Summer School on Algorithmic and Enumerative Combinatorics
Hagenberg, Austria, July 29 – August 2, 2019

Continued fractions and Hankel-total positivity
Alan Sokal

SOME EXERCISES ON CONTINUED FRACTIONS

Problem 1. Recall that the *Stieltjes–Rogers polynomial* $S_n(\alpha_1, \ldots, \alpha_n)$ is the generating polynomial for Dyck paths of length $2n$ in which each rise gets weight 1 and each fall from height i gets weight α_i.

(a) Compute, by hand, the Stieltjes–Rogers polynomials $S_n(\boldsymbol{\alpha})$ for $0 \le n \le 4$. To be sure that you haven't forgotten any paths, check that $S_n(\boldsymbol{\alpha})$ specialized to $\boldsymbol{\alpha} = \mathbf{1}$ is the Catalan number C_n.

(b) Define the $n \times n$ Hankel matrix $H_n(\boldsymbol{S}) = (S_{i+j}(\boldsymbol{\alpha}))_{0 \le i,j \le n-1}$ and its determinant $\Delta_n(\boldsymbol{S}) = \det H_n(\boldsymbol{S})$. Compute $\Delta_n(\boldsymbol{S})$ for $0 \le n \le 3$. Do you see a pattern? Can you conjecture the general formula?

Later we will give two proofs of this general formula: a combinatorial proof based on the Lindström–Gessel–Viennot lemma, and an algebraic proof (due to Stieltjes [8]) based on the LDL^{T} factorization of the Hankel matrix.

Problem 2. Recall the Euler–Gauss method for proving continued fractions: Let $(g_k(t))_{k \ge -1}$ be a sequence of formal power series (with coefficients in some commutative ring R) with constant term 1, and suppose that this sequence satisfies a linear three-term recurrence of the form

$$g_k(t) - g_{k-1}(t) \;=\; \alpha_{k+1} t \, g_{k+1}(t) \qquad \text{for } k \ge 0 \tag{1}$$

for some coefficients $\boldsymbol{\alpha} = (\alpha_i)_{i \ge 1}$ in R. If we define $f_k(t) = g_k(t)/g_{k-1}(t)$ for $k \ge 0$, then (1) can be rewritten as

$$f_k(t) \;=\; \frac{1}{1 \,-\, \alpha_{k+1} t \, f_{k+1}(t)} \,, \tag{2}$$

which implies by iteration the continued fraction

$$f_k(t) \;=\; \cfrac{1}{1 - \cfrac{\alpha_{k+1} t}{1 - \cfrac{\alpha_{k+2} t}{1 - \cfrac{\alpha_{k+3} t}{1 - \cdots}}}} \tag{3}$$

and hence in particular

$$f_0(t) \;=\; \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cdots}}}} \;. \tag{4}$$

(a) Let us use this method, following Euler [4, section 21][1], to prove the continued fraction

$$\sum_{n=0}^{\infty} n!\, t^n \;=\; \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cdots}}}}} \tag{5}$$

with coefficients $\alpha_{2k-1} = \alpha_{2k} = k$ for $k \ge 1$. You will need to guess all the series $g_k(t)$ and then verify the recurrence (1). [*Hint:* You will take $g_{-1} = 1$ and hence $g_0(t) = \sum_{n=0}^{\infty} n!\, t^n$; and you will need slightly different formulae for $g_{2j-1}(t)$ and $g_{2j}(t)$.] I can see two ways of guessing the $g_k(t)$:

- Produce numerically the first few terms of the first few series $g_k(t)$ and then try to guess the general pattern. I have attached the relevant pages of Euler's paper (translated from Latin into English!), so that you can try to reverse-engineer it and guess the series $g_k(t)$.

- An even better method (when it works): Use the recurrence (1) to successively compute $g_1(t)$, $g_2(t)$, … *explicitly to all orders*, extracting at each stage the factor $\alpha_{k+1} t$ that makes $g_{k+1}(t)$ have constant term 1. After a few steps of this computation, you may be able to *guess* the general formulae for α_k and $g_k(t)$.

Once you have the formulae for $g_k(t)$, it is straightforward to verify the recurrence (1). At the end of this problem sheet (so as not to spoil the fun) I have given the needed formulae for $g_k(t)$.

(b) In section 26 of the same paper [4], Euler says that the same method can be

[1] The paper [4], which is E247 in Eneström's [3] catalogue, was probably written circa 1746; it was presented to the St. Petersburg Academy in 1753, and published in 1760.

applied to prove the more general continued fraction

$$\sum_{n=0}^{\infty} a(a+1)\cdots(a+n-1)\, t^n \;=\; \cfrac{1}{1-\cfrac{at}{1-\cfrac{1t}{1-\cfrac{(a+1)t}{1-\cfrac{2t}{1-\cfrac{(a+2)t}{1-\cfrac{3t}{1-\cdots}}}}}}} \,, \tag{6}$$

which reduces to (5) when $a = 1$; but he does not provide the details, and he instead proves (6) by an alternative method. Three decades later, however, Euler [5] returned to his original method and presented the details of the derivation of (6).[2] Now

$$\alpha_{2j-1} = a+j-1, \quad \alpha_{2j} = j \;. \tag{7}$$

Can you guess how your formulae for $g_{2j-1}(t)$ and $g_{2j}(t)$ should be generalized? (You will continue to take $g_{-1} = 1$.) Do this, and verify the recurrence (1). The answer is again at the end.

(c) We can, in fact, carry this process one step farther, by introducing an additional parameter b. Let

$$\alpha_{2j-1} = a+j-1, \quad \alpha_{2j} = b+j-1 \;. \tag{8}$$

Can you guess how your formulae for $g_{2j-1}(t)$ and $g_{2j}(t)$ should be further generalized? Now you will no longer have $g_{-1} = 1$ (unless $b = 1$), but no matter; we can still conclude that $g_0(t)/g_{-1}(t)$ is given by the continued fraction with coefficients (8). What you will prove in this way is the continued fraction for ratios of contiguous hypergeometric series ${}_2F_0$:

$$\frac{{}_2F_0\!\left(\begin{matrix} a,b \\ - \end{matrix}\,\middle|\, t\right)}{{}_2F_0\!\left(\begin{matrix} a,b-1 \\ - \end{matrix}\,\middle|\, t\right)} \;=\; \cfrac{1}{1-\cfrac{at}{1-\cfrac{bt}{1-\cfrac{(a+1)t}{1-\cfrac{(b+1)t}{1-\cfrac{(a+2)t}{1-\cfrac{(b+2)t}{1-\cdots}}}}}}} \,, \tag{9}$$

where as usual

$${}_2F_0\!\left(\begin{matrix} a,b \\ - \end{matrix}\,\middle|\, t\right) \;=\; \sum_{n=0}^{\infty} \frac{a^{\overline{n}}\, b^{\overline{n}}}{n!}\, t^n \tag{10}$$

---

[2] The paper [5], which is E616 in Eneström's [3] catalogue, was apparently presented to the St. Petersburg Academy in 1776, and published posthumously in 1788.

and I have used the Knuth notation $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$. The recurrence (1) is simply the contiguous relation

$$
{}_2F_0\!\left(\begin{matrix} a,b \\ \text{—} \end{matrix} \,\middle|\, t \right) \;-\; {}_2F_0\!\left(\begin{matrix} a,b-1 \\ \text{—} \end{matrix} \,\middle|\, t \right) \;=\; at\,{}_2F_0\!\left(\begin{matrix} a+1,b \\ \text{—} \end{matrix} \,\middle|\, t \right) , \tag{11}
$$

applied with interchanges $a \leftrightarrow b$ at alternate levels. It seems to me, in fact, that the reasoning is somewhat more transparent in this greater generality!

Remarks. 1. If we expand the ratio (9) as a power series,

$$
\frac{{}_2F_0\!\left(\begin{matrix} a,b \\ \text{—} \end{matrix} \,\middle|\, t \right)}{{}_2F_0\!\left(\begin{matrix} a,b-1 \\ \text{—} \end{matrix} \,\middle|\, t \right)} \;=\; \sum_{n=0}^{\infty} P_n(a,b)\, t^n \;, \tag{12}
$$

it follows easily from the continued fraction that $P_n(a,b)$ is a polynomial of total degree n in a and b, with nonnegative integer coefficients. It is therefore natural to ask: What do these nonnegative integers count?

Euler's continued fraction (5) tells us that $P_n(1,1) = n!$; and there are $n!$ permutations of an n-element set. It is therefore reasonable to guess that $P_n(a,b)$ enumerates permutations of an n-element set according to some natural bivariate statistic. This is indeed the case; and Dumont and Kreweras [2] have identified the statistic. Given a permutation σ of $\{1,2,\ldots,n\}$, let us say that an index $i \in \{1,2,\ldots,n\}$ is a

- *record* (or *left-to-right maximum*) if $\sigma(j) < \sigma(i)$ for all $j < i$ [note in particular that the index 1 is always a record];
- *antirecord* (or *right-to-left minimum*) if $\sigma(j) > \sigma(i)$ for all $j > i$ [note in particular that the index n is always an antirecord];
- *exclusive record* if it is a record and not also an antirecord;
- *exclusive antirecord* if it is an antirecord and not also a record.

Dumont and Kreweras [2] then showed that

$$
P_n(a,b) \;=\; \sum_{\sigma \in \mathfrak{S}_n} a^{\mathrm{rec}(\sigma)} b^{\mathrm{earec}(\sigma)} \tag{13}
$$

where $\mathrm{rec}(\sigma)$ [resp. $\mathrm{earec}(\sigma)$] is the number of records (resp. exclusive antirecords) in σ.

2. By an argument similar to the one we have used for ${}_2F_0$, Gauss [6] found in 1812 a continued fraction for the ratio of two contiguous hypergeometric functions ${}_2F_1$. Moreover, the formula for ${}_2F_0$, as well as analogous formulae for ratios of ${}_1F_1$, ${}_1F_0$ or ${}_0F_1$, can be deduced from Gauss' formula by specialization or taking limits.

See [10, Chapter XVIII] for details. In fact, one of the special cases of the ${}_0F_1$ formula is Lambert's [7] continued fraction for the tangent function

$$\frac{\tan t}{t} = \cfrac{1}{1 - \cfrac{\frac{1}{1\cdot 3}t^2}{1 - \cfrac{\frac{1}{3\cdot 5}t^2}{1 - \cfrac{\frac{1}{5\cdot 7}t^2}{1 - \cfrac{\frac{1}{7\cdot 9}t^2}{1 - \cdots}}}}} \, , \tag{14}$$

which he used to prove the irrationality of π.

Problem 3. The goal of this exercise is to prove the very important *contraction formulae* that allow an S-fraction to be rewritten as a J-fraction (and sometimes but not always conversely). These formulae are classical [10, p. 21], but it was only in the 1980s that Viennot [9, section V.5] gave them a beautiful combinatorial interpretation, based on grouping pairs of steps in a Dyck path. I will therefore ask you to find two proofs of each identity: one algebraic and one combinatorial.

(a) The formula for *even contraction* states that, as an identity in $\mathbb{Z}[\boldsymbol{\alpha}][[t]]$,

$$\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cdots}}}} = \cfrac{1}{1 - \alpha_1 t - \cfrac{\alpha_1\alpha_2 t^2}{1 - (\alpha_2 + \alpha_3)t - \cfrac{\alpha_3\alpha_4 t^2}{1 - (\alpha_4 + \alpha_5)t - \cfrac{\alpha_5\alpha_6 t^2}{1 - \cdots}}}} \, . \tag{15}$$

Prove this:

- Algebraically by using repeatedly the identity

$$\cfrac{a}{1 - \cfrac{b}{1 - c}} = a + \frac{ab}{1 - b - c} \, . \tag{16}$$

- Combinatorially by grouping steps (in a Dyck path of length $2n$) in pairs, and then suitably mapping these pairs onto steps of a Motzkin path of length n.

(b) The formula for *odd contraction* states that, as an identity in $\mathbb{Z}[\boldsymbol{\alpha}][[t]]$,

$$\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cdots}}}} = 1 + \cfrac{\alpha_1 t}{1 - (\alpha_1 + \alpha_2)t - \cfrac{\alpha_2\alpha_3 t^2}{1 - (\alpha_3 + \alpha_4)t - \cfrac{\alpha_4\alpha_5 t^2}{1 - \cdots}}} \, . \tag{17}$$

Once again prove this both algebraically and combinatorially. [*Hint:* This time your Motzkin path should have length $n-1$.]

(c) For use in the next problem, let us prove a slight generalization of these two contraction formulae. Consider the generic S-fraction

$$f(t) \;=\; \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}} \;, \tag{18}$$

and let w be an additional indeterminate. Then, as identities in $\mathbb{Z}[\boldsymbol{\alpha}, w][[t]]$, we have

$$\frac{1}{1-wt}\, f\Big(\frac{t}{1-wt}\Big) \;=\; \cfrac{1}{1 - (\alpha_1 + w)t - \cfrac{\alpha_1\alpha_2 t^2}{1 - (\alpha_2 + \alpha_3 + w)t - \cfrac{\alpha_3\alpha_4 t^2}{1 - \cdots}}} \tag{19}$$

and

$$f\Big(\frac{t}{1-wt}\Big) \;=\; 1 \;+\; \cfrac{\alpha_1 t}{1 - (\alpha_1 + \alpha_2 + w)t - \cfrac{\alpha_2\alpha_3 t^2}{1 - (\alpha_3 + \alpha_4 + w)t - \cfrac{\alpha_4\alpha_5 t^2}{1 - \cdots}}} \;. \tag{20}$$

Problem 4. The Bell number B_n is, by definition, the number of partitions of an n-element set into nonempty blocks; by convention we set $B_0 = 1$. The Stirling subset number (also called Stirling number of the second kind) $\left\{ n \atop k \right\}$ is, by definition, the number of partitions of an n-element set into k nonempty blocks; for $n = 0$ we make the convention $\left\{ 0 \atop k \right\} = \delta_{k0}$. Now define the Bell polynomials

$$B_n(x) \;=\; \sum_{k=0}^{n} \left\{ n \atop k \right\} x^k \tag{21}$$

and their homogenized version

$$B_n(x,y) \;=\; y^n B_n(x/y) \;=\; \sum_{k=0}^{n} \left\{ n \atop k \right\} x^k y^{n-k} \;, \tag{22}$$

so that $B_n = B_n(1) = B_n(1,1)$. Then define the ordinary generating functions

$$\mathcal{B}(t) \;=\; \sum_{n=0}^{\infty} B_n t^n \tag{23a}$$

$$\mathcal{B}_x(t) \;=\; \sum_{n=0}^{\infty} B_n(x)\, t^n \tag{23b}$$

$$\mathcal{B}_{x,y}(t) \;=\; \sum_{n=0}^{\infty} B_n(x,y)\, t^n \tag{23c}$$

(a) Prove, by a combinatorial argument, that the Stirling subset numbers satisfy the recurrence

$$\left\{ {n \atop k} \right\} \;=\; k \left\{ {n-1 \atop k} \right\} \;+\; \left\{ {n-1 \atop k-1} \right\} \qquad \hbox{for } n \ge 1 \tag{24}$$

with initial conditions $\left\{ {0 \atop k} \right\} = \delta_{k0}$ and $\left\{ {n \atop -1} \right\} = 0$.

(b) Prove the "vertical" generating function for the Stirling subset numbers:

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} t^n \;=\; \frac{t^k}{(1-t)(1-2t)\cdots(1-kt)} \;. \tag{25}$$

[*Hint:* Use the recurrence (24) and induction on k.] Deduce from this the factorial series

$$\mathcal{B}_{x,y}(t) \;=\; \sum_{k=0}^{\infty} \frac{x^k t^k}{(1-yt)(1-2yt)\cdots(1-kyt)} \;. \tag{26}$$

(c) Prove the functional equation

$$\mathcal{B}_{x,y}(t) \;=\; 1 + \frac{xt}{1-yt} \, \mathcal{B}_{x,y}\Bigl(\frac{t}{1-yt} \Bigr) \;. \tag{27}$$

(d) Prove the continued fraction

$$\mathcal{B}_{x,y}(t) \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{yt}{1 - \cfrac{xt}{1 - \cfrac{2yt}{1 - \cfrac{xt}{1 - \cfrac{3yt}{1 - \cdots}}}}}}} \tag{28}$$

with coefficients $\alpha_{2k-1} = x$ and $\alpha_{2k} = ky$.

[*Hint:* Consider a generic S-fraction (18). Rewrite $f(t)$ using odd contraction (17), and rewrite $1 + \frac{xt}{1-yt} f\Bigl(\frac{t}{1-yt}\Bigr)$ using the transformed even contraction (19). Compare the two formulae to show that the S-fraction (18) satisfies the functional equation (27) if and only if $\alpha_{2k-1} = x$ and $\alpha_{2k} = ky$.]

This elegant method of proving the continued fraction for the Bell polynomials is due to the late Dominique Dumont [1]. Also, Zeng [11, Lemma 3] has given two different q-generalizations of all four parts of this exercise.

(e) You can also prove the continued fraction (28) by the Euler–Gauss recurrence method. Once again you can take $g_{-1} = 1$; then use the recurrence (1) to successively compute $g_1(t)$, $g_2(t)$, … *to all orders*, extracting at each stage the factor $\alpha_{k+1} t$ that makes $g_{k+1}(t)$ have constant term 1. After a few steps of this computation, you may be able to *guess* the general formulae for $g_{2j-1}(t)$ and $g_{2j}(t)$. (The answer is again at the end.) Once you have done this, it is easy to verify the recurrence (1) by using the recurrence (24) for the Stirling subset numbers.

References

[1] D. Dumont, A continued fraction for Stirling numbers of the second kind, unpublished note (1989), cited in [11].

[2] D. Dumont and G. Kreweras, Sur le développement d'une fraction continue liée à la série hypergéométrique et son interprétation en termes de records et anti-records dans les permutations, European J. Combin. **9**, 27–32 (1988).

[3] G. Eneström, *Die Schriften Eulers chronologisch nach den Jahren geordnet, in denen sie verfaßt worden sind*, Jahresbericht der Deutschen Mathematiker-Vereinigung (Teubner, Leipzig, 1913).

[4] L. Euler, De seriebus divergentibus, Novi Commentarii Academiae Scientiarum Petropolitanae **5**, 205–237 (1760); reprinted in *Opera Omnia*, ser. 1, vol. 14, pp. 585–617. [Latin original and English and German translations available at `http://eulerarchive.maa.org/pages/E247.html`]

[5] L. Euler, De transformatione seriei divergentis $1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + \text{etc.}$ in fractionem continuam, Nova Acta Academiae Scientarum Imperialis Petropolitanae **2**, 36–45 (1788); reprinted in *Opera Omnia*, ser. 1, vol. 16, pp. 34–46. [Latin original and English and German translations available at `http://eulerarchive.maa.org/pages/E616.html`]

[6] C.F. Gauss, Disquisitiones generales circa seriem infinitam $1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.}$, Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores, Classis Mathematicae **2** (1813). [Reprinted in C.F. Gauss, *Werke*, vol. 3 (Cambridge University Press, Cambridge, 2011), pp. 123–162.] Available on-line at `http://gdz.sub.uni-goettingen.de/dms/load/toc/?PPN=PPN235999628`

[7] J.H. Lambert, Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques, Mémoires de l'Académie Royale des Sciences de Berlin **17**, 265–322 (1768). Available on-line at `http://www.kuttaka.org/~JHL/L1768b.html`

[8] T.J. Stieltjes, Sur la réduction en fraction continue d'une série procédant selon les puissances descendantes d'une variable, Ann. Fac. Sci. Toulouse **3**, H1–H17 (1889).

[9] G. Viennot, Une théorie combinatoire des polynômes orthogonaux généraux, Notes de conférences données à l'Université du Québec à Montréal, septembre-octobre 1983. Available on-line at `http://www.xavierviennot.org/xavier/polynomes_orthogonaux.html`

[10] H.S. Wall, *Analytic Theory of Continued Fractions* (Van Nostrand, New York, 1948).

[11] J. Zeng, The q-Stirling numbers, continued fractions and the q-Charlier and q-Laguerre polynomials, J. Comput. Appl. Math. **57**, 413–424 (1995).

ANSWERS TO SELECTED PROBLEMS

Problem 1(a):

$$g_{2j-1}(t) \;=\; \sum_{n=0}^{\infty} \binom{n+j}{n} \binom{n+j-1}{n} n!\, t^n \tag{29a}$$

$$g_{2j}(t) \;=\; \sum_{n=0}^{\infty} \binom{n+j}{n}^2 n!\, t^n \tag{29b}$$

for $j \geq 0$ (as Euler himself may well have known).

Problem 1(b):

$$g_{2j-1}(t) \;=\; \sum_{n=0}^{\infty} (a+j)^{\overline{n}} \binom{n+j-1}{n} t^n \tag{30a}$$

$$g_{2j}(t) \;=\; \sum_{n=0}^{\infty} (a+j)^{\overline{n}} \binom{n+j}{n} t^n \tag{30b}$$

where I have used the Knuth notation $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$.

Problem 4(e):

$$g_{2j-1}(t) \;=\; \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{k+j-1}{k} \left\{ {n+j \atop k+j} \right\} x^k y^{n-k}\, t^n \tag{31a}$$

$$g_{2j}(t) \;=\; \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{k+j}{k} \left\{ {n+j \atop k+j} \right\} x^k y^{n-k}\, t^n \tag{31b}$$

the abscissa $v = 1$ up into ten parts again, and the ordinates in the single points of the division will behave in this way:

if v is	y will be	if v is	y will be
$v = \frac{0}{10}$,	$y = 0$;	$v = \frac{5}{10}$,	$y = \frac{1}{(1+\log 10 - \log 5)}$;
$v = \frac{1}{10}$,	$y = \frac{1}{(1+\log 10 - \log 1)}$;	$v = \frac{6}{10}$,	$y = \frac{1}{(1+\log 10 - \log 6)}$;
$v = \frac{2}{10}$,	$y = \frac{1}{(1+\log 10 - \log 2)}$;	$v = \frac{7}{10}$,	$y = \frac{1}{(1+\log 10 - \log 7)}$;
$v = \frac{3}{10}$,	$y = \frac{1}{(1+\log 10 - \log 3)}$;	$v = \frac{8}{10}$,	$y = \frac{1}{(1+\log 10 - \log 8)}$;
$v = \frac{4}{10}$,	$y = \frac{1}{(1+\log 10 - \log 4)}$;	$v = \frac{9}{10}$,	$y = \frac{1}{(1+\log 10 - \log 9)}$;
$v = \frac{5}{10}$,	$y = \frac{1}{(1+\log 10 - \log 5)}$;	$v = \frac{10}{10}$,	$y = 1$.

And therefore by approximation of the area one will again obtain the value of the letter A to a high enough degree of accuracy.

§21 But there is another method, derived from the nature of continued fractions, to inquire into the sum of this series, which completes the task a lot easier and faster; hence let, by the expressing the formula more generally, be

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B};$$

it will be

$$B = \frac{1x - 2x^2 + 6x^3 - 24x^4 + 120x^5 - 720x^6 + 5040x^7 - \text{etc.}}{1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.}} = \frac{x}{1+C}$$

and

$$1 + C = \frac{1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.}}{1 - 2x + 6x^2 - 24x^3 + 120x^4 - 720x^5 + 5040x^6 - \text{etc.}}.$$

Therefore

$$C = \frac{x - 4x^2 + 18x^3 - 96x^4 + 600x^5 - 4320x^6 + \text{etc.}}{1 - 2x + 6x^2 - 24x^3 + 120x^4 - 720x^5 + \text{etc.}} = \frac{x}{1+D}$$

hence

$$D = \frac{2x - 12x^2 + 72x^3 - 480x^4 + 3600x^5 - \text{etc.}}{1 - 4x + 18x^2 - 96x^3 + 600x^4 - \text{etc.}} = \frac{2x}{1+E}$$

Further

$$E = \frac{2x - 18x^2 + 144x^3 - 1200x^4 + \text{etc.}}{1 - 6x + 36x^2 - 240x^3 + \text{etc.}} = \frac{2x}{1-F}$$

and

$$F = \frac{3x - 36x^2 + 360x^3 - \text{etc.}}{1 - 9x + 72x^2 - 600x^3 + \text{etc.}} = \frac{3x}{1+G}.$$

It will be

$$G = \frac{3x - 48x^2 + \text{etc.}}{1 - 12x + 120x^2 - \text{etc.}} = \frac{3x}{1+H}.$$

So

$$H = \frac{4x - \text{etc}}{1 - 16x + \text{etc}} = \frac{4x}{1+I}.$$

And therefore it will become clear, that it will analogously be

$$I = \frac{4x}{1+K'}, \quad K = \frac{5x}{1+L'}, \quad L = \frac{5x}{1+M} \quad \text{etc. to infinity,}$$

so that the structure of these formulas is easily perceived. Having substituted these values one after another it will be

$$1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.}$$

$$A = \cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1+\cfrac{2x}{1+\cfrac{3x}{1+\cfrac{3x}{1+\cfrac{4x}{1+\cfrac{4x}{1+\cfrac{5x}{1+\cfrac{5x}{1+\cfrac{6x}{1+\cfrac{6x}{1+\cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§22 But how the value of continued fractions of this kind are to be investigated, I showed elsewhere. Because the integer parts of the single denominators are unities of course, only the numerators are important for the calculation; hence let $x = 1$ and the investigation of the sum A will be performed as follows:

$$\begin{array}{lcccccccccccc} A & = & \frac{0}{1}, & \frac{1}{1}, & \frac{1}{2}, & \frac{2}{3}, & \frac{4}{7}, & \frac{8}{13}, & \frac{20}{34}, & \frac{44}{73}, & \frac{124}{209}, & \frac{300}{501} & \text{etc.} \\ \text{Numerators:} & & 1, & 1, & 2, & 2, & 3, & 3, & 4, & 4, & 5, & 5 & \text{etc.} \end{array}$$

The fractions, exhibited here, get continuously closer to the true value of A of course and they are alternately too great and too small, so that it is

$$\begin{array}{llllll} A > \frac{0}{1}, & A > \frac{1}{2}, & A > \frac{4}{7}, & A > \frac{20}{34}, & A > \frac{124}{209} & \text{etc.} \\ A < \frac{1}{1}, & A < \frac{2}{3}, & A < \frac{8}{13}, & A < \frac{44}{73}, & A < \frac{300}{501} & \text{etc.} \end{array}$$