Exercises for the course on planar maps
AEC Summer School, Hagenberg, 2018

Exercise 1: Whitney's theorem

Our aim is to show here Whitney's theorem (assuming Menger's theorem for 3-connected graphs), which states that a 3-connected planar graph has a unique embedding on the sphere (up to a mirror). A graph G is called 3-connected if it simple, has at least 4 vertices and for any pair u, v of vertices, the graph $G\backslash\{u, v\}$ is connected.

Q1. Let M be a planar map. Show that if a cycle C is not the contour of a face, then C either has a chord (i.e., an edge not on C with its two extremities on C), or C is a separating cycle (i.e., $G\backslash C$ is not connected).

Q2. If M is a 3-connected planar map, show that every face-contour is a non-separating chordless cycle.

(To show non-separation we need Menger's theorem, which states that for G a 3-connected graph and u, v any two vertices of G, there are 3 paths connecting u to v that are vertex-disjoint except at their extremities.)

Q3. Deduce from it that a 3-connected planar graph has exactly two embeddings on the sphere, which differ by a mirror.

Exercise 2: Tutte's method (cf Brown) for simple triangulations

We define a quasi-triangulation as a rooted planar map that is simple (no loops nor multiple edges), with all inner faces of degree 3 and such that the outer face contour is a simple cycle (no pinch point).

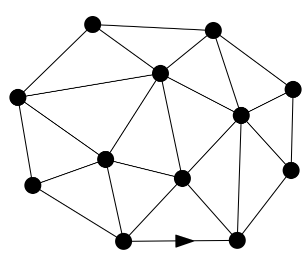


Figure 1: A quasi-triangulation with 4 internal vertices and 8 external vertices.

For $n \geq 0$ and $m \geq 0$ we let $d_{n,m}$ be the number of quasi-triangulations with n internal vertices and $m + 3$ external vertices (those incident to the outer face), and let $D \equiv D(t, x) = \sum_{n,m} d_{n,m} t^n x^m$ be the associated generating function;

and let $D_0 \equiv D(t,0) = \sum_{n\geq 0} d_{n,0} t^n$, which is the generating function of rooted simple triangulations.

Q1. Show (using deletion of the root-edge) that $D \equiv D(t,x)$ satisfies the equation

$$(E): \qquad D = (1+xD)^2 + tx^{-1}(D - D_0) - tDD_0 \quad \text{where } D_0 = D(t,0).$$

Q2. Show that this equation has a unique solution that is a power series in t and x (hint: there is a certain way to order the coefficients such that the equation determines the coefficients iteratively)

Q3. Let $K \equiv K(t)$ be an arbitrary power series in t. Consider the equation, in the unknown $D \equiv D(t,x)$:

$$(E_K): \qquad D = (1+xD)^2 + tx^{-1}(D-K) - tDK.$$

Show that if there is a solution D to this equation that is a power series in (t,x), then $D(t,0) = K(t)$ and thus D is the unique solution of (E).

Q4. (To be done with a computer algebra software) From (E) one can quickly compute the coefficients of $D(t,x)$ and conjecture that

$$D(t,0) = \sum_{n\geq 0} \frac{2(4n+1)!}{(3n+2)!(n+1)!} t^n,$$

One can check (e.g. using the Lagrange inversion formula) that this is parametrized as $D(t,0) = \frac{1-2u}{(1-u)^3}$, where $u = u(t)$ is the power series given by $u = t/(1-u)^3$.

Check that if we consider the power series $K \equiv K(t)$ given by

$$\begin{cases} K &= (1-2u)/(1-u)^3, \\ t &= u(1-u)^3, \end{cases}$$

then there is a solution D to (E_K) that is a power series in (t,x).

Note: This is Brown's proof ('Enumeration of triangulations of the disk', 1963) for the number of rooted simple triangulations, where he more generally obtains the following nice explicit formula for $d_{n,m}$:

$$d_{n,m} = \frac{2(2m+3)!(4n+2m+1)!}{(m+2)!m!n!(3n+2m+3)!}.$$

Exercise 3: counting tree-rooted d-regular maps

A *tree-rooted map* is a pair (M, T) where M is a rooted planar map and T is a spanning tree of M. A map is called d-regular if all its vertices have degree d. Our aim here is to find (bijectively) a formula for the number $r_n^{(d)}$ of tree-rooted d-regular maps with n vertices.

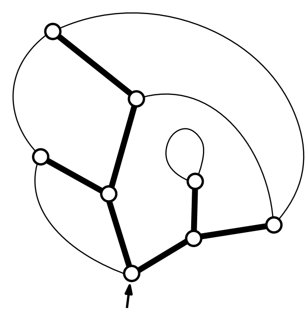


Figure 2: A tree-rooted 3-regular map with 8 vertices (the root is here indicated as a marked corner).

Q1. For $d \geq 3$ and $n \geq 1$, let $\mathcal{A}_n^{(d)}$ be the set of rooted (embedded) trees with n nodes (in a tree, a node is a vertex that is not a leaf), such that the root-node has d children and all the other nodes have $d-1$ children. Let $a_n^{(d)} = \mathrm{Card}(\mathcal{A}_n^{(d)})$ and let $F^{(d)}(x) = \sum_{n\geq 1} a_n^{(d)} x^n$. Show that

$$F^{(d)}(x) = x(1+u)^d, \quad \text{where } u \equiv u(x) \text{ is given by } u = x(1+u)^{d-1}.$$

Deduce from it a formula for $a_n^{(d)}$.

(We recall the Lagrange inversion formula: if $F(x)$ is given by $F(x) = \psi(u(x))$ where $u(x)$ is given by $u(x) = x\phi(u(x))$, then $[x^n]F(x) = \frac{1}{n}[u^{n-1}]\psi'(u)\phi(u)^n$.)

Q2. Let f be the number of leaves of a tree in $\mathcal{A}_n^{(d)}$. Express f in terms of n and d.

Q3. Show that (with $\mathrm{Cat}_m = \frac{(2m)!}{m!(m+1)!}$ the mth Catalan number)

$$r_n^{(d)} = a_n^{(d)} \cdot \mathrm{Cat}_{f/2}.$$

and deduce from it a closed-form formula for $r_n^{(d)}$.

Q4. Check that the asymptotic for $r_n^{(d)}$ is always of the form $c\gamma^n n^{-3}$ for some constants c and γ (and for n such that $r_n^{(d)} \neq 0$).

Exercise 4: local characterization of geodesic labellings

For a connected graph G with a marked vertex v_0, consider the geodesic labelling with respect to v_0, that is, every vertex $v \in G$ gets label $d(v_0, v)$ (the length of a shorted path connecting v_0 to v).

Q1. Show that this labelling is characterized as the unique labelling $\ell(v)$ of the vertices of G (with labels in $\mathbb{Z}$) satisfying the following properties:

- for every edge $\{u, v\}$ of G we have $|\ell(u) - \ell(v)| \leq 1$,
- every vertex $v \neq v_0$ has a neighbour of smaller label
- the label of v_0 is 0.

Q2. Show that the graph is bipartite if and only if there is no edge $\{u, v\}$ with $\ell(u) = \ell(v)$.

Exercise 5: relation expected radius $\leftrightarrow$ expected 2-point distance

Q1. Let $\mathcal{Q}_n$ be the set of pairs (M, v_0) where M is a rooted quadrangulation with n faces, and v_0 is a vertex of M. For Q_n a random element from $\mathcal{Q}_n$, with $e = \{u, w\}$ the root-edge, let X_n be the distance from v_0 to e, i.e., $X_n = \min(\mathrm{dist}(v_0, u), \mathrm{dist}(v_0, w))$. And let Y_n be the radius of Q_n centered at v_0, i.e., $Y_n = \max_{v \in Q_n} \mathrm{dist}(v_0, v)$. Show that

$$\mathbb{E}(Y_n) = 2\mathbb{E}(X_n) + 1.$$

Q2. A similar feature occurs for plane trees. Let $\mathcal{T}_n$ be the set of pairs (T, v_0) where T is a rooted plane tree (embedded tree with a marked corner) with n edges, and v_0 is a vertex of T. Let v_1 be the vertex incident to the root-corner (possibly $v_0 = v_1$). For T_n a random element from $\mathcal{T}_n$, let $X_n = \mathrm{dist}(v_0, v_1)$, and let Y_n be the radius of T_n centered at v_0, i.e., $Y_n = \max_{v \in T_n} \mathrm{dist}(v_0, v)$. Show that

$$\mathbb{E}(Y_n) = 2\mathbb{E}(X_n).$$

Exercise 6: an invariant for well-labelled trees

Recall that a well-labelled tree is a rooted plane tree T (plane tree with a marked corner) where each vertex v has a label $\ell(v) \in \mathbb{Z}$ such that the root-vertex has label 0 and for every edge $e = \{u, v\}$ of T we have $|\ell(v) - \ell(u)| \leq 1$.

The generating function of well-labelled trees by edges is denoted $R \equiv R(t)$, and for $i \geq 0$, $R_i \equiv R_i(t)$ denotes the generating function of well-labelled trees where all the vertex-labels are strictly larger than $-i$ (so we have $R(t) = \lim_{i \to \infty} R_i(t)$

for the coefficientwise convergence in power series). Recall that we have the equations

$$R = 1 + 3tR^2, \qquad R_0 = 0, \qquad R_i = 1 + tR_i(R_{i-1} + R_i + R_{i+1}) \quad \text{for } i \geq 1.$$

Q1. Let $S_i = R_i - R_{i-1}$ be the generating function of well-labelled trees where the smallest vertex-label is $-i+1$. Show that

$$S_i = R_{i-1}t(S_{i-1} + S_i + S_{i+1})R_i.$$

Q2. Deduce from it that the quantity $R_i - tR_{i-1}R_iR_{i+1}$ is an invariant (does not depend on i), and that it gives $R_1 = R - tR^3$.

Q3. We now consider the generating functions $r_i \equiv r_i(s)$ defined by the relation

$$r_i(s) = \frac{R_i(t)}{R_1(t)} \quad \text{with the change of variable relation } s = tR_1(t)^2.$$

Show that the generating functions r_i satisfy the equations

$$r_0 = 0, \quad r_i = 1 + s \cdot (r_{i-1}\, r_i\, r_{i+1}) \;\text{ for } i \geq 1.$$

Give a combinatorial interpretation of objects counted by $r_i(s)$.

Q4. It is possible to find an exact expression for $r_i(s)$ by a guessing/checking approach as seen in the course. But we can also proceed by a substitution approach starting from the known expression of $R_i(t)$. Recall that $R_i(t)$ has the explicit expression

$$R_i = R\frac{(1-x^i)(1-x^{i+3})}{(1-x^{i+1})(1-x^{i+2})},$$

where $R \equiv R(t)$ is given by $R = 1+3tR^2$ and $x \equiv x(t)$ is given by $x+\frac{1}{x}+1 = \frac{1}{tR^2}$. Show that r_i is expressed as

$$r_i = r\frac{(1-y^i)(1-y^{i+3})}{(1-y^{i+1})(1-y^{i+2})},$$

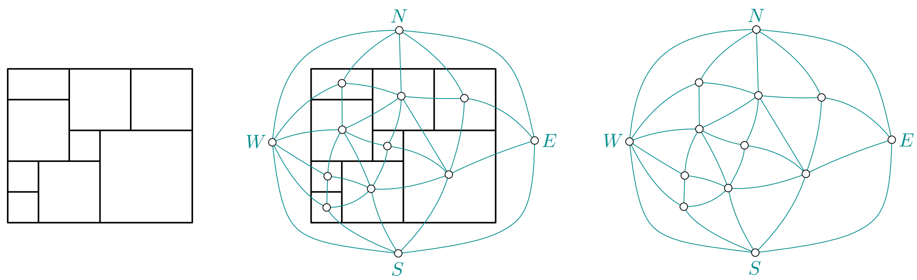
where $r \equiv r(s) = \lim_{i\to\infty} r_i(s)$ is given by $r = 1 + sr^3$ and $y \equiv y(s)$ is given by $y + \frac{1}{y} + 1 = \frac{1}{sr^2}$.

(Hint: check that under the change of variable relation $s = tR_1^2$, one has $tR(t)^2 = sr(s)^2$ and $\frac{R_i(t)}{R(t)} = \frac{r_i(s)}{r(s)}$.)

Note: The combinatorial proof of the invariant $R_i - tR_{i-1}R_iR_{i+1}$ in Q1&2 is due to Guillaume Chapuy. Another combinatorial proof relying on local operations on quadrangulations is given in Section 3.3 of the article 'Planar maps and continued fractions' by Bouttier and Guitter. The substitution approach in Q4 to obtain the exact expression of $r_i(s)$ is given in Section 2.3 of the article 'Distance statistics in quadrangulations with no multiple edges and the geometry of minbus' by Bouttier and Guitter.

Exercise 7: uniqueness of square tilings with a prescribed combinatorics

As shown in the figure below, a square tiling of a rectangle has naturally a dual map, which is a triangulation of the 4-gon (we consider square tilings with no degeneracy, i.e., no point belonging to four squares).



In this exercise we consider the inverse problem. Given T a triangulation of the 4-gon, with W, N, E, S its four outer vertices, realize T as the dual of a square tiling. As we will see here, the square tiling is unique (up to scaling) and well characterized (it also always exists, up to allowing for degeneracies, but the proof is more involved).

Let V be the set of internal vertices of T. A *metric* on T is a mapping $m(\cdot)$ from V to non-negative values that are not all zero. The norm of m is $||m||$ given by $||m||^2 = \sum_{v\in V} m(v)^2$. An (N, S)-path is a path γ from N to S passing by internal vertices only, and the m-length of γ, denoted $\ell_\gamma(m)$, is the sum of the m-values of the visited internal vertices. The (N, S)-length of m is then defined as

$$\ell_m := \min \ell_\gamma(m),$$

where the minimum is taken over all (N, S)-paths. A metric s is called optimal if it satisfies $\frac{\ell_s^2}{||s||^2} \geq \frac{\ell_m^2}{||m||^2}$ for every metric m.

Q1. Show that there exists an optimal metric and that it is unique up to scaling, i.e., up to multiplication of all values by a same constant.

(Hint: consider the problem of minimizing $||m||$ under the constraint $\ell_m \geq 1$)

Q2. Assume there is a square tiling with T as its dual, and of unit total area. Let $s(\cdot)$ be the metric assigning to each internal vertex the length of the associated square. Show that s is the optimal metric of T.

(Hint: let h be the height and h^{-1} the width of the tiling. Let m be an arbitrary metric on T. For $t \in [0, h^{-1}]$ consider the (N, S)-path γ_t consisting of vertices that are dual to the squares intersected by the vertical line $\{x = t\}$. Integrate $\ell_m(\gamma_t)$ on $[0, h^{-1}]$.)

Note: This exercise is extracted from the article 'Square tilings with prescribed combinatorics' by Oded Schramm, which also proves the existence of such tilings (if degeneracies are allowed).