paths or other Catalan objects such as binary trees, among which the well known Tamari lattice and the inclusion order on Dyck paths, that we will call the Dyck lattice (or Stanley lattice). In this work, we are interested in the simplest possible intervals, namely the linear intervals. These are those which are totally ordered. We count them according to their height.

To do so, we first study the structure of linear intervals, then we give a combinatorial description and we finally solve the equations that we found.

The Tamari lattice can be defined on rooted planar binary trees (we will simply call them trees) with left rotations as covering relations. The Dyck lattice can be defined on Dyck paths with the inclusion order. In this case, covering relations are changes of a valley into a peak.

We are interested in the linear intervals, that is to say the totally ordered ones. In both lattices, we can define a subset of intervals that we call left and right intervals, and they are linear.

Proposition 1. *In the Tamari lattice and the Dyck lattice :*

- *All linear intervals of height* 0 *are trivial.*
- *All linear intervals of height* 1 *are covering relations.*
- *All linear intervals of height* $k \geq 2$ *are either left or right intervals.*

Then, we can study these linear intervals in both lattices.

Proposition 2 (Linear intervals in the Tamari lattice)**.** *A trivial interval can be described as a non empty tree.*

A covering relation in the Tamari lattice can be described as a tree with a marked node and another tree plugged in the right edge for the bottom element and the left edge for the top element.

A linear interval of height $k \geq 2$ *can be described as a tree with a marked node, a choice of direction (left or right) and a sequence of* k *trees.*

Proposition 3 (Linear intervals in the Dyck lattice)**.** *A trivial interval can be described as a non empty Dyck path.*

A covering relation in the Dyck lattice can be described as a Dyck path with a marked down step and another Dyck path.

A linear interval of height $k \geq 2$ *can be described as a Dyck path with a marked step (either up or down), and a sequence of* k *Dyck paths.*

Surprisingly, those two combinatorial descriptions lead to the same equations, which implies both lattices have the same number of linear intervals.

Proposition 4. *Let B be the generating series of (possibly empty) rooted planar binary trees (resp. Dyck paths) and L_k be the generating series of linear intervals of height k in the Tamari lattice (resp. Dyck lattice). We have the following equations:*

$$\begin{aligned} L_0(t) &= B - 1, \\ L_1(t) &= t^2 B' B, \\ L_k(t) &= t^{k+1} B' B^k. \end{aligned}$$

Using the Lagrange inversion, we can compute the coefficients of L_k, which gives the number of intervals of a given height in both lattices.

Theorem 5. *In the Tamari lattice of size n and in the Dyck lattice of size n, there are:*

- $\frac{1}{n+1}\binom{2n}{n}$ *linear intervals of height* 0,
- $\binom{2n-1}{n-2}$ *linear intervals of height* 1,
- $2\binom{2n-k}{n-k-1}$ *linear intervals of height k, for $2 \leq k < n$.*

Furthermore, there are no linear interval of height $k \geq n$.

References

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