Lecture 5

Finishing Up : Missing Pieces

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Outline

Preamble

In lecture 5 we clean up some of the missing material that was left over from the previous lecture (pages 17-19) and previous material covered in the exercises (pages 4-16).

I also include some material originally planned for this lecture on problems where the input power series are not the power bases. This material starts on page 20. Thus we consider cases where the power series are described by interpolation bases (Newton or Lagrange) or if the power series are actually representative of matrix power series. The main point is that the problem is again reduced to a problem of structured linear algebra but with a Krylov rather than Sylvester structure.

The material on computing with altenate bases is again done with Bernhard Beckermann.

Block Inverse

Given square matrices a_i consider the block Hankel matrix:

$$H_n = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix}$$

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H_n is invertible if and only if:

$$H_n[v_n, \cdots, v_1]^T = -[a_{n+1}, \cdots, a_{2n}]^T, \quad H_n[q_{n-1}, \cdots, q_0]^T = [0, \cdots, 0, 1]^T,$$

or if and only if

$$[v_n^*, \cdots, v_1^*]H_n = -[a_{n+1}, \cdots, a_{2n}], \quad [q_{n-1}^*, \cdots, q_0^*]H_n = [0, \cdots, 0, I].$$

Inverse Formula

Inverse of H_n given by

$$\begin{bmatrix} v_{n-1} & \cdots & v_0 \\ \vdots & & \\ v_0 & & \end{bmatrix} \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & & \vdots \\ & \ddots & \\ & & q_{n-1}^* \end{bmatrix} - \begin{bmatrix} q_{n-2} & \cdots q_0 & 0 \\ \vdots & & \\ q_0 & & \\ 0 & & \end{bmatrix} \begin{bmatrix} v_n^* & \cdots & v_1^* \\ & \ddots & \vdots \\ & & v_n^* \end{bmatrix}$$

where

$$H_n[v_n, \cdots, v_1]^T = -[a_{n+1}, \cdots, a_{2n}]^T, \quad H_n[q_{n-1}, \cdots, q_0]^T = [0, \cdots, 0, 1]^T,$$

and

$$[v_n^*, \cdots, v_1^*]H_n = -[a_{n+1}, \cdots, a_{2n}], \quad [q_{n-1}^*, \cdots, q_0^*]H_n = [0, \cdots, 0, I].$$

Why?

Notice:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix} \cdot \begin{bmatrix} q_{n-1} \\ \vdots \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

same as

$$(a_0 + a_1 z + a_2 z^2 + \cdots)(q_0 + \cdots + q_{n-1} z^{n-1}) - (p_0 + \cdots + p_{n-1} z^{n-1}) = z^{2n-1} + z^{2n} r_1 + \cdots$$

i.e. $a(z)q(z) - p(z) = z^{2n-1} r(z)$ with $r_0 = r(0) = I$.

Notice:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix} \cdot \begin{bmatrix} q_{n-1} \\ \vdots \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

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i.e. $a(z)q(z) - p(z) = z^{2n-1} r(z)$ with $r_0 = r(0) = I$.

Also

$$[q^*_{n-1}, \ldots, q^*_0] \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix} \cdot = [0, \ldots, 0, I]$$

same as

$$(q^*_0 + \cdots + q^*_{n-1} z^{n-1})(a_0 + a_1 z + a_2 z^2 + \cdots) - (p^*_0 + \cdots + p^*_{n-1} z^{n-1}) = z^{2n-1} + z^{2n} r^*_1 + \cdots$$

i.e. $q^*(z)a(z) - p^*(z) = z^{2n-1} r^*(z)$ with $r^*_0 = r^*(0) = I$.

Notice:

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same as

$$(a_0 + a_1 z + a_2 z^2 + \cdots)(v_0 + \cdots + v_n z^n) - (u_0 + \cdots + u_n z^n) = z^{2n+1} w_0 + z^{2n+2} w_1 + \cdots$$

i.e. $a(z)v(z) - u(z) = z^{2n+1} w(z)$ with $v_0 = v(0) = I$.

Notice:

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i.e. $a(z)v(z) - u(z) = z^{2n+1} w(z)$ with $v_0 = v(0) = I$.

Also

$$[v_n^*, \ldots, v_1^*] \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix} \cdot = -[a_{n+1}, \ldots, a_{2n}]$$

same as

$$(v_0^* + \cdots + v_n^* z^n)(a_0 + a_1 z + a_2 z^2 + \cdots) - (u_0^* + \cdots + u_n^* z^n) = z^{2n+1} w_0^* + z^{2n+2} w_1^* + \cdots$$

i.e. $v^*(z)a(z) - u^*(z) = z^{2n+1} w^*(z)$ with $v_0^* = v^*(0) = I$.

Observe

Suppose $a(z)v(z) - u(z) = z^{2n+1}w(z)$ with $v_0 = 1$.

Let $n = 4$ so $u(z)$ and $v(z)$ are both of degree at most 4.

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{array}\right] \left[\begin{array}{cccc} 0 & 0 & 0 & v_4 \\ 0 & 0 & v_4 & v_3 \\ 0 & v_4 & v_3 & v_2 \\ v_4 & v_3 & v_2 & v_1 \\ \hline v_3 & v_2 & v_1 & v_0 \\ v_2 & v_1 & v_0 & 0 \\ v_1 & v_0 & 0 & 0 \\ v_0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc} u_4 & u_3 & u_2 & u_1 \\ & u_4 & u_3 & u_2 \\ & & u_4 & u_3 \\ & & & u_4 \end{array}\right]$$

Observe

Suppose $a(z)v(z) - u(z) = z^{2n+1}w(z)$ with $v_0 = 1$.

Let $n = 4$ so $u(z)$ and $v(z)$ are both of degree at most 4.

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{array}\right] \left[\begin{array}{cccc} 0 & 0 & 0 & v_4 \\ 0 & 0 & v_4 & v_3 \\ 0 & v_4 & v_3 & v_2 \\ v_4 & v_3 & v_2 & v_1 \\ \hline v_3 & v_2 & v_1 & v_0 \\ v_2 & v_1 & v_0 & 0 \\ v_1 & v_0 & 0 & 0 \\ v_0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc} u_4 & u_3 & u_2 & u_1 \\ & u_4 & u_3 & u_2 \\ & & u_4 & u_3 \\ & & & u_4 \end{array}\right]$$

that is:

$$H_4 \cdot \left[\begin{array}{cccc} v_3 & v_2 & v_1 & v_0 \\ v_2 & v_1 & v_0 & 0 \\ v_1 & v_0 & 0 & 0 \\ v_0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc} u_4 & u_3 & u_2 & u_1 \\ & u_4 & u_3 & u_2 \\ & & u_4 & u_3 \\ & & & u_4 \end{array}\right] - H_0 \cdot \left[\begin{array}{cccc} 0 & 0 & 0 & v_4 \\ 0 & 0 & v_4 & v_3 \\ 0 & v_4 & v_3 & v_2 \\ v_4 & v_3 & v_2 & v_1 \end{array}\right]$$

Observe

Suppose $a(z)q(z) - p(z) = z^{2n-1}r(z)$ with $r_0 = 1$.

Use $n = 4$ so $q(z)$ and $p(z)$ are both of degree at most 3.

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{array}\right] \left[\begin{array}{cccc} 0 & 0 & 0 & q_3 \\ 0 & 0 & q_3 & q_2 \\ 0 & q_3 & q_2 & q_1 \\ q_3 & q_2 & q_1 & q_0 \\ \hline q_2 & q_1 & q_0 & 0 \\ q_1 & q_0 & 0 & 0 \\ q_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc} p_3 & p_2 & p_1 & p_0 \\ & p_3 & p_2 & p_1 \\ & & p_3 & p_2 \\ & & & p_3 \end{array}\right]$$

that is:

$$H_4 \cdot \left[\begin{array}{cccc} q_2 & q_1 & q_0 & 0 \\ q_1 & q_0 & 0 & 0 \\ q_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc} p_3 & p_2 & p_1 & p_0 \\ & p_3 & p_2 & p_1 \\ & & p_3 & p_2 \\ & & & p_3 \end{array}\right] - H_0 \cdot \left[\begin{array}{cccc} 0 & 0 & 0 & q_3 \\ 0 & 0 & q_3 & q_2 \\ 0 & q_3 & q_2 & q_1 \\ q_3 & q_2 & q_1 & q_0 \end{array}\right]$$

In general

Notice:

$$H_n \cdot \begin{bmatrix} v_{n-1} & \cdots & v_0 \\ \vdots & & \\ v_0 & & \end{bmatrix} = \begin{bmatrix} u_n & \cdots & u_1 \\ & \ddots & \\ & & u_n \end{bmatrix} - H_0 \begin{bmatrix} & & v_n \\ & & \vdots \\ v_n & \cdots & v_1 \end{bmatrix}$$

$$H_n \cdot \begin{bmatrix} q_{n-2} & \cdots q_0 & 0 \\ \vdots & & \\ q_0 & & \\ 0 & & \end{bmatrix} = \begin{bmatrix} p_{n-1} & \cdots & p_0 \\ & \ddots & \\ & & p_{n-1} \end{bmatrix} - H_0 \begin{bmatrix} & & q_{n-1} \\ & & \vdots \\ q_{n-1} & \cdots & q_0 \end{bmatrix}$$

In general

Notice:

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$$H_n \cdot \begin{bmatrix} q_{n-2} & \cdots q_0 & 0 \\ \vdots & & \\ q_0 & & \\ 0 & & \end{bmatrix} = \begin{bmatrix} p_{n-1} & \cdots & p_0 \\ & \ddots & \\ & & p_{n-1} \end{bmatrix} - H_0 \begin{bmatrix} & & q_{n-1} \\ & & \vdots \\ q_{n-1} & \cdots & q_0 \end{bmatrix}$$

For scalar case we know $q(z)v(z) = v(z)q(z)$ and played with

$$\begin{aligned} a(z)q(z) - p(z) &= z^{2n-1}r(z) \text{ with } r(0) = 1 \\ a(z)v(z) - u(z) &= z^{2n+1}w(z) \text{ with } v(0) = 1 \end{aligned}$$

to get $u(z)q(z) - p(z)v(z) = z^{2n-1}$. This gave the result.

Block Cases

Recall: In block case H_n is nonsingular *iff* there exist matrix polynomials $q(z)$, $v(z)$, $q^*(z)$, $v^*(z)$ such that

$$
\begin{aligned}
a(z)q(z) - p(z) &= z^{2n-1} r(z) \text{ with } r(0) = I \\
a(z)v(z) - u(z) &= z^{2n+1} w(z) \text{ with } v(0) = I
\end{aligned}
$$

$$
\begin{aligned}
q^*(z)a(z) - p^*(z) &= z^{2n-1} r^*(z) \text{ with } r^*(0) = I \\
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Unfortunately no obvious reason why one would get that

$$\begin{aligned} q(z)v^*(z) - v(z)q^*(z) &= 0 \\ u(z)q^*(z) - p(z)v^*(z) &= z^{2n-1} I. \end{aligned}$$

Recall

Remember

$$\begin{aligned} q(z)v^*(z) - v(z)q^*(z) &= 0 \\ u(z)q^*(z) - p(z)v^*(z) &= z^{2n-1}I. \end{aligned}$$

combined with

$$H_n \cdot \begin{bmatrix} v_{n-1} & \cdots & v_0 \\ \vdots & & \\ v_0 & & \end{bmatrix} = \begin{bmatrix} u_n & \cdots & u_1 \\ & \ddots & \\ & & u_n \end{bmatrix} - H_0 \begin{bmatrix} & & v_n \\ & & \vdots \\ v_n & \cdots & v_1 \end{bmatrix}$$

$$H_n \cdot \begin{bmatrix} q_{n-2} & \cdots q_0 & 0 \\ \vdots & & \\ q_0 & & \\ 0 & & \end{bmatrix} = \begin{bmatrix} p_{n-1} & \cdots & p_0 \\ & \ddots & \\ & & p_{n-1} \end{bmatrix} - H_0 \begin{bmatrix} & & q_{n-1} \\ & & \vdots \\ q_{n-1} & \cdots & q_0 \end{bmatrix}$$

would give our inverse formula.

Matrix Polynomial Commutative Formulas

Notice that

$$\begin{array}{rcrcl} a(z)q(z) & - & p(z) & = & z^{2n-1}r(z) \\ v^*(z)a(z) & - & u^*(z) & = & z^{2n+1}w^*(z) \end{array}$$

Intelligent elimination gives

$$\begin{aligned} u^*(z)q(z) - v^*(z)p(z) &= z^{2n-1}(v^*(z)r(z) - z^2w^*(z)q(z)) \\ &= z^{2n-1}(I + \cdots) \\ &= z^{2n-1}I \end{aligned}$$

Similarly

$$p^*(z)q(z) - q^*(z)p(z) = 0.$$

Matrix Polynomial Commutative Formulas

Notice that

$$\begin{aligned} a(z)v(z) &- u(z) = z^{2n+1}r(z) \\ q^*(z)a(z) &- p^*(z) = z^{2n-1}r^*(z) \end{aligned}$$

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$$\begin{aligned} u^*(z)q(z) &- & v_n^*(z)p(z) &= z^{2n-1}I \\ p^*(z)q(z) &- & q^*(z)p(z) &= 0 \\ u^*(z)v(z) &- & v^*(z)u(z) &= 0 \\ q^*(z)u(z) &- & p^*(z)v(z) &= z^{2n-1}I \end{aligned}$$

$$\rightarrow \quad \begin{bmatrix} u*(z) & v^*(z) \\ p^*(z) & q^*(z) \end{bmatrix} \begin{bmatrix} q(z) & -v(z) \\ -p(z) & u(z) \end{bmatrix} = \begin{bmatrix} z^{2n-1}I & 0 \\ 0 & z^{2n-1}I \end{bmatrix}$$

Reverse coefficients in matrix polynomials (denoted by hat):

$$\rightarrow \quad \begin{bmatrix} \hat{u}^*(z) & \hat{v}^*(z) \\ \hat{p}^*(z) & \hat{q}^*(z) \end{bmatrix} \begin{bmatrix} \hat{q}(z) & -\hat{v}(z) \\ -\hat{p}(z) & \hat{u}(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Continued

So:

$$\begin{bmatrix} \hat{u}^*(z) & \hat{v}^*(z) \\ \hat{p}^*(z) & \hat{q}^*(z) \end{bmatrix} \begin{bmatrix} \hat{q}(z) & -\hat{v}(z) \\ -\hat{p}(z) & \hat{u}(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\rightarrow \quad \begin{bmatrix} \hat{q}(z) & -\hat{v}(z) \\ -\hat{p}(z) & \hat{u}(z) \end{bmatrix} \begin{bmatrix} \hat{u}^*(z) & \hat{v}^*(z) \\ \hat{p}^*(z) & \hat{q}^*(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\rightarrow \quad \begin{bmatrix} q(z) & -v(z) \\ -p(z) & u(z) \end{bmatrix} \begin{bmatrix} u^*(z) & v^*(z) \\ p^*(z) & q^*(z) \end{bmatrix} = \begin{bmatrix} z^{2n-1}I & 0 \\ 0 & z^{2n-1}I \end{bmatrix}$$

This gives the desired identities

$$\begin{aligned} u(z)v^*(z) &- p(z)q^*(z) &= z^{2n-1}I \\ q(z)v^*(z) &- v(z)q^*(z) &= 0 \end{aligned}$$

and so ultimately our inverse formula.

What was I trying to describe yesterday?

Given $\mathbf{A}(z) = [a_1(z), \ldots, a_m(z)]$ a vector of power series from $\mathbb{K}[[z]]$, $\vec{n} = (n_1, \ldots, n_m)$ a degree bound, $\mathbb{K}$ integral domain.

We wanted to describe:

- Determine a sequence of degree values $\vec{\nu}_k$ and orders σ_k,

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- Structured linear algebra our guide

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- Elements of $\mathbf{M}_{(\sigma_k,\vec{v}_k)}(z)$ are all determinant polynomials
- Coefficients of $M_{(\sigma_k,\vec{v}_k)}(z)$ grow linearly
- FF algorithm essentially same as Sigma Basis algorithm with an additional normalization step at each iteration

Theorem (Properties of the algorithm FFFG)

For each k let σ_k be the order at step k, a degree bound $\vec{n}_k$ and $\vec{\nu}_{\sigma_k}$ the closest normal point to $\vec{n}_k$. Then

(a) *For all k, σ_k, the vector space of solutions of the linear system of type $(\sigma_k, \vec{n}_k)$ is spanned by the vectors associated with*

$$\mathbf{M}^{(j)}_{(\sigma,\nu_k)}(z),\ z\cdot\mathbf{M}^{(j)}_{(\sigma,\nu_k)}(z),\cdots,\ z^{\vec{n}_k^{(j)}-\vec{\nu}_\sigma^{(j)}-1}\mathbf{M}^{(j)}_{(\sigma,\nu_k)}(z)\quad j=1,...,m.$$

(b) *Any $\mathbf{P}(z)$ of type $(\sigma, \vec{n}_k)$ satisfies*

$$\mathbf{P}(z)=\alpha_1(z)\mathbf{M}^{(1)}_{(\sigma,\nu_k)}(z)+\cdots+\alpha_m(z)\mathbf{M}^{(m)}_{(\sigma,\nu_k)}(z)$$

with deg $\alpha_j(z) < \vec{n}_k^{(j)} - \vec{\nu}_\sigma^{(j)}$.

(c) *For all $k, \sigma \geq 0$ we have* rank $\underline{K}(\vec{n}_k, \sigma) = |\min(\vec{\nu}_\sigma, \vec{n}_k)|$.

Our Problem

- Given $\mathcal{V} \equiv$ domain of power series given in alternate basis
- Input : $f_1, \ldots, f_m \in \mathcal{V}, \qquad n_1, \ldots, n_m,\ \sigma \in \mathbb{Z}$
- Output : *all* solutions of identity :

$$p_1(z) \cdot f_1 + \cdots + p_m(z) \cdot f_m = r$$

  - Here $p_1(z), \ldots, p_m(z) \in \mathbb{K}[z]$
  - $p_i(z)$ of degree at most n_i
  - $r \in \mathcal{V}$ with special properties :

    "Order" of $r \geq \sigma$

The Approach

- Express

$$p_1(z) \cdot f_1 + \cdots + p_m(z) \cdot f_m = r$$

as linear system in unknowns

  - unknowns are the coefficients of $p_1(z), \ldots, p_m(z)$

- Notice : For any polynomial $q(z) = q_0 + q_1 z + \cdots + q_k z^k$:

$$q(z) \cdot f = q_0 \cdot (f) + q_1 \cdot (zf) + q_2 \cdot \left(z^2 f\right) + \cdots$$

$$c_k(\, q(z) \cdot f \,) = q_0 \cdot c_k(f) + q_1 \cdot c_k(zf) + q_2 \cdot c_k(z^2 f) + \cdots$$

- Linear system depends on coefficients of

$$f, \; z \cdot f, \; z^2 \cdot f, \; z^3 \cdot f, \; \ldots$$

The Formal Setting

- $\mathcal{V}$ infinite dimensional vector space over a field $\mathbb{K}$.
  - Basis $\{\ \omega_i\ \}_{i=0,1,\ldots}$ and dual basis $\{\ c_i\ \}_{i=0,1,\ldots}$ for $\mathcal{V}$.

$$f = f_0 \cdot \omega_0 + f_1 \cdot \omega_1 + f_2 \cdot \omega_2 + \cdots \qquad \text{with} \qquad c_k(f) = f_k$$

  - c_i represents a coefficient function.

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- z is a 'special' element which acts on $\mathcal{V}$ via

$$c_k(z \cdot f) = c_{k0} \cdot c_0(f) + c_{k1} \cdot c_1(f) + \cdots + c_{kk} \cdot c_k(f)$$

with each $c_{k,j} \in \mathbb{K}$.

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  with each $c_{k,j} \in \mathbb{K}$.

- Sometimes want to extend to :

$$c_k(z \cdot f) = c_{k0} \cdot c_0(f) + c_{k1} \cdot c_1(f) + \cdots + c_{kk} \cdot c_k(f) + c_{k,k+1} \cdot c_{k+1}(f)$$

The goals?

(1) Describe general framework for all such problems

(2) Find *all* solutions to such problems

(i.e. a module basis over $\mathbb{K}[z]$)

(3) Give formulas for bases of such problems

(e.g. subresultant-like determinants)

(4) Find an efficient algorithm for the computation of the basis.

(without fractions, linear growth of coefficients)

References

(1)* B. Beckermann and G. Labahn,
Fraction-free Computation of Matrix Rational Interpolants and Matrix GCD's, SIAM J. Matrix Analysis and Applications, 22(1) (2000) 114-144.

(2) H. Cheng and G. Labahn,
On Computing Polynomial GCDs in Alternate Bases, Proceedings of ISSAC'06, Genoa, Italy, ACM Press, (2006) 47-54.

Example

- Let
$$f(x) = c_0\omega_0(x) + c_1\omega_1(x) + c_2\omega_2(x) + \cdots$$
be a Newton expansion of $f(x)$ at interpolating points x_0, x_1, etc

- Find polynomials $p(x)$ and $q(x)$, degrees at most 2 and 4 such that
$$q(x) \cdot f(x) - p(x) = r_7\omega_7(x) + r_8\omega_8(x) + \cdots$$

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- Then
$$f(x_0) = \frac{p(x_0)}{q(x_0)}$$

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- Then

$$f(x_0) = \frac{p(x_0)}{q(x_0)}, \; f(x_1) = \frac{p(x_1)}{q(x_1)}, \cdots, \; f(x_6) = \frac{p(x_6)}{q(x_6)}$$

Rational interpolation. Note issue if $q(x_i) = 0$ for some i.

Rational Interpolation (example)

Given 6 points from $\mathbb{Q}[a]$ and integers $n_1 = 2, \quad n_2 = 3$.

$$\left(2, \frac{1-a}{21}\right), \left(3, \frac{11-9a}{371}\right), \left(4, \frac{9-6a}{416}\right), \left(5, \frac{9-5a}{527}\right), \left(6, \frac{19-9a}{1345}\right), \left(7, \frac{51-21a}{4231}\right)$$

and $\quad f(x) = c_0 \cdot \omega_0(x) + c_1 \cdot \omega_1(x) + \cdots$ through the points.

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and $\quad f(x) = c_0 \cdot \omega_0(x) + c_1 \cdot \omega_1(x) + \cdots$ through the points. Then

$$(12\,x^3 + 17\,x - 4) \cdot f(x) - (x^2 - 3\,ax + 2) \quad = \quad r_6 \cdot \omega_6(x) + r_7 \cdot \omega_7(x) + \cdots$$

so

$$A(x) = \frac{p(x)}{q(x)} = \frac{x^2 - 3\,ax + 2}{12\,x^3 + 17\,x - 4}$$

with $A(2) = \frac{1-a}{21}, \quad \ldots, \; A(7) = \frac{51-21a}{4231}$.

Example 1a : Simultaneous Rational Interpolation

Given:

- Points :
  $(x_0, y_0^{(1)}),\ \ldots\ ,(x_N,\ y_N^{(1)}),\ \ \ldots\ ,\ (x_0, y_0^{(m)}),\ \ldots\ ,(x_N,\ y_N^{(m)})$

- Integers $n_0,\ n_1,\ \ldots\ ,\ n_m$ with $N = n_0 + \cdots + n_m$

Example 1a : Simultaneous Rational Interpolation

Given:

- Points :
  $(x_0, y_0^{(1)}), \; \ldots \;, (x_N, \; y_N^{(1)}), \;\; \ldots \;, \;\; (x_0, y_0^{(m)}), \; \ldots \;, (x_N, \; y_N^{(m)})$

- Integers $n_0, \; n_1, \; \ldots \;, \; n_m$ with $N = n_0 + \cdots + n_m$

Find rational functions $\frac{p_1(x)}{q(x)}, \ldots, \frac{p_m(x)}{q(x)}$ such that

- $p_i(x)$ and $q(x)$ have degrees at most $N - n_i$ and $N - n_0$

- $\frac{p_1(x_i)}{q(x_i)} = y_i^{(1)}, \quad \ldots, \frac{p_m(x_i)}{q(x_i)} = y_i^{(m)}$ for $i = 1, \ldots, N$

Possible even that $x_0, \; \ldots, \; x_N$ have repeated values.

Simultaneous Interpolation (example)

Given points:

$$(0,0)\,,\;\left(1,\frac{a}{25}\right),\;\left(2,\frac{a}{63}\right),\;\left(3,\frac{3a}{371}\right),\;\left(4,\frac{a}{208}\right),\left(5,\frac{5a}{1581}\right),\;\left(6,\frac{3a}{1345}\right)$$

and

$$\left(0,\frac{a}{4}\right),\left(1,\frac{1-a}{25}\right),\left(2,\frac{4-a}{63}\right),\left(3,\frac{9-a}{371}\right),\left(4,\frac{16-a}{832}\right),\left(5,\frac{25-a}{1581}\right),\left(6,\frac{36-a}{2690}\right)$$

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$$(0,0)\,,\ \left(1,\frac{a}{25}\right),\ \left(2,\frac{a}{63}\right),\ \left(3,\frac{3a}{371}\right),\ \left(4,\frac{a}{208}\right),\left(5,\frac{5a}{1581}\right),\ \left(6,\frac{3a}{1345}\right)$$

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Then

$$\frac{p_1(x)}{q(x)} = \frac{ax}{12x^3+17x-4} \quad \text{and} \quad \frac{p_2(x)}{q(x)} = \frac{x^2-a}{12x^3+17x-4}$$

is a simultaneous interpolant with deg bounds $3,\ 1,\ 2$ and “order” 7.

Example 2 : Simplification

"Simplify" the expression (in Maple)

$$x^{m-1} J_{\mu+m}(x) - x^n J_{\nu+n}(x)$$

Here $J_\ell(x)$ denotes a Bessel function, m and n are integers

Example 2 : Simplification

"Simplify" the expression (in Maple)

$$x^{m-1}J_{\mu+m}(x) - x^n J_{\nu+n}(x)$$

Here $J_\ell(x)$ denotes a Bessel function, m and n are integers

In Maple this "simplify" means : rewrite this expression as

$$p_1(x) \cdot J_\mu(x) + p_2(x) \cdot J_{\mu+1}(x) + q_1(x) \cdot J_\nu(x) + q_2(x) \cdot J_{\nu+1}(x)$$

where $p_1(x)$, $p_2(x)$, $q_1(x)$, $q_2(x)$ are polynomials.

Simplification (example)

Expression

$$x^9 J_{10}(x) - x^8 J_{p+8}(x)$$

simplifies to

$$\begin{aligned}
&\left(-92897280\,x + 9031680\,x^3 - 201600\,x^5 + 1200\,x^7 - x^9\right) \cdot J_0(x)\\
&\left(185794560 + 1693440\,x^4 - 41287680\,x^2 + 50\,x^8 - 19200\,x^6\right) \cdot J_1(x)\\
&-x\,\Big(28800\,x^3 - 106560\,p^3x - 322560\,x + x^7 - 480\,x^5 - 18880\,p^4x - 1728\,p^5x + 9520\,p^2x^3\\
&+1440\,p^3x^3 - 216\,px^5 - 326656\,p^2x - 513792\,px - 64\,p^6x + 27360\,px^3 + 80\,p^4x^3 - 24\,p^2x^5\Big) \cdot J_p(x)\\
&-x\,\Big(-32\,x^6 + 3584\,p^6 + 645120 + 1672704\,p - 138240\,x^2 + 1680896\,p^2 + 250880\,p^4\\
&-192\,p^5x^2 + 866432\,p^3 + 4800\,x^4 - 3840\,p^4x^2 + 3760\,px^4 + 41216\,p^5\\
&-8\,px^6 - 200448\,px^2 - 111360\,p^2x^2 + 128\,p^7 + 960\,p^2x^4 - 29760\,p^3x^2 + 80\,p^3x^4\Big) J_{p+1}(x)
\end{aligned}$$

Key Ideas

(1) Use of linear functionals

- *linearly dependent* vs *linear independent*

(2) Linear algebra with structured matrices

(3) Mahler Systems as a module basis for all solutions.

- Closest normal point
- Shifted Popov form

(4) Structured solve of structured linear system

- Fraction-free, fast, good coefficient growth

Examples

$\mathcal{V}$ and special rule

Example : Standard power basis

$\mathcal{V} = \mathbb{K}[[x]]$ is space of formal power series

(i) Standard basis : $1, \; x, \; \ldots, \; x^i, \; \ldots$ (i.e. $\omega_i = x^i$)

- Linear functional : $c_k(\omega_i) = \delta_{ik}$
- Special element : $z = x$.
- Special rule : $c_k(\, z \cdot f \,) = c_{k-1}(\, f \,)$ so $c_{i,j} = \delta_{i-1,j}$

Special rule basically says

$$\text{coeff}(\, x \cdot f, \; x, \; k \,) = \text{coeff}(\, f, \; x, \; k-1 \,)$$

Example : Newton basis (distinct points)

$\mathcal{V}$ is space of formal power series, interpolation points $x_0,\ x_1,\ \ldots$

(ii) Newton basis : $1,\ x-x_0,\ (x-x_0)(x-x_1),\ \ldots$.

- c_k is k^{th} divided difference $[x_0,\ldots,x_k]$
- Special element $z = x$.

$$\begin{aligned} z \cdot (x-x_0) \quad \cdots \quad (x-x_{k-1}) \\ &= x_k(x-x_0)\cdots(x-x_{k-1}) + (x-x_0)\cdots(x-x_k) \end{aligned}$$

i.e.

$$z \cdot \omega_{k-1} = x_k \omega_{k-1} + \omega_k$$

⋆ Gives special rule $c_k(z \cdot f) = x_k \cdot c_{k-1}(f) + c_k(f)$

⋆ Can also get special rule from divided differences.

Example : Lagrange basis (distinct points)

$\mathcal{V}$ is space of formal power series, interpolation points $x_0, x_1, \ldots$.

(iii) Lagrange basis : $L_0(x), \ldots, L_i(x), \ldots$.

- Linear functional : $c_k(\omega_i) = \delta_{ik} x_k$
- Special element $z = x$.
- Special rule $\quad c_k(z \cdot f) = x_k \cdot c_{k-1}(f)$ so $c_{i,j} = \delta_{i,j} x_i$

Example : Interpolation basis (confluent case)

$\mathcal{V}$ is space of formal power series,

– interpolation points $x_0, x_1, \ldots$, repeats allowed

(iv) Interpolation basis : $\omega_0(x), \ldots, \omega_i(x), \ldots$ (confluent case)

- Linear functional : $c_k(f) = \frac{f^{(\rho_k)}(x_k)}{(\rho_k!)}$

  (where ρ_k is multiplicity of x_k in $(x_0, \ldots, x_{k-1})$.)

- Special element : $z = x$.

- Matrix $\underline{C} = [c_{ij}]$ is permutation of Jordan normal form matrix.

- e.g. order as x_0 $(m_0$ times $), \cdots,$ x_i $(m_i$ times $), \cdots$

Then $\mathbf{C} = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_\ell \end{bmatrix}$ with $J_i = \begin{bmatrix} x_i & & & \\ 1 & x_i & & \\ & \ddots & \ddots & \\ & & 1 & x_i \end{bmatrix}_{m_i \times m_i}$.

Example : Specific Interpolation basis (confluent) I

$\mathcal{V}$ is space of formal power series,

– e.g. repeated interpolation points $x_0, x_0, x_1, x_1, \ldots$.

(v) Interpolation basis : $\omega_0(x), \ldots, \omega_i(x), \ldots$ (confluent case)

- Linear functional : $c_{2k}(f) = f(x_k), \;\; c_{2k+1}(f) = f'(x_k)$
- Special element : $z = x$.

$$\mathbf{C} = \begin{bmatrix} x_0 & & & & & \\ 1 & x_0 & & & & \\ 0 & 0 & x_1 & & & \\ 0 & 0 & 1 & x_1 & & \\ & & & \ddots & & \\ & & & & & \ddots \end{bmatrix}.$$

Example : Specific Interpolation basis (confluent) II

$\mathcal{V}$ is space of formal power series,

– e.g. repeated interpolation points $x_0,\ x_0,\ x_1,\ x_1,\ x_1,\ x_2,\ldots$.

(vi) Interpolation basis : $\omega_0(x),\ldots,\omega_i(x),\ldots$ (confluent case)

- Linear functional : $c_0(f) = f(x_0),\ \ c_1(f) = f'(x_0)$, etc
- Special element : $z = x$.

$$\mathbf{C} = \begin{bmatrix} x_0 & & & & & \\ 1 & x_0 & & & & \\ 0 & 0 & x_1 & & & \\ 0 & 0 & 1 & x_1 & & \\ 0 & 0 & 0 & 1 & x_1 & \\ & & & \ddots & & \\ & & & & & \ddots \end{bmatrix}.$$

Example : Matrix rational approximants

$\mathcal{V} = \mathbb{K}[[x]]^n$ is space of n vectors of formal power series

(vii) Basis : $\omega_k = \omega_{nq+r} = x^q \, [0, \ldots, 1, \ldots, 0] = x^q \, \vec{e}_{r+1}$.

- Linear functional : $c_{nq+r}(f) =$ coeff$(\, f_{r+1},\ x,\ q)$
- Special element : $z = x$.
- Special rule : $c_k(z \cdot f) = c_{k-n}(f)$ so $c_{i,j} = \delta_{i-n,j}$

(viii) Basis : $\bar{\omega}_k = \omega_k(x^n) \cdot [1, x, \ldots, x^{n-1}]$.

- Linear functional : $c_{nq+r}(f) =$ coeff$(\, x^{k-1} f_q,\ x,\ r\ )$
- Special element : $z = x^n$.
- Special rule : $c_k(z \cdot f) = c_{k-n}(f)$ so $c_{i,j} = \delta_{i-n,j}$

Specific Vector Example

Input looks like $f = [f_1,\, f_2,\, f_3]$.

Vectors represented in 2 distinct forms:

$$
\begin{aligned}
\text{(i)} \quad f \;&=\; (f_1)_0 \cdot \vec{e}_1 + (f_2)_0 \cdot \vec{e}_2 + (f_3)_0 \cdot \vec{e}_n \\
&\quad + \; (f_1)_1 \cdot x\, \vec{e}_1 + (f_2)_2 \cdot x + (f_3)_1 \cdot x\, \vec{e}_3 + \cdots
\end{aligned}
$$

or

$$
\begin{aligned}
\text{(ii)} \quad f \;&=\; f_1(x^3) + x \cdot f_2(x^3) + x^2 \cdot f_n(x^3) \\
&=\; (f_1)_0 + (f_2)_0 x + (f_3)_0 x^2 + (f_1)_1 x^3 + (f_2)_1 x^4 + (f_3)_1 x^5 + \cdots
\end{aligned}
$$

Can do something similar for $\mathcal{V} = \mathbb{K}[[x]]^{m \times n}$ (i.e. matrices)

Useful since it makes vectors and matrices look like scalars.

More Examples : Orthogonal Polynomials

Table: Common orthogonal polynomial bases. ($c_{k,k+1} \neq 0$)

Basis	$\omega_k(x)$	z	$c_{k,k+1}$	$c_{k,k}$	$c_{k,k-1}$
Chebyshev	$T_k(x)$	$2x$	1	0	1
Chebyshev	$U_k(x)$	$2x$	1	0	1
Shifted Chebyshev	$T_k^*(x)$	$4x-2$	1	0	1
Shifted Chebyshev	$U_k^*(x)$	$4x-2$	1	0	1
Hermite	$H_k(x)$	$2x$	$2k+2$	0	1
Generalized Laguerre	$L_k^{(\alpha)}(x)$	x	$-k-\alpha-1$	$2k+\alpha+1$	$-k$
Legendre	$P_k(x)$	x	$\frac{k+1}{2k+3}$	0	$\frac{k}{2k-1}$
Ultraspherical	$C_k^{(\alpha)}(x)$	$2x$	$\frac{k+2\alpha}{k+\alpha+1}$	0	$\frac{k}{k+\alpha-1}$
Meixner	$m_k(x;a,b)$	$(b-1)x$	$(k+1)(k+a)$	$-k-bk-ab$	b
Charlier	$c_k(x;a)$	x	$-k-1$	$k+a$	$-a$

Linear Algebra

Recall

- Express

$$p_1(z) \cdot f_1 + \cdots + p_m(z) \cdot f_m = r$$

as linear system in unknowns (the coefficients of the $p_i(z)$)

- For any polynomial $q(z) = q_0 + q_1 z + \cdots + q_k z^k$:

$$q(z) \cdot f = q_0 \cdot (f) + q_1 \cdot (zf) + q_2 \cdot \left(z^2 f\right) + \cdots$$

$$c_k(\ q(z) \cdot f\ ) = q_0 \cdot c_k(f) + q_1 \cdot c_k(zf) + q_2 \cdot c_k(z^2 f) + \cdots$$

Matrix of such a linear system depends on coeffs of f, zf, $z^2 f, \ldots$

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Matrix of such a linear system depends on coeffs of f, zf, $z^2 f, \ldots$

- Matrix of such a linear system has *special structure*

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Matrix of such a linear system depends on coeffs of f, zf, $z^2 f, \ldots$

- Matrix of such a linear system has *special structure*

- Can limit linear system so it is always square ?????

Linear System I

Identity

$$p^{(1)}(z) \cdot f_1 + \cdots + p^{(m)}(z) \cdot f_m = r$$

same as linear system in unknowns (the coefficients of the $p_i(z)$) :

$$\left[\begin{array}{ccc|cc|ccc}
c_0(f_1) & \cdot\cdot & c_0(z^{n_1-1}f_1) & \cdots & \cdots & c_0(f_m) & \cdot\cdot & c_0(z^{n_m-1}f_m) \\
c_1(f_1) & \cdot\cdot & c_1(z^{n_1-1}f_1) & \cdots & \cdots & c_1(f_m) & \cdot\cdot & c_1(z^{n_m-1}f_m) \\
\vdots & & \vdots & & & \vdots & & \vdots \\
\vdots & & \vdots & & & \vdots & \vdots & \\
\vdots & & \vdots & & & \vdots & & \vdots \\
c_\sigma(f_1) & \cdots & c_\sigma(z^{n_1-1}f_1) & \cdots & \cdots & c_\sigma(f_m) & \cdots & c_\sigma(z^{n_m-1}f_m)
\end{array}\right]
\left[\begin{array}{c}
p_0^{(1)} \\ \vdots \\ p_{n_1-1}^{(1)} \\ \hline \vdots \\ \hline p_0^{(m)} \\ \vdots \\ p_{n_m-1}^{(m)}
\end{array}\right]
=
\left[\begin{array}{c}
c_0(r) \\ c_1(r) \\ \vdots \\ \vdots \\ \vdots \\ c_\sigma(r)
\end{array}\right]$$

When $\sigma + 1 = n_1 + \cdots + n_m = |\vec{n}|$ then system is square.

Linear System II

Identity

$$p^{(1)}(z)\cdot f_1 + \cdots + p^{(m)}(z)\cdot f_m = r$$

when order is σ same as linear system in unknowns (coeffs of $p_i(z)$) :

$$\left[\begin{array}{ccc|cc|ccc}
c_0(f_1) & \cdots & c_0(z^{n_1-1}f_1) & \cdots & \cdots & c_0(f_m) & \cdots & c_0(z^{n_m-1}f_m) \\
c_1(f_1) & \cdots & c_1(z^{n_1-1}f_1) & \cdots & \cdots & c_1(f_m) & \cdots & c_1(z^{n_m-1}f_m) \\
\vdots & & \vdots & & & \vdots & & \vdots \\
\vdots & & \vdots & & & \vdots & \vdots & \\
\vdots & & \vdots & & & \vdots & & \vdots \\
c_\sigma(f_1) & \cdots & c_\sigma(z^{n_1-1}f_1) & \cdots & \cdots & c_\sigma(f_m) & \cdots & c_\sigma(z^{n_m-1}f_m)
\end{array}\right]
\left[\begin{array}{c}
p_0^{(1)} \\ \vdots \\ p_{n_1-1}^{(1)} \\ \hline \vdots \\ \hline p_0^{(m)} \\ \vdots \\ p_{n_m-1}^{(m)}
\end{array}\right]
=
\left[\begin{array}{c}
c_0(r) \\ c_1(r) \\ \vdots \\ \vdots \\ \vdots \\ c_\sigma(r)
\end{array}\right]$$

When $\sigma + 1 = n_1 + \cdots + n_m = |\vec{n}|$ then system is square.

Linear System III

Identity

$$p^{(1)}(z)\cdot f_1 + \cdots + p^{(m)}(z)\cdot f_m = r$$

when order is σ same as linear system in unknowns (coeffs of $p_i(z)$) :

$$\left[\begin{array}{ccc|cc|ccc}
c_0(f_1) & \cdots & c_0(z^{n_1-1}f_1) & \cdots & \cdots & c_0(f_m) & \cdots & c_0(z^{n_m-1}f_m) \\
c_1(f_1) & \cdots & c_1(z^{n_1-1}f_1) & \cdots & \cdots & c_1(f_m) & \cdots & c_1(z^{n_m-1}f_m) \\
\vdots & & \vdots & & & \vdots & & \vdots \\
\vdots & & \vdots & & & \vdots & \vdots & \\
\vdots & & \vdots & & & \vdots & & \vdots \\
c_\sigma(f_1) & \cdots & c_\sigma(z^{n_1-1}f_1) & \cdots & \cdots & c_\sigma(f_m) & \cdots & c_\sigma(z^{n_m-1}f_m)
\end{array}\right]
\left[\begin{array}{c}
p_0^{(1)} \\ \vdots \\ p_{n_1-1}^{(1)} \\ \hline \vdots \\ \hline p_0^{(m)} \\ \vdots \\ p_{n_m-1}^{(m)}
\end{array}\right]
=
\left[\begin{array}{c}
c_0(r) \\ c_1(r) \\ \vdots \\ \vdots \\ \vdots \\ c_\sigma(r)
\end{array}\right]$$

Aside : When $c_{|\vec{n}|}(r) =$ det $\underline{K}(\vec{n}, \underline{F})$ then no fractions (Cramer's rule)

Linear System IV

Identity

$$p^{(1)}(z) \cdot f_1 + \cdots + p^{(m)}(z) \cdot f_m = r$$

when order is σ same as linear system in unknowns (coeffs of $p_i(z)$) :

$$\left[\begin{array}{ccc|cc|ccc} c_0(f_1) & \cdots & c_0(z^{n_1-1}f_1) & \cdots & \cdots & c_0(f_m) & \cdots & c_0(z^{n_m-1}f_m) \\ c_1(f_1) & \cdots & c_1(z^{n_1-1}f_1) & \cdots & \cdots & c_1(f_m) & \cdots & c_1(z^{n_m-1}f_m) \\ \vdots & & \vdots & & & \vdots & & \vdots \\ \vdots & & \vdots & & & \vdots & \vdots & \\ \vdots & & \vdots & & & \vdots & & \vdots \\ c_\sigma(f_1) & \cdots & c_\sigma(z^{n_1-1}f_1) & \cdots & \cdots & c_\sigma(f_m) & \cdots & c_\sigma(z^{n_m-1}f_m) \end{array}\right] \left[\begin{array}{c} p_0^{(1)} \\ \vdots \\ p_{n_1-1}^{(1)} \\ \hline \vdots \\ \hline p_0^{(m)} \\ \vdots \\ p_{n_m-1}^{(m)} \end{array}\right] = \left[\begin{array}{c} c_0(r) \\ c_1(r) \\ \vdots \\ \vdots \\ \vdots \\ c_\sigma(r) \end{array}\right]$$

Notice : When $c_{|\vec{n}|}(r) =$ det $\underline{K}(\vec{n}, \underline{F})$ then no fractions (Cramer's rule).

Example : Standard Basis : $c_n(z \cdot f) = c_{n-1}(f)$.

$$\mathbf{F} = (A, B, C), \qquad \mathbf{C}_\sigma = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \qquad \vec{n} = (2, 3, 3) \qquad \sigma = 7$$

$$\left[A_\sigma \quad \underset{\sim}{C}_\sigma A_\sigma \;\middle|\; B_\sigma \quad \underset{\sim}{C}_\sigma B_\sigma \quad \underset{\sim}{C}_\sigma^2 B_\sigma \;\middle|\; C_\sigma \quad \underset{\sim}{C}_\sigma C_\sigma \quad \underset{\sim}{C}_\sigma^2 C_\sigma \right] [p_0^{(1)}, \cdots, p_2^{(2)}]^T \quad = \quad r[0, \cdots 0, 1]^T$$

$$\left[\begin{array}{cc|ccc|ccc} a_0 & & b_0 & & & c_0 & & \\ a_1 & a_0 & b_1 & b_0 & & c_1 & c_0 & \\ a_2 & a_1 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ A & zA & B & zB & z^2B & C & zC & z^2C \end{array}\right] \cdot \begin{bmatrix} p_0 \\ p_1 \\ q_0 \\ q_1 \\ q_2 \\ s_0 \\ s_1 \\ s_2 \end{bmatrix} = r \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Same as : $\quad p(z) \cdot A \; + \; q(z) \cdot B \; + \; s(z) \cdot C \quad = \quad r$

where $r_0 = r_1 = \cdots = r_6 = 0$.

Example : Lagrange Basis : $c_n(z \cdot f) = x_n c_n(f)$.

$$\mathbf{F} = (A, B, C), \qquad \mathbf{C}_\sigma = \begin{bmatrix} x_0 & & & \\ & x_1 & & \\ & & \ddots & \\ & & & x_{\sigma-1} \end{bmatrix}, \qquad \vec{n} = (2, 3, 3) \qquad \sigma = 7$$

$$\left[A_\sigma \quad \underline{C}_\sigma A_\sigma \;\middle|\; B_\sigma \quad \underline{C}_\sigma B_\sigma \quad \underline{C}_\sigma^2 B_\sigma \;\middle|\; C_\sigma \quad \underline{C}_\sigma C_\sigma \quad \underline{C}_\sigma^2 C_\sigma \right] [p_0^{(1)}, \cdots, p_2^{(2)}]^T \quad = \quad r[0, \cdots 0, 1]^T$$

$$\left[\begin{array}{cc|ccc|ccc} a_0 & x_0a_0 & b_0 & x_0b_0 & x_0^2b_0 & c_0 & x_0c_0 & x_0^2c_0 \\ a_1 & x_1a_1 & b_1 & x_1b_1 & x_1^2b_1 & c_1 & x_1c_1 & x_1^2c_1 \\ a_2 & x_2a_2 & b_2 & x_2b_2 & x_2^2b_2 & c_2 & x_2c_2 & x_2^2c_2 \\ a_3 & x_3a_3 & b_3 & x_3b_3 & x_3^2b_3 & c_3 & x_3c_3 & x_3^2c_3 \\ a_4 & x_4a_4 & b_4 & x_4b_4 & x_4^2b_4 & c_4 & x_4c_4 & x_4^2c_4 \\ a_5 & x_5a_5 & b_5 & x_5b_5 & x_5^2b_5 & c_5 & x_5c_5 & x_5^2c_5 \\ a_6 & x_6a_6 & b_6 & x_6b_6 & x_6^2b_6 & c_6 & x_6c_6 & x_6^2c_6 \\ A & zA & B & zB & z^2B & C & zC & z^2C \end{array}\right] \cdot \begin{bmatrix} p_0 \\ p_1 \\ q_0 \\ q_1 \\ q_2 \\ s_0 \\ s_1 \\ s_2 \end{bmatrix} \quad = \quad r \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Same as : $\quad p(z) \cdot A \; + \; q(z) \cdot B \; + \; s(z) \cdot C \quad = \quad r$

with *ord* $r \geq 7$, that is, $r(x_0) = \cdots = r(x_6) = 0$.

Example : Confluent Basis : $c_n(f) = \frac{f^{(n)}(x_0)}{n!}$.

$\mathbf{F} = (A, B, C)$, Points : $x_0, x_0; x_1, x_1, x_1, x_2, \cdots ,$ $\vec{n} = (2, 3, 2)$ $\sigma = 6$

$$\left[A_\sigma \quad \underline{C}_\sigma A_\sigma \;\middle|\; B_\sigma \quad \underline{C}_\sigma B_\sigma \quad \underline{C}^2_\sigma B_\sigma \;\middle|\; C_\sigma \quad \underline{C}_\sigma C_\sigma \right] [p_0^{(1)}, \cdots, p_2^{(2)}]^T = r[0, \cdots 0, 1]^T$$

$$\left[\begin{array}{cc|ccc|cc} a_0 & x_0 a_0 & b_0 & x_0 b_0 & x_0^2 b_0 & c_0 & x_0 c_0 \\ a_1 & x_0 a_1 + a_0 & b_1 & x_0 b_1 + b_0 & x_0^2 b_1 + 2x_0 b_0 & c_1 & x_0 c_1 + c_0 \\ a_2 & x_1 a_2 & b_2 & x_1 b_2 & x_1^2 b_2 & c_2 & x_1 c_2 \\ a_3 & x_1 a_3 + a_2 & b_3 & x_1 b_3 + b_2 & x_1^2 b_3 + 2x_1 b_2 & c_3 & x_1 c_3 + c_2 \\ a_4 & x_1 a_4 + a_3 & b_4 & x_1 b_4 + b_3 & x_1^2 b_4 + 2x_1 b_3 + b_2 & c_4 & x_1 c_4 + c_3 \\ a_5 & x_2 a_5 & b_5 & x_2 b_5 & x_2^2 b_5 & c_5 & x_2 c_5 \\ A & zA & B & zB & z^2 B & C & zC \end{array} \right] \cdot \begin{bmatrix} p_0 \\ p_1 \\ q_0 \\ q_1 \\ q_2 \\ s_0 \\ s_1 \end{bmatrix} = r \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Same as : $p(z) \cdot A + q(z) \cdot B + s(z) \cdot C = r$

with *ord* $r \geq 6$, that is, $r(x_0) = r'(x_0) = r(x_1) = r'(x_1) = r''(x_1) = r(x_2) = 0$

Linear System V

Identity

$$p^{(1)}(z) \cdot f_1 + \cdots + p^{(m)}(z) \cdot f_m = r$$

when order is σ same as linear system in unknowns (coeffs of $p_i(z)$) :

$$\left[\begin{array}{ccc|cc|ccc} c_0(f_1) & \cdot\cdot & c_0(z^{n_1-1}f_1) & \cdots & \cdots & c_0(f_m) & \cdot\cdot & c_0(z^{n_m-1}f_m) \\ c_1(f_1) & \cdot\cdot & c_1(z^{n_1-1}f_1) & \cdots & \cdots & c_1(f_m) & \cdot\cdot & c_1(z^{n_m-1}f_m) \\ \vdots & & \vdots & & & \vdots & & \vdots \\ \vdots & & \vdots & & & \vdots & & \vdots \\ \vdots & & \vdots & & & \vdots & & \vdots \\ c_{\sigma-1}(f_1) & \cdot\cdot & c_{\sigma-1}(z^{n_1-1}f_1) & \cdots & \cdots & c_{\sigma-1}(f_m) & \cdot\cdot & c_{\sigma-1}(z^{n_m-1}f_m) \\ f_1 & \cdot\cdot & z^{n_1-1}f_1 & \cdots & \cdots & f_m & \cdot\cdot & z^{n_m-1}f_m \end{array}\right] \left[\begin{array}{c} p_0^{(1)} \\ \vdots \\ p_{n_1-1}^{(1)} \\ \hline \vdots \\ \hline p_0^{(m)} \\ \vdots \\ p_{n_m-1}^{(m)} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ r \end{array}\right]$$

Notice : When $c_{|\vec{n}|}(r) = \det \underline{K}(\vec{n}, \underline{F})$ then no fractions (Cramer's rule).

Linear System VI

Identity

$$p^{(1)}(z)\cdot f_1+\cdots+p^{(m)}(z)\cdot f_m=r$$

when order is σ same as linear system in unknowns (coeffs of $p_i(z)$) :

$$\left[\begin{array}{ccc|cc|ccc} c_0(f_1) & \cdot\cdot & c_0(z^{n_1-1}f_1) & \cdots & \cdots & c_0(f_m) & \cdot\cdot & c_0(z^{n_m-1}f_m) \\ c_1(f_1) & \cdot\cdot & c_1(z^{n_1-1}f_1) & \cdots & \cdots & c_1(f_m) & \cdot\cdot & c_1(z^{n_m-1}f_m) \\ \vdots & & \vdots & & & \vdots & & \vdots \\ \vdots & & \vdots & & & \vdots & & \vdots \\ \vdots & & \vdots & & & \vdots & & \vdots \\ c_{\sigma-1}(f_1) & \cdot\cdot & c_{\sigma-1}(z^{n_1-1}f_1) & \cdots & \cdots & c_{\sigma-1}(f_m) & \cdot\cdot & c_{\sigma-1}(z^{n_m-1}f_m) \end{array}\right] \left[\begin{array}{c} p_0^{(1)} \\ \vdots \\ p_{n_1-1}^{(1)} \\ \hline \vdots \\ \hline p_0^{(m)} \\ \vdots \\ p_{n_m-1}^{(m)} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 0 \end{array}\right]$$

Notice : When $c_{|\vec{n}|}(r) =$ det $\underline{K}(\vec{n},\underline{F})$ then no fractions (Cramer's rule).

Linear System VII

Identity

$$p^{(1)}(z)\cdot f_1+\cdots+p^{(m)}(z)\cdot f_m=r$$

when order is σ same as linear system in unknowns (coeffs of $p_i(z)$) :

$$\left[\begin{array}{ccc|c|ccc} c_0(f_1) & \cdot\cdot & c_0(z^{n_1-2}f_1) & \cdots & c_0(f_m) & \cdot\cdot & c_0(z^{n_m-1}f_m) \\ c_1(f_1) & \cdot\cdot & c_1(z^{n_1-2}f_1) & \cdots & c_1(f_m) & \cdot\cdot & c_1(z^{n_m-1}f_m) \\ \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots & & \vdots \\ c_{\sigma-1}(f_1) & \cdot\cdot & c_{\sigma-1}(z^{n_1-2}f_1) & \cdots & c_{\sigma-1}(f_m) & \cdot\cdot & c_{\sigma-1}(z^{n_m-1}f_m) \end{array}\right] \left[\begin{array}{c} p_0^{(1)} \\ \vdots \\ p_{n_1-2}^{(1)} \\ \hline \vdots \\ \hline p_0^{(m)} \\ \vdots \\ p_{n_m-1}^{(m)} \end{array}\right] = -p_{n_1-1}^{(1)} \left[\begin{array}{c} c_0(\\ c_1(\\ \\ \\ c_{\sigma-1} \end{array}\right.$$

Notice : When $p_{n_1-1}^{(1)} =$ det $\underline{K}(\vec{n}-\vec{e}_1,\underline{F})$ then no fractions (Cramer).