Continued fractions and Hankel-total positivity

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Key references:

1. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. **32**, 125–161 (1980).
2. Viennot, *Une théorie combinatoire des polynômes orthogonaux généraux* (UQAM, 1983).
3. Pétréolle–Sokal–Zhu, Lattice paths and branched continued fractions, arXiv:1807.03271

Total positivity

A (finite or infinite) matrix of real numbers is called *totally positive* if all its minors are nonnegative.

Applications:

- Mechanics of oscillatory systems
- Zeros of polynomials and entire functions
- Numerical linear algebra
- Approximation theory
- Stochastic processes
- Statistics
- Lie theory and cluster algebras
- Representation theory of the infinite symmetric group
- Theory of immanants
- Planar discrete potential theory and the planar Ising model
- **Stieltjes moment problem**
- **Enumerative combinatorics**

⋮

Hankel-total positivity

Given a sequence $\boldsymbol{a} = (a_n)_{n \ge 0}$, we define its *Hankel matrix*

$$H_\infty(\boldsymbol{a}) \;=\; (a_{i+j})_{i,j \ge 0} \;=\; \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- We say that the sequence $\boldsymbol{a}$ is *Hankel-totally positive* if its Hankel matrix $H_\infty(\boldsymbol{a})$ is totally positive.
- This implies that the sequence is *log-convex*, but is much stronger.

Fundamental Characterization (Stieltjes 1894, Gantmakher–Krein 1937):

For a sequence $\boldsymbol{a} = (a_n)_{n \ge 0}$ of real numbers, the following are equivalent:

(a) $\boldsymbol{a}$ is Hankel-totally positive.

(b) There exists a positive measure μ on $[0, \infty)$ such that $a_n = \int x^n \, d\mu(x)$ for all $n \ge 0$.
[That is, $(a_n)_{n \ge 0}$ is a **Stieltjes moment sequence**.]

(c) There exist numbers $\alpha_0, \alpha_1, \ldots \ge 0$ such that

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}$$

in the sense of formal power series.

[**Stieltjes-type continued fraction** with nonnegative coefficients]

From numbers to polynomials
[or, From counting to counting-with-weights]

Some simple examples:

1. Counting subsets of $[n]$: $a_n = 2^n$

   Counting subsets of $[n]$ by cardinality: $P_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$

2. Counting permutations of $[n]$: $a_n = n!$

   Counting permutations of $[n]$ by number of cycles:
   $$P_n(x) = \sum_{k=0}^{n} {n \brack k} x^k \quad \text{(Stirling cycle polynomial)}$$

   Counting permutations of $[n]$ by number of descents:
   $$P_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k \quad \text{(Eulerian polynomial)}$$

3. Counting partitions of $[n]$: $a_n = B_n$ (Bell number)

   Counting partitions of $[n]$ by number of blocks:
   $$P_n(x) = \sum_{k=0}^{n} {n \brace k} x^k \quad \text{(Bell polynomial)}$$

4. Counting non-crossing partitions of $[n]$: $a_n = C_n$ (Catalan number)

   Counting non-crossing partitions of $[n]$ by number of blocks:
   $$P_n(x) = \sum_{k=0}^{n} N(n,k)\, x^k \quad \text{(Narayana polynomial)}$$

These polynomials can also be **multivariate**!
(count with many simultaneous statistics)

An industry in combinatorics: q-Narayana polynomials, p, q-Bell polynomials, ...

Coefficientwise total positivity

- Consider sequences and matrices whose entries are *polynomials* with real coefficients in one or more indeterminates $\mathbf{x}$.
- A matrix is *coefficientwise totally positive* if every minor is a polynomial with nonnegative coefficients.
- A sequence is *coefficientwise Hankel-totally positive* if its Hankel matrix is coefficientwise totally positive.
- More generally, can consider sequences and matrices with entries in a *partially ordered commutative ring*.

But now there is no analogue of the Fundamental Characterization!

Coefficientwise Hankel-TP is **combinatorial**, not analytic.

Coefficientwise Hankel-TP *implies* that $(P_n(\mathbf{x}))_{n\ge 0}$ is a Stieltjes moment sequence for all $\mathbf{x} \ge 0$, but it is *stronger*.

Coefficientwise Hankel-TP in combinatorics

Many interesting sequences of combinatorial polynomials $(P_n(x))_{n\geq 0}$ have been proven in recent years to be *coefficientwise log-convex*:

- Bell polynomials $B_n(x) = \sum\limits_{k=0}^{n} {n \brace k} x^k$
  (Liu–Wang 2007, Chen–Wang–Yang 2011)
- Narayana polynomials $N_n(x) = \sum\limits_{k=0}^{n} N(n,k)\, x^k$
  (Chen–Wang–Yang 2010)
- Narayana polynomials of type B: $W_n(x) = \sum\limits_{k=0}^{n} \binom{n}{k}^2 x^k$
  (Chen–Tang–Wang–Yang 2010)
- Eulerian polynomials $A_n(x) = \sum\limits_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k$
  (Liu–Wang 2007, Zhu 2013)

Might these sequences actually be *coefficientwise Hankel-totally positive*?

- In many cases I can prove that the answer is **yes**, by using the Flajolet–Viennot method of *continued fractions*.
- In several other cases I have strong **empirical evidence** that the answer is **yes**, but no proof.
- The continued-fraction approach gives a *sufficient but not necessary* condition for coefficientwise Hankel-total positivity.

Combinatorics of continued fractions (Flajolet 1980)

We consider two types of continued fractions:

- Stieltjes type (S-fractions):

$$f(t) \;=\; \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cdots}}}}$$

- Jacobi type (J-fractions):

$$f(t) \;=\; \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \gamma_2 t - \cfrac{\beta_3 t^2}{1 - \gamma_3 t - \cdots}}}}$$

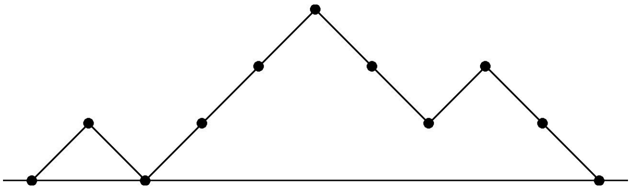
If time permits, I will discuss also a third type:

- Thron type (T-fractions):

$$f(t) \;=\; \cfrac{1}{1 - \delta_1 t - \cfrac{\alpha_1 t}{1 - \delta_2 t - \cfrac{\alpha_2 t}{1 - \delta_3 t - \cfrac{\alpha_3 t}{1 - \delta_4 t - \cdots}}}}$$

Combinatorics of Stieltjes-type continued fractions

A *Dyck path* of length $2n$ is a path in the right quadrant $\mathbb{N} \times \mathbb{N}$ from $(0,0)$ to $(2n,0)$ using steps $(1,1)$ ["rise"] and $(1,-1)$ ["fall"]:



Theorem (Flajolet 1980): As an identity in $\mathbb{Z}[\boldsymbol{\alpha}][[t]]$, we have

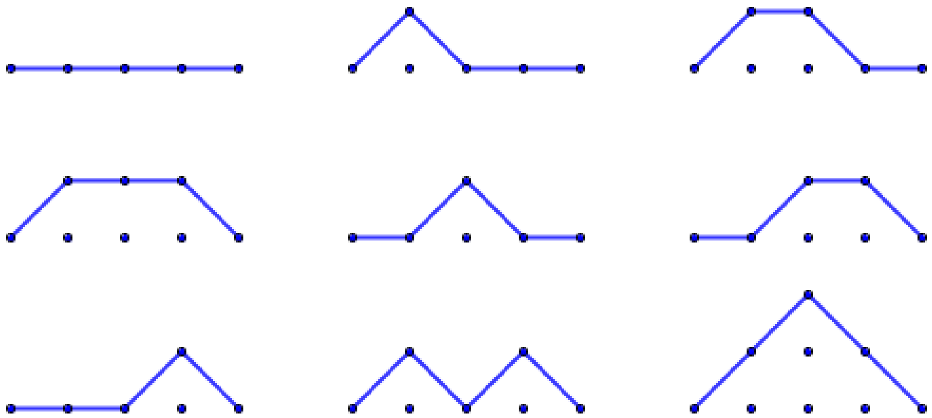
$$\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}} \;=\; \sum_{n=0}^{\infty} S_n(\alpha_1, \ldots, \alpha_n)\, t^n$$

where $S_n(\alpha_1, \ldots, \alpha_n)$ is the generating polynomial for Dyck paths of length $2n$ in which each fall starting at height i gets weight α_i.

$S_n(\boldsymbol{\alpha})$ is called the *Stieltjes–Rogers polynomial* of order n.

Combinatorics of Jacobi-type continued fractions

A *Motzkin path* of length n is a path in the right quadrant $\mathbb{N} \times \mathbb{N}$ from $(0,0)$ to $(n,0)$ using steps $(1,1)$ ["rise"], $(1,-1)$ ["fall"] and $(1,0)$ ["level"]:



All the Motzkin paths of length $n = 4$.

Theorem (Flajolet 1980): As an identity in $\mathbb{Z}[\boldsymbol{\beta},\boldsymbol{\gamma}][[t]]$, we have

$$\cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \gamma_2 t - \cdots}}} \;=\; \sum_{n=0}^{\infty} J_n(\boldsymbol{\beta},\boldsymbol{\gamma})\, t^n$$

where $J_n(\boldsymbol{\beta},\boldsymbol{\gamma})$ is the generating polynomial for Motzkin paths of length n in which each level step at height i gets weight γ_i and each fall starting at height i gets weight β_i.

$J_n(\boldsymbol{\beta},\boldsymbol{\gamma})$ is called the *Jacobi–Rogers polynomial* of order n.

Hankel matrix of Stieltjes–Rogers polynomials

Now form the infinite Hankel matrix corresponding to the sequence $\boldsymbol{S} = (S_n(\boldsymbol{\alpha}))_{n \ge 0}$ of Stieltjes–Rogers polynomials:

$$H_\infty(\boldsymbol{S}) \;=\; \big(S_{i+j}(\boldsymbol{\alpha})\big)_{i,j \ge 0}$$

And consider any minor of $H_\infty(\boldsymbol{S})$:

$$\Delta_{IJ}(\boldsymbol{S}) \;=\; \det H_{IJ}(\boldsymbol{S})$$

where $I = \{i_1, i_2, \ldots, i_k\}$ with $0 \le i_1 < i_2 < \ldots < i_k$
and $J = \{j_1, j_2, \ldots, j_k\}$ with $0 \le j_1 < j_2 < \ldots < j_k$

Theorem (Viennot 1983): The minor $\Delta_{IJ}(\boldsymbol{S})$ is the generating polynomial for families of disjoint Dyck paths $P_1, \ldots, P_k$ where path P_r starts at $(-2i_r, 0)$ and ends at $(2j_r, 0)$, in which each fall starting at height i gets weight α_i.

The proof uses the Karlin–McGregor–Lindström–Gessel–Viennot lemma on families of nonintersecting paths.

Corollary (A.S. 2014): The sequence $\boldsymbol{S} = (S_n(\boldsymbol{\alpha}))_{n \ge 0}$ is a Hankel-totally positive sequence in the polynomial ring $\mathbb{Z}[\boldsymbol{\alpha}]$ equipped with the coefficientwise partial order.

Now specialize $\boldsymbol{\alpha}$ to nonnegative elements in any partially ordered commutative ring:

Corollary: Let $\boldsymbol{\alpha} = (\alpha_n)_{n \ge 1}$ be a sequence of nonnegative elements in a partially ordered commutative ring R. Then $(S_n(\boldsymbol{\alpha}))_{n \ge 0}$ is a Hankel-totally positive sequence in R.

Hankel matrix of Stieltjes–Rogers polynomials (bis)

Can also get explicit formulae for the Hankel determinants $\Delta_n^{(m)}(\boldsymbol{S}) = \det H_n^{(m)}(\boldsymbol{S})$ for small m:

Theorem:

$$\Delta_n^{(0)}(\boldsymbol{S}) = (\alpha_1\alpha_2)^{n-1}(\alpha_3\alpha_4)^{n-2} \cdots (\alpha_{2n-3}\alpha_{2n-2})$$

$$\Delta_n^{(1)}(\boldsymbol{S}) = \alpha_1^n(\alpha_2\alpha_3)^{n-1}(\alpha_4\alpha_5)^{n-2} \cdots (\alpha_{2n-2}\alpha_{2n-1})$$

These formulae are classical in the theory of continued fractions, but Viennot 1983 gives a beautiful combinatorial interpretation.

See also Ishikawa–Tagawa–Zeng 2009 for extensions to $m = 2, 3$.

Hankel matrix of Jacobi–Rogers polynomials

Form the Hankel matrix

$$H_\infty(\boldsymbol{J}) \;=\; \big(J_{i+j}(\boldsymbol{\beta},\boldsymbol{\gamma})\big)_{i,j\ge 0}$$

But the story is more complicated than for S-type fractions, because:

- The matrix $H_\infty(\boldsymbol{J})$ is **not** totally positive in $\mathbb{Z}[\boldsymbol{\beta},\boldsymbol{\gamma}]$.
- It is not even totally positive in $\mathbb{R}$ for all $\boldsymbol{\beta},\boldsymbol{\gamma} \ge 0$.
- Rather, the total positivity of $H_\infty(\boldsymbol{J})$ holds only when $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ satisfy suitable **inequalities**.

Form the infinite tridiagonal matrix ("production matrix")

$$P(\boldsymbol{\beta},\boldsymbol{\gamma}) \;=\; \begin{pmatrix} \gamma_0 & 1 & 0 & 0 & \cdots \\ \beta_1 & \gamma_1 & 1 & 0 & \cdots \\ 0 & \beta_2 & \gamma_2 & 1 & \cdots \\ 0 & 0 & \beta_3 & \gamma_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Theorem: If $P(\boldsymbol{\beta},\boldsymbol{\gamma})$ is totally positive, then so is $H_\infty(\boldsymbol{J})$.
(special case of general result on production matrices;
works in a partially ordered commutative ring)

So we will need to test the production matrix for total positivity.

Luckily, there is a simple criterion:

> A tridiagonal matrix is totally positive if and only if all its off-diagonal elements and all its contiguous principal minors are nonnegative.

Classical for real-valued matrices; proof extends easily to matrices with values in a partially ordered commutative ring.

Example 1: Narayana polynomials

- Narayana numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$
- Count numerous objects of combinatorial interest:
  - Dyck paths of length $2n$ with k peaks
  - Non-crossing partitions of $[n]$ with k blocks
  - Non-nesting partitions of $[n]$ with k blocks
- Narayana polynomials $N_n(x) = \sum_{k=0}^{n} N(n,k)\, x^k$
- Ordinary generating function $\mathcal{N}(t,x) = \sum_{n=0}^{\infty} N_n(x)\, t^n$
- Elementary "renewal" argument on Dyck paths implies

$$\mathcal{N} \;=\; \frac{1}{1 - tx - t(\mathcal{N} - 1)}$$

  which can be rewritten as

$$\mathcal{N} \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - t\mathcal{N}}}$$

- Leads immediately to S-type continued fraction

$$\sum_{n=0}^{\infty} N_n(x)\, t^n \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{t}{1 - \cfrac{xt}{1 - \cfrac{t}{1 - \cdots}}}}}$$

Conclusion: The sequence $(N_n(x))_{n \ge 0}$ of Narayana polynomials is coefficientwise Hankel-totally positive.

Example 2: Bell polynomials

- Stirling number $\left\{ {n \atop k} \right\}$ = # of partitions of $[n]$ with k blocks
- Bell polynomials $B_n(x) = \sum\limits_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$
- Ordinary generating function $\mathcal{B}(t,x) = \sum\limits_{n=0}^{\infty} B_n(x)\, t^n$
- Flajolet (1980) expressed $\mathcal{B}(t,x)$ as a J-type continued fraction
- Can be transformed to an S-type continued fraction

$$\sum_{n=0}^{\infty} B_n(x)\, t^n \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{1t}{1 - \cfrac{xt}{1 - \cfrac{2t}{1 - \cdots}}}}}$$

Conclusion: The sequence $(B_n(x))_{n \ge 0}$ of Bell polynomials is coefficientwise Hankel-totally positive.

- Can extend to polynomial $B_n(x,p,q)$ that enumerates set partitions w.r.t. blocks (x), crossings (p) and nestings (q):

$$\sum_{n=0}^{\infty} B_n(x,p,q)\, t^n \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{[1]_{p,q} t}{1 - \cfrac{xt}{1 - \cfrac{[2]_{p,q} t}{1 - \cdots}}}}}$$

where $[n]_{p,q} = \dfrac{p^n - q^n}{p - q} = \sum\limits_{j=0}^{n-1} p^j q^{n-1-j}$

- Implies coefficientwise Hankel-TP jointly in x, p, q

Example 3: Narayana polynomials of type B

The polynomials

$$W_n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 x^k$$

- Grand Dyck paths with weight x for each peak
- Coordinator polynomial of the classical root lattice A_n
- Rank generating function of the lattice of noncrossing partitions of type B on $[n]$

- There is no S-type continued fraction *in the ring of polynomials*: we have

$$\alpha_1, \alpha_2, \ldots \;=\; 1+x,\; \frac{2x}{1+x},\; \frac{1+x^2}{1+x},\; \frac{x+x^2}{1+x^2},\; \frac{1+x^3}{1+x^2},\; \frac{x+x^3}{1+x^3},\; \ldots$$

- However, there *is* a nice *J-type* continued fraction:

$$\sum_{n=0}^{\infty} W_n(x)\, t^n \;=\; \cfrac{1}{1-(1{+}x)t - \cfrac{2xt^2}{1-(1{+}x)t - \cfrac{xt^2}{1-(1{+}x)t - \cfrac{xt^2}{1-\cdots}}}}$$

  with coefficients $\gamma_n = 1+x$, $\beta_1 = 2x$, $\beta_n = x$ for $n \ge 2$.

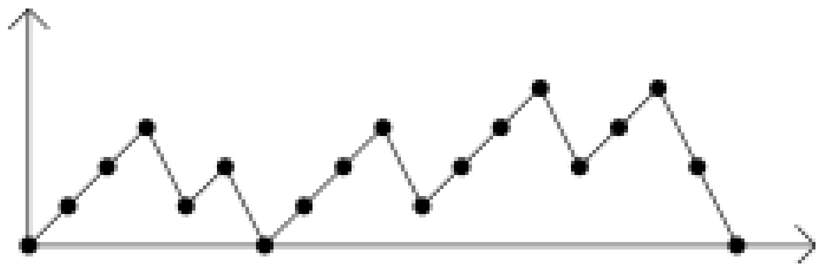
- The tridiagonal production matrix is totally positive.

- **Theorem** (A.S. unpublished 2014, Wang–Zhu 2016):
  The sequence $(W_n(x))_{n\ge 0}$ is coefficientwise Hankel-TP.

A new tool: Branched continued fractions

Generalize Dyck paths: Fix an integer $m \geq 1$.

An *m-Dyck path* of length $(m+1)n$ is a path in the right quadrant $\mathbb{N} \times \mathbb{N}$ from $(0,0)$ to $((m+1)n, 0)$ using steps $(1,1)$ ["rise"] and $(1,-m)$ ["m-fall"]:



A 2-Dyck path of length 18.

Let $S_n^{(m)}(\boldsymbol{\alpha})$ be the generating polynomial for m-Dyck paths of length $(m+1)n$ in which each m-fall starting at height i gets weight α_i.

We call $S_n^{(m)}(\boldsymbol{\alpha})$ the *m-Stieltjes–Rogers polynomial* of order n.

Theorem (Pétréolle–A.S.–Zhu 2018): The sequence $(S_n^{(m)}(\boldsymbol{\alpha}))_{n\geq 0}$ of m-Stieltjes–Rogers polynomials is coefficientwise Hankel-TP in the polynomial ring $\mathbb{Z}[\boldsymbol{\alpha}]$.

Proof is essentially identical to the one for $m = 1$!

Remark: $S_n^{(m)}(\boldsymbol{\alpha})$ are the Taylor coefficients of (extremely ugly) *branched continued fractions*.

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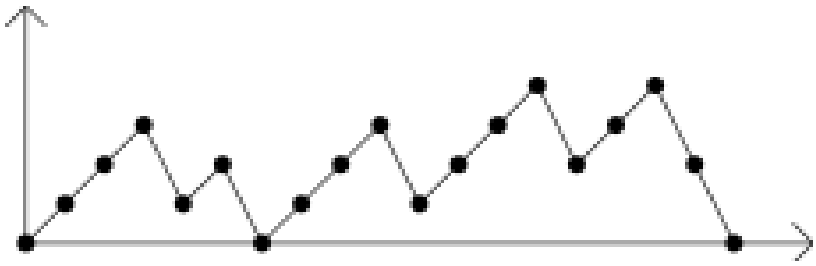

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Proof is essentially identical to the one for $m = 1$!

Remark: $S_n^{(m)}(\boldsymbol{\alpha})$ are the Taylor coefficients of (extremely ugly) *branched continued fractions:*

$$f(t) = \cfrac{1}{1 - \cfrac{\alpha_m t}{\left(1 - \cfrac{\alpha_{m+1} t}{\left(1 - \frac{\alpha_{m+2} t}{(\cdots)\cdots(\cdots)}\right)\cdots\left(1 - \frac{\alpha_{2m+1} t}{(\cdots)\cdots(\cdots)}\right)}\right) \cdots \left(1 - \cfrac{\alpha_{2m} t}{\left(1 - \frac{\alpha_{2m+1} t}{(\cdots)\cdots(\cdots)}\right)\cdots\left(1 - \frac{\alpha_{3m} t}{(\cdots)\cdots(\cdots)}\right)}\right)}}$$

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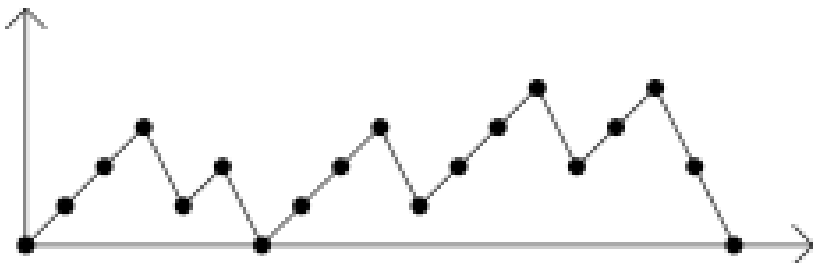

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Proof is essentially identical to the one for $m = 1$!

Remark: $S_n^{(m)}(\boldsymbol{\alpha})$ are the Taylor coefficients of (extremely ugly) *branched continued fractions*.

Non-obvious fact: The $S_n^{(m)}(\boldsymbol{\alpha})$ get more general as m grows.

Branched continued fractions: An example

- $n! = \int_0^\infty x^n e^{-x}\, dx$ is a Stieltjes moment sequence.
- Euler showed in 1746 that

$$\sum_{n=0}^{\infty} n!\, t^n \;=\; \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cdots}}}}}$$

- The entrywise product of Stieltjes moment sequences is also one.
- So $(n!)^2$ is a Stieltjes moment sequence.
- Straightforward computation gives for $(n!)^2$

$$\alpha_1, \alpha_2, \ldots \;=\; 1,\ 3,\ \tfrac{20}{3},\ \tfrac{164}{15},\ \tfrac{3537}{205},\ \tfrac{127845}{5371},\ \tfrac{4065232}{124057},\ \tfrac{244181904}{5868559},\ \tfrac{38418582575}{721944303},\ \ldots$$

- The $\boldsymbol{\alpha}$ are indeed positive, but what the hell are they???

- $(n!)^2$ has a nice m-branched continued fraction with $m = 2$:

$$\boldsymbol{\alpha} \;=\; 1, 1, 2, 4, 4, 6, 9, 9, 12, \ldots$$

- Similar results hold for $(n!)^m$, $(2n-1)!!^m$, $(mn)!$ and much more general things.
- But these are special cases of something vastly more general …

Branched continued fractions for ratios of contiguous hypergeometric functions

- Euler also showed in 1746 that

$$\sum_{n=0}^{\infty} a(a+1)\cdots(a+n-1)\, t^n \;=\; \cfrac{1}{1-\cfrac{at}{1-\cfrac{1t}{1-\cfrac{(a+1)t}{1-\cfrac{2t}{1-\cdots}}}}}$$

- And this is the $b=1$ special case of

$$\frac{{}_2F_0\!\left(\begin{matrix} a,b \\ - \end{matrix}\,\middle|\, t\right)}{{}_2F_0\!\left(\begin{matrix} a,b-1 \\ - \end{matrix}\,\middle|\, t\right)} \;=\; \cfrac{1}{1-\cfrac{at}{1-\cfrac{bt}{1-\cfrac{(a+1)t}{1-\cfrac{(b+1)t}{1-\cdots}}}}}$$

($_2F_0$ limiting case of Gauss continued fraction for $_2F_1$)

- We generalize this to ratios of contiguous $_{m+1}F_0$: the result is an m-branched continued fraction ...

Branched continued fractions for ratios of contiguous hypergeometric functions (bis)

Theorem (Pétréolle–A.S.–Zhu 2018): For each $m \geq 1$,

$$\frac{{}_{m+1}F_0\left(\begin{matrix} a_1, \ldots, a_{m+1} \\ - \end{matrix} \,\middle|\, t\right)}{{}_{m+1}F_0\left(\begin{matrix} a_1, \ldots, a_m, a_{m+1}-1 \\ - \end{matrix} \,\middle|\, t\right)} \;=\; \sum_{n=0}^{\infty} S_n^{(m)}(\boldsymbol{\alpha})\, t^n$$

where the $\boldsymbol{\alpha}$ are very simple polynomials in $a_1, \ldots, a_{m+1}$:

$$\boldsymbol{\alpha} \;=\; a_1 \cdots a_m,\; a_2 \cdots a_{m+1},\; a_3 \cdots a_{m+1}(a_1+1),\; a_4 \cdots a_{m+1}(a_1+1)(a_2+1),\; \ldots$$

Corollary: The polynomials $P_n^{(m)}(a_1, \ldots, a_m; a_{m+1}) \;=\; S_n^{(m)}(\boldsymbol{\alpha})$ arising as the Taylor coefficients of this ratio are coefficientwise Hankel-TP jointly in $a_1, \ldots, a_{m+1}$.

Can obtain many examples by specialization of $a_1, \ldots, a_{m+1}$.

Even more generally: For every $r, s \geq 0$ we find an m-branched continued fraction with $m = \max(r-1, s)$ for ratios of contiguous ${}_rF_s$.

- Generalizes Gauss continued fraction for ${}_2F_1$.
- Can further generalize to q-hypergeometric functions ${}_r\phi_s$.
- But corollaries for Hankel-TP are more subtle than for $s = 0$. (Already this was the case for ${}_2F_1$ compared to ${}_2F_0$.)

Coefficientwise Hankel-TP seems to be very common ... but not so easy to prove

There are *many* cases where:

- I find **empirically** that a sequence $(P_n(x))_{n \ge 0}$ is coefficientwise Hankel-TP ...
- But I am **unable to prove it** because there is neither an S-type nor a J-type continued fraction in the ring of polynomials (and maybe no branched continued fraction, either?).

- Rook polynomials
- Domb polynomials
- Apéry polynomials
- Boros–Moll polynomials
- Ramanujan polynomials
- Inversion enumerators for trees
- Reduced binomial discriminant polynomials

$\vdots$

Example 1: Rook polynomials

- Non-attacking rooks on $n \times n$ chessboard with weight x per rook:
  $$R_n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 k!\, x^k$$
- Can *prove*: Stieltjes moment sequence for each $x \geq 0$.
  (Can find explicit moment representation)
- *Empirical*: Hankel matrix is coefficientwise TP up to 11×11.
- **Conjecture**: Hankel matrix is coefficientwise TP.
  (Special case of more general conjecture for Laguerre polynomials)
- We have conjectural (but unproven) branched continued fraction.

Example 2: Apéry polynomials

- Apéry numbers $A_n \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$
- **Theorem** (conjectured by me, 2014; proven G. Edgar, unpub. 2016):
  $(A_n)_{n\geq 0}$ is a Stieltjes moment sequence.
- Define Apéry polynomials $A_n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k$
- **Conjecture 1**: $(A_n(x))_{n\geq 0}$ is a Stieltjes moment sequence for all $x \geq 1$ (but not for $0 < x < 1$).
- **Conjecture 2**: $(A_n(1+y))_{n\geq 0}$ is coefficientwise Hankel-TP in y.
  (Tested up to 12×12)
- Don't know (even conjecturally) any continued fraction.

(Tentative) Conclusion

- Many interesting sequences $(P_n(\mathbf{x}))_{n\geq 0}$ of combinatorial polynomials are (or appear to be) coefficientwise Hankel-totally positive.
- In some cases this can be proven by the Flajolet–Viennot method of continued fractions.
  - When S-fractions exist, they give the simplest proofs.
  - Sometimes S-fractions don't exist, but J-fractions can work.
  - Sometimes neither S-fractions nor J-fractions exist, but branched S-fractions do.
  - Branched S-fractions are a powerful (but not universal) tool.
- Alas, in many cases *none* of these methods work!
- **New methods of proof will be needed:**
  - Differential operators?
  - Direct study of Hankel minors?
  - ... ???
- Coefficientwise Hankel-TP is a big phenomenon that we understand, at present, only very incompletely.

Dedicated to the memory of Philippe Flajolet (1948–2011)