Planar maps: bijections and applications

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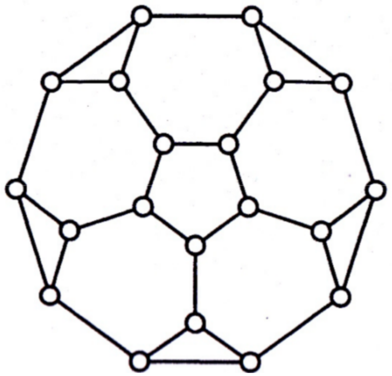
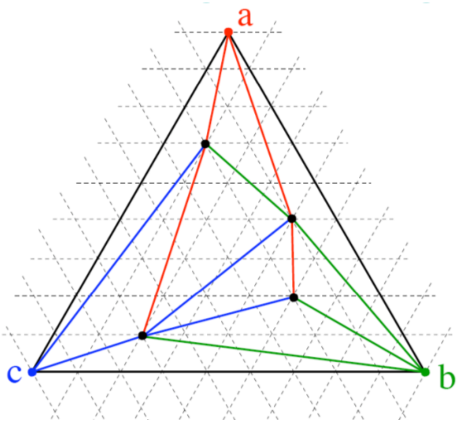
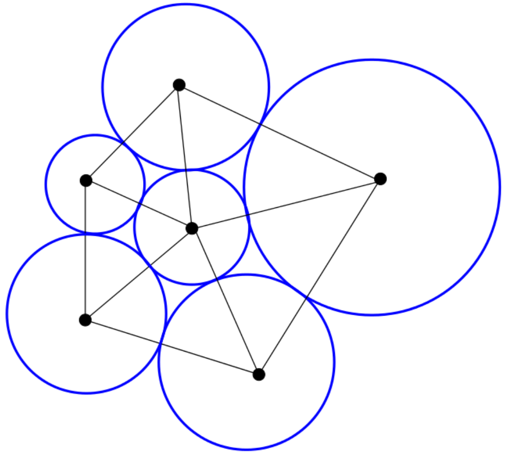
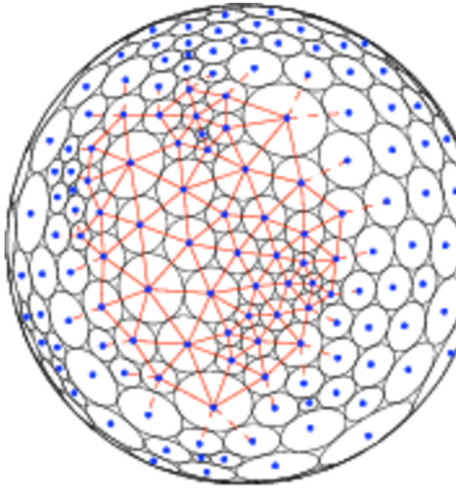
Geometric representation of planar maps

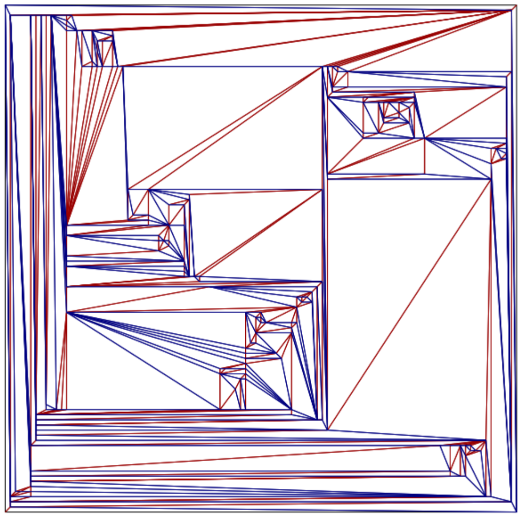
Various methods can be used to draw a map on the plane/sphere

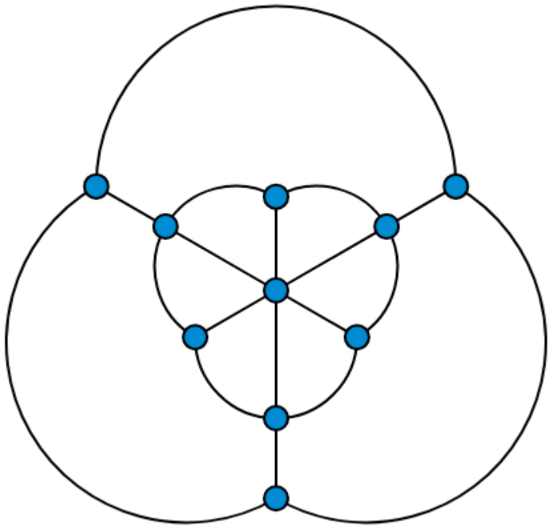


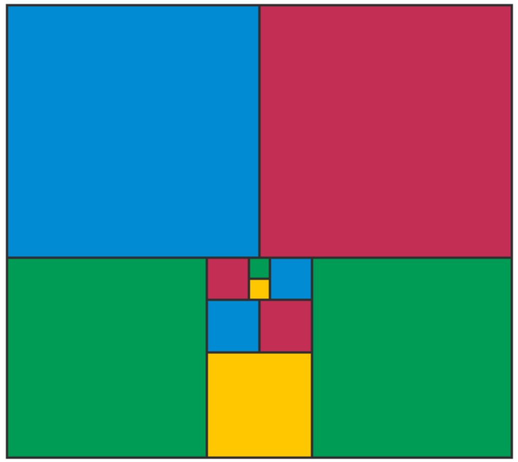






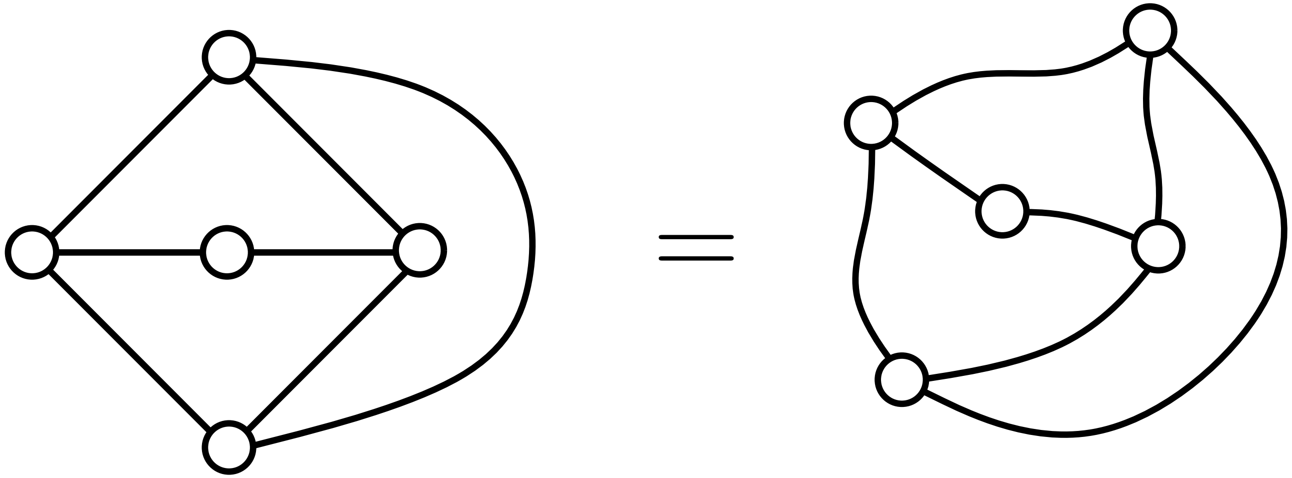






Existence question

planar map (with outer face) = equivalence class of planar drawings of graphs up to continuous deformation



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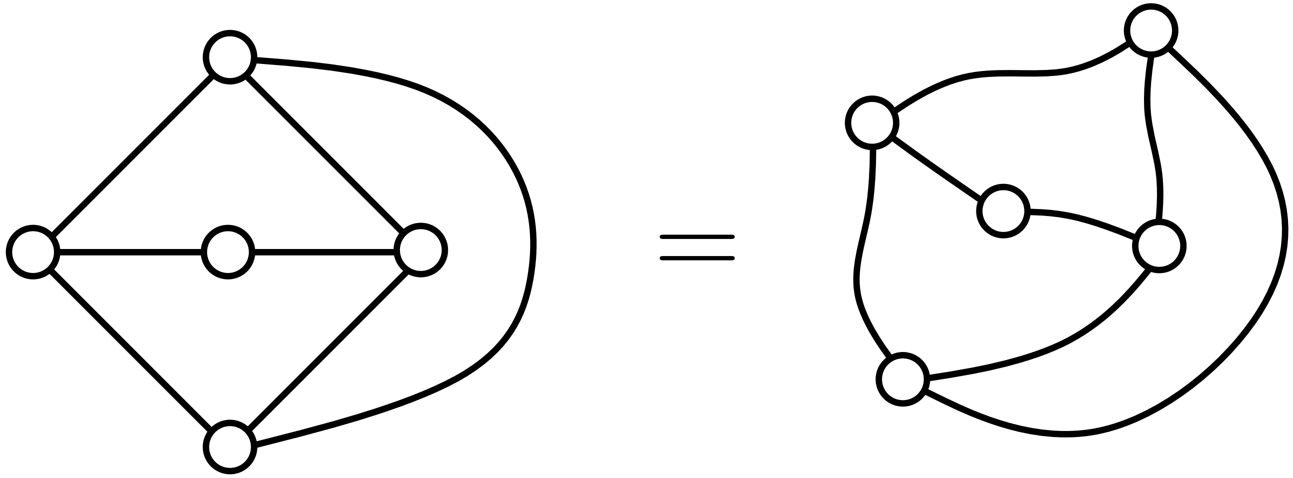

Question: Does there always exist an equivalent planar drawing such that all edges are drawn as segments ?

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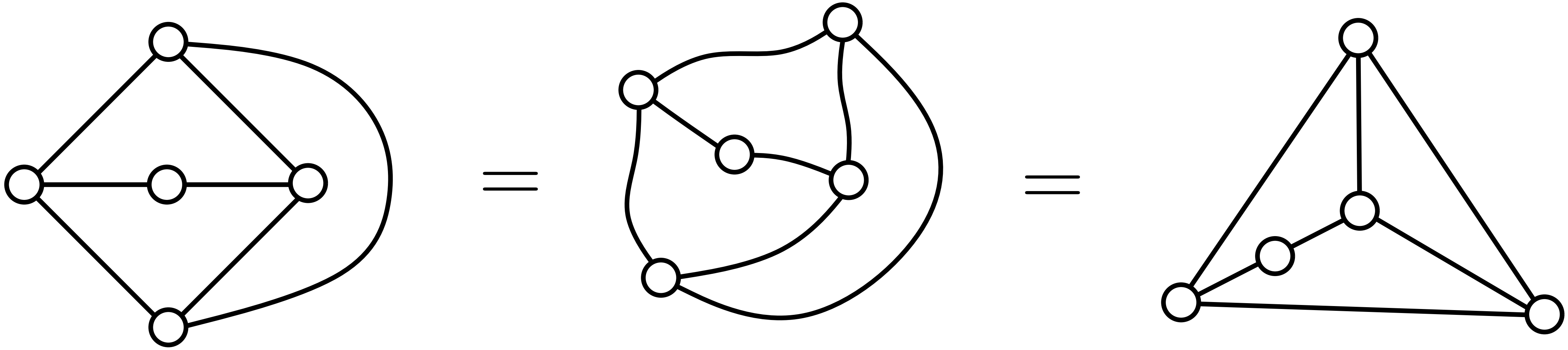

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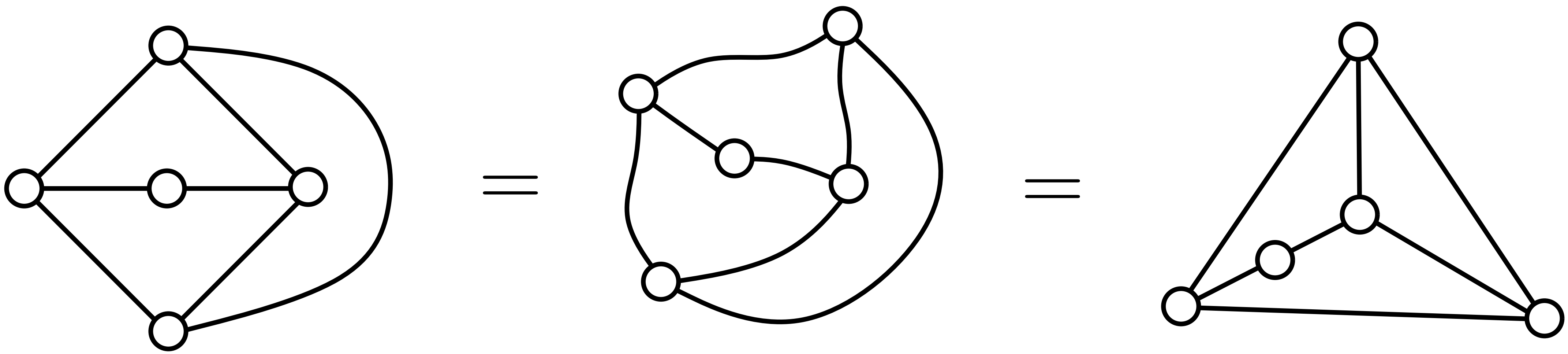

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(such as drawing is called a (planar) **straight-line drawing**)

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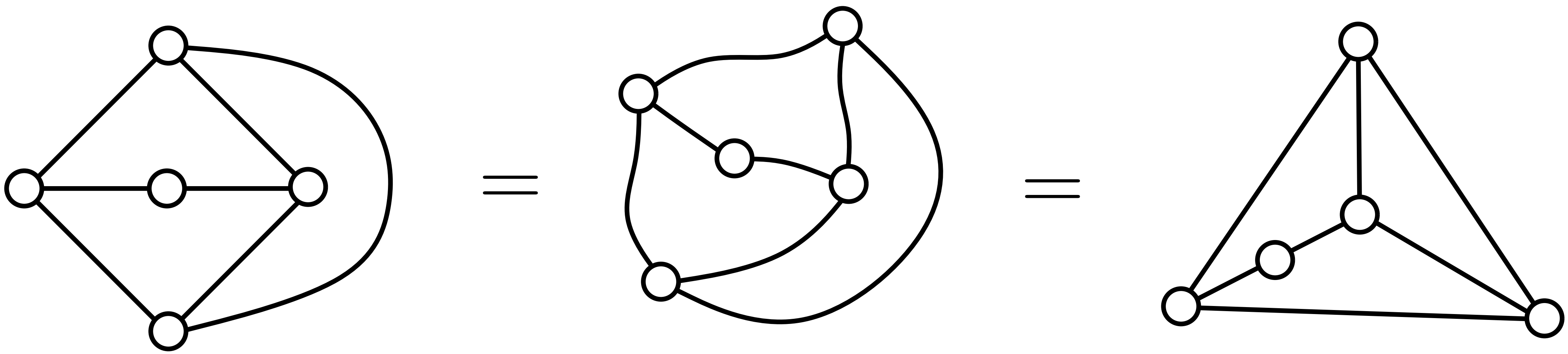

Question: Does there always exist an equivalent planar drawing such that all edges are drawn as segments ?

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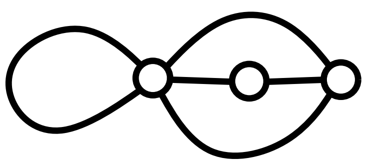
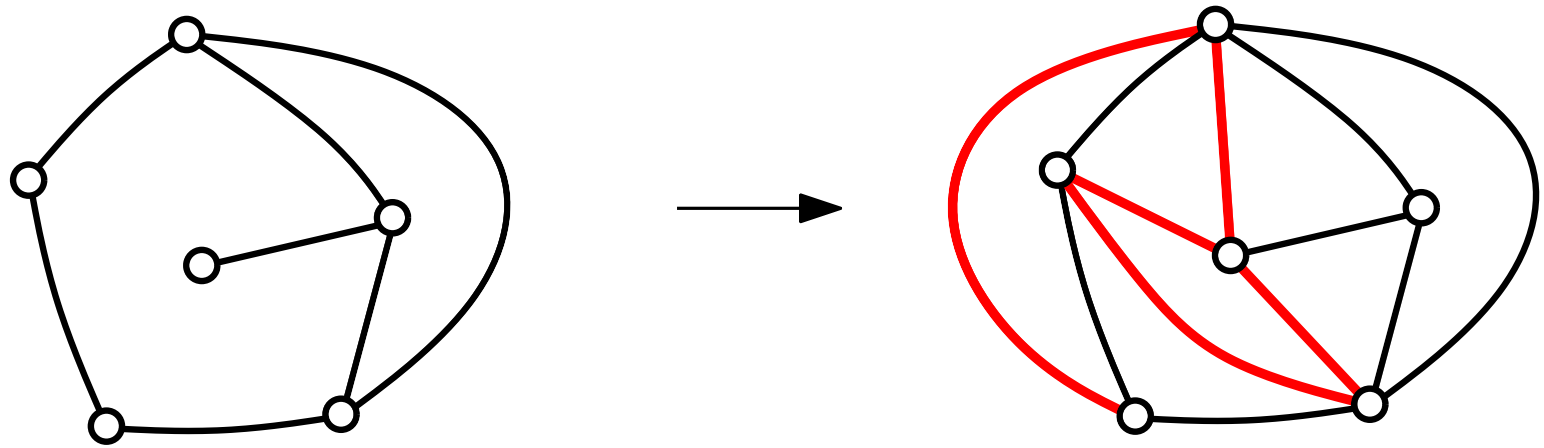
Remark: For such a drawing to exist, the map needs to be simple



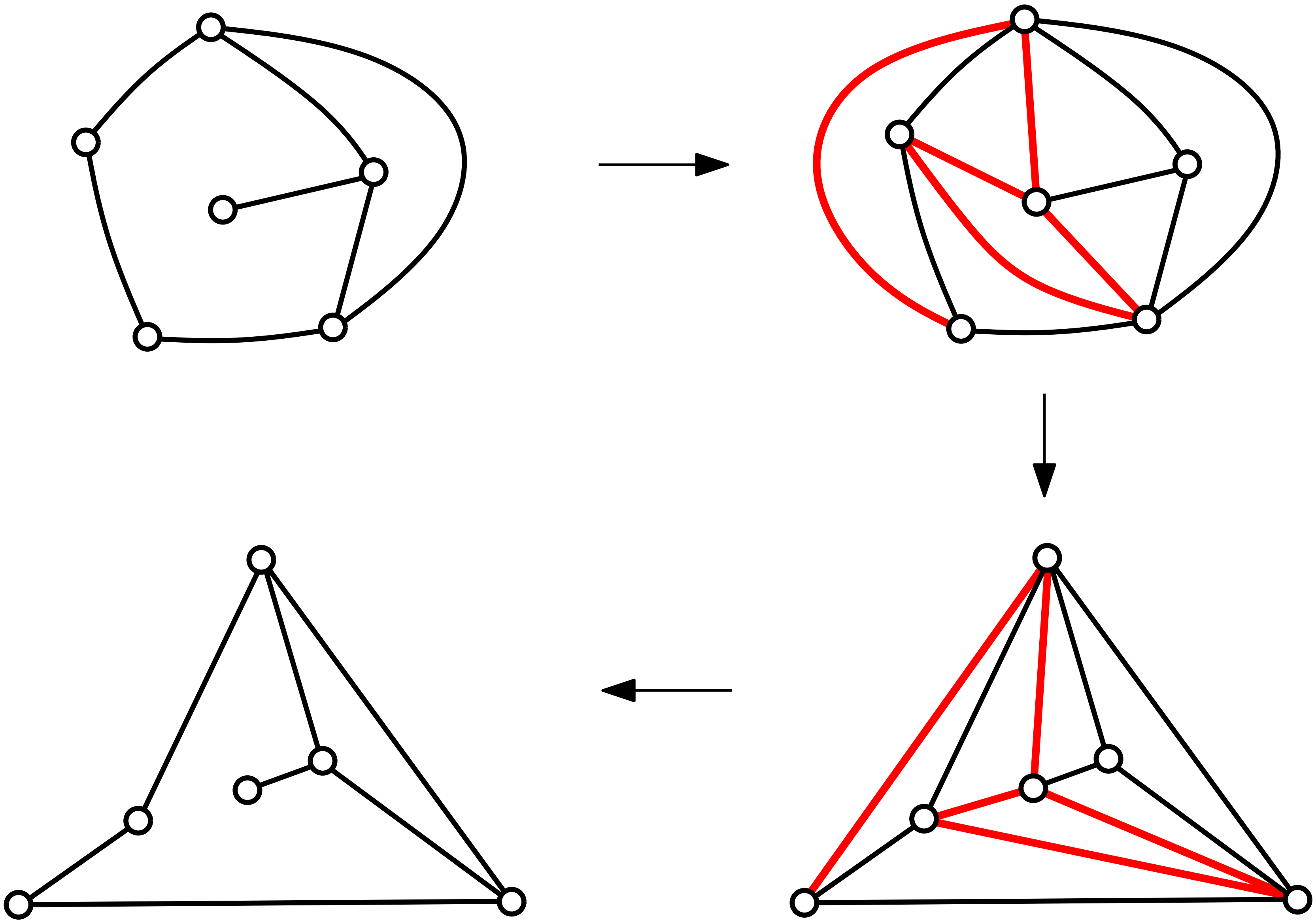
Existence proof (reduction to triangulations)

- Any simple planar map M can be completed to a simple triangulation T



Existence proof (reduction to triangulations)

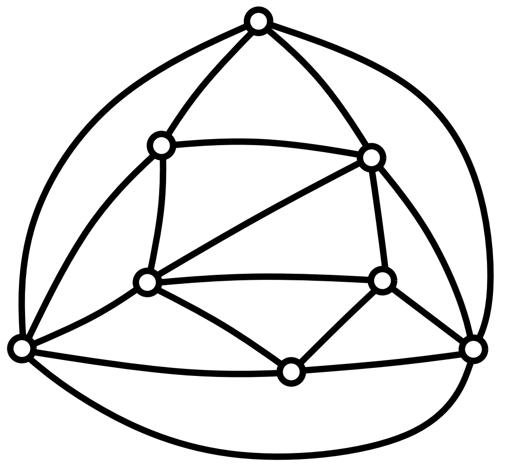
- Any simple planar map M can be completed to a simple triangulation T
- A straight-line drawing of T yields a straight-line drawing of M



Existence proof (for triangulations)

First proof: induction on the number of vertices

Let T be a triangulation with n vertices



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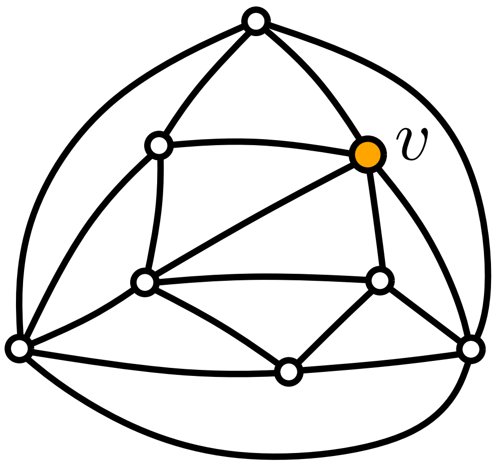
Exercise: T has at least one inner vertex v of degree ≤ 5



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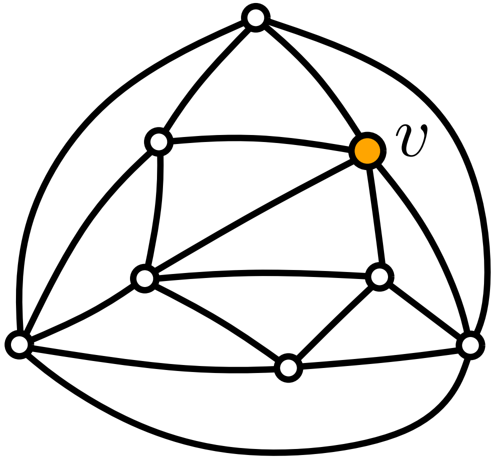
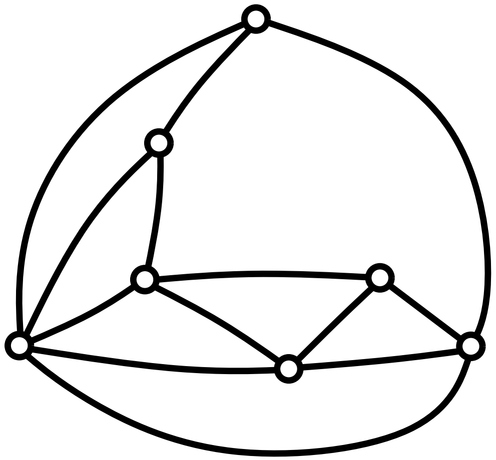
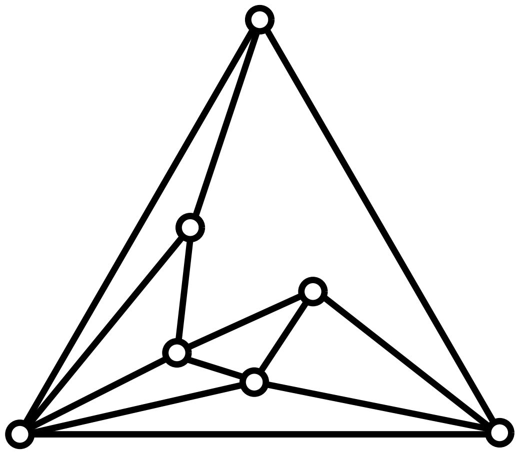


$\Downarrow$

$T \backslash v$



$\Rightarrow$



induction

$T \backslash v$ has a straight-line drawing

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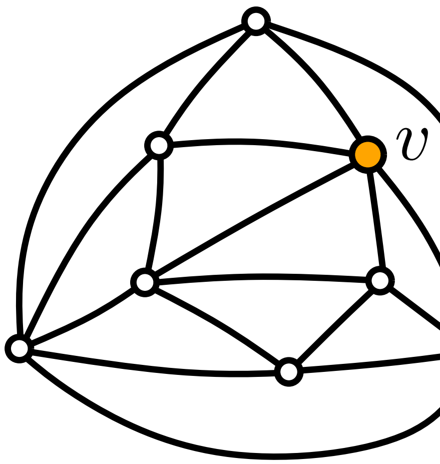

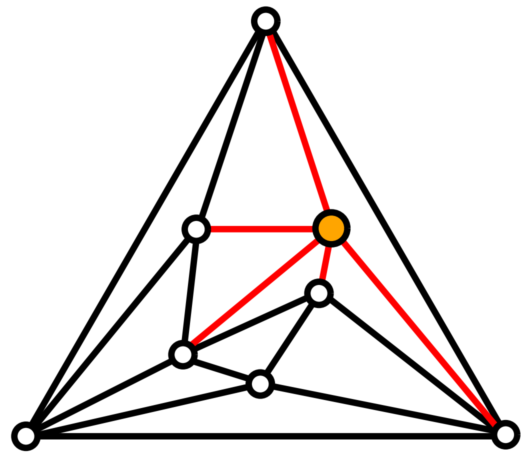

$\Downarrow$ $\Uparrow$

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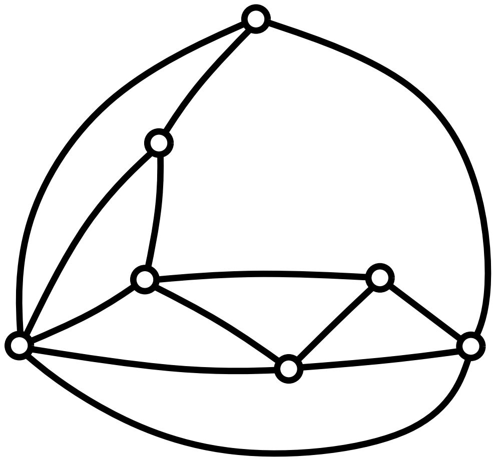

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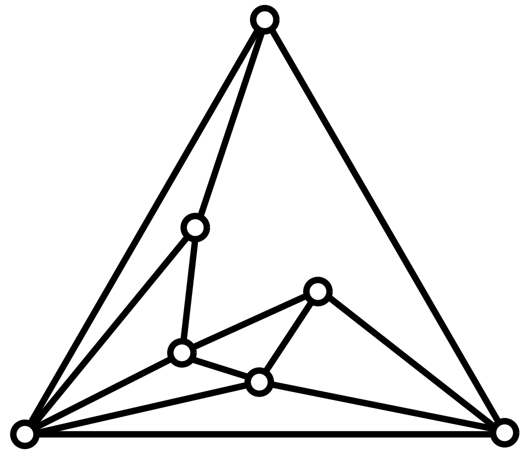


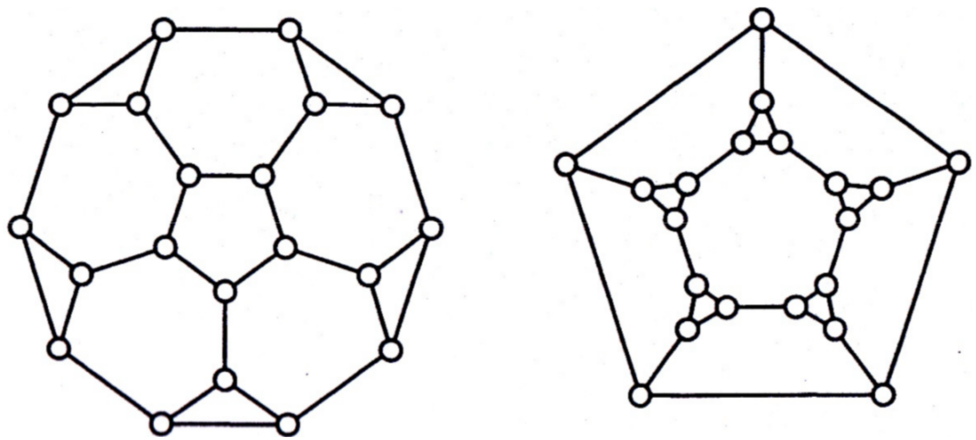
induction

$T \backslash v$ has a straight-line drawing

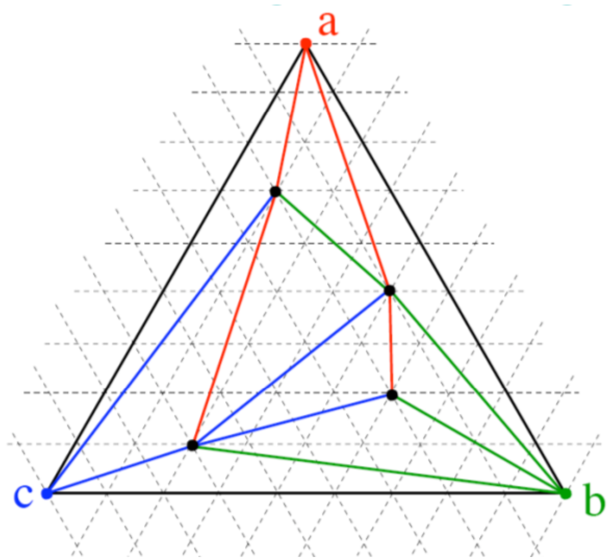
Straight-line drawing algorithms

We present two classical algorithms

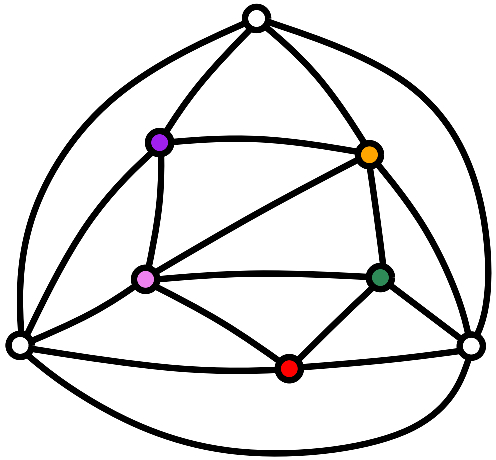
- Tutte's barycentric method

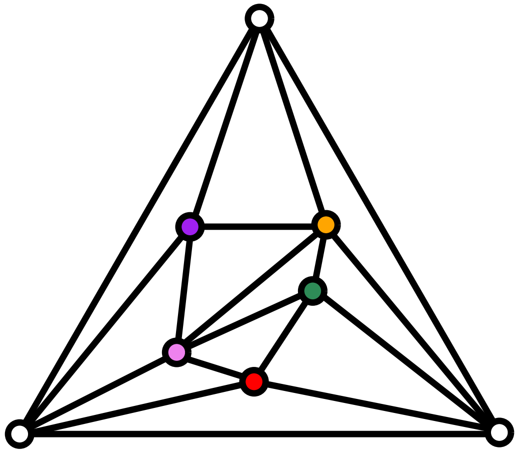


- Schnyder's face-counting algorithm

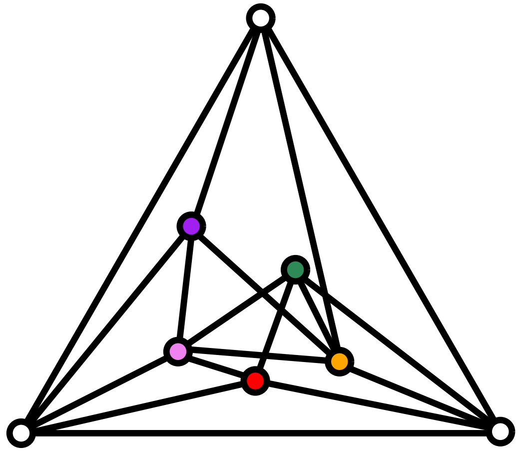


Planarity criterion for straight-line drawings



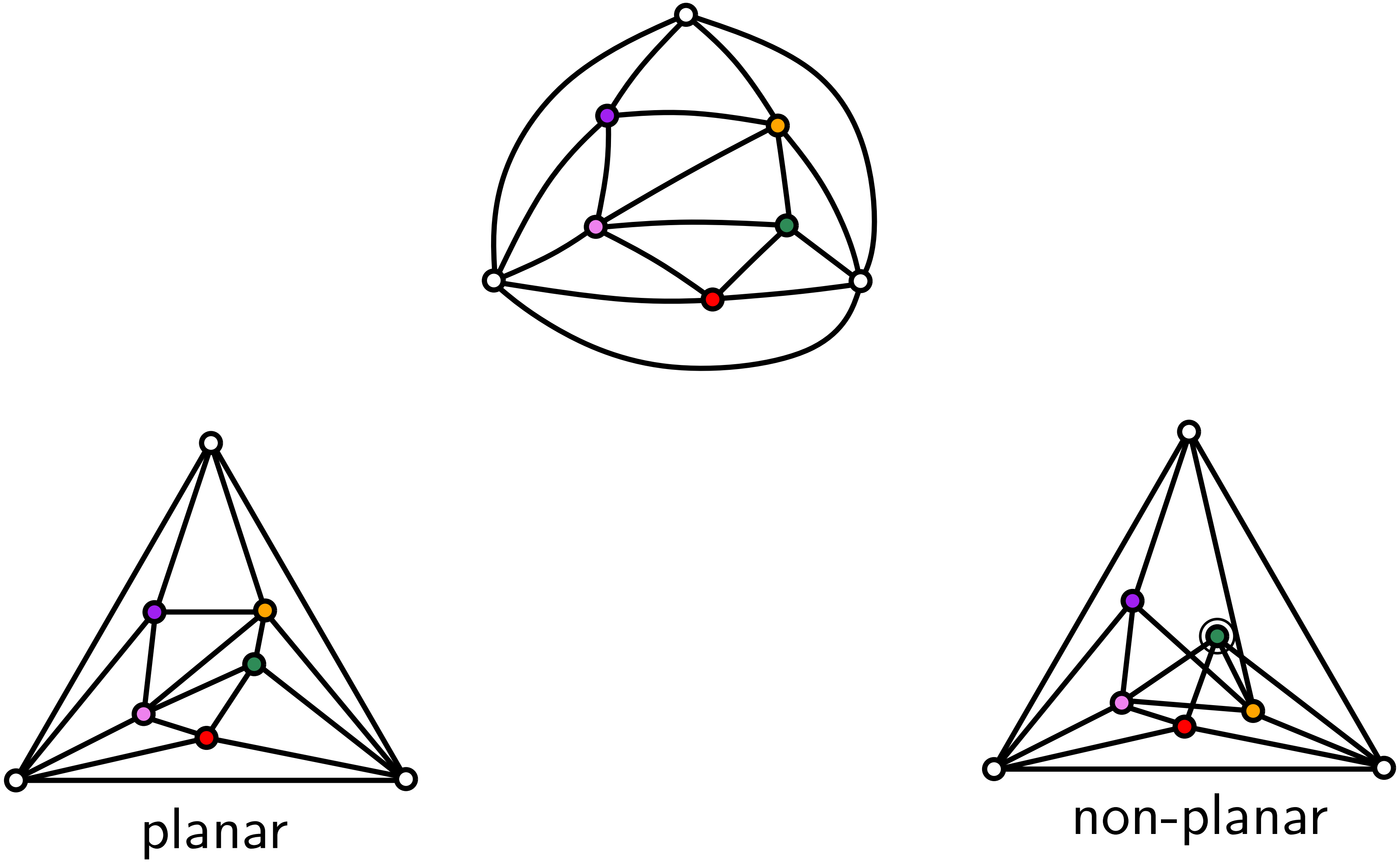


planar



non-planar

Planarity criterion for straight-line drawings

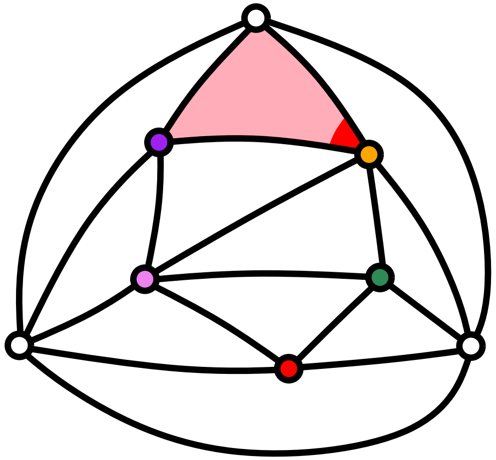


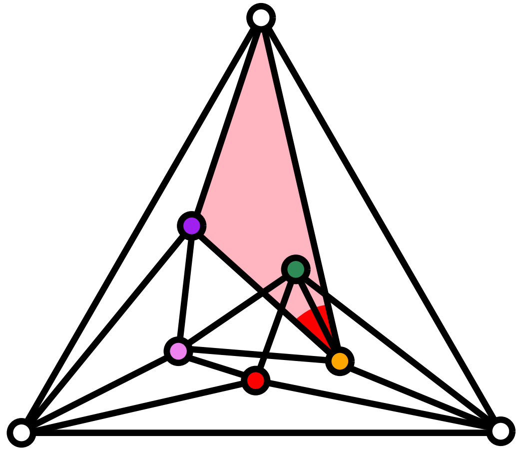
Theorem: a straight-line drawing is planar iff every inner vertex is inside the **convex hull** of its neighbours

(works for triangulations and more generally for 3-connected planar graphs)

Proof idea

- For each corner $c \in T$ let $\theta(c)$ be the angle of c in the drawing





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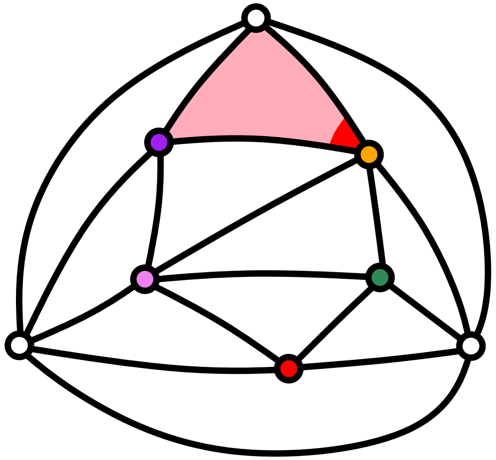

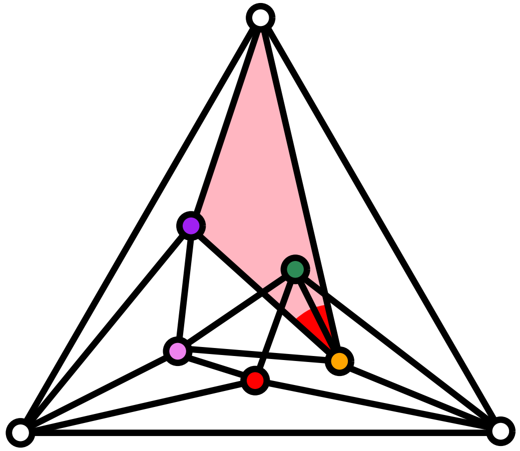

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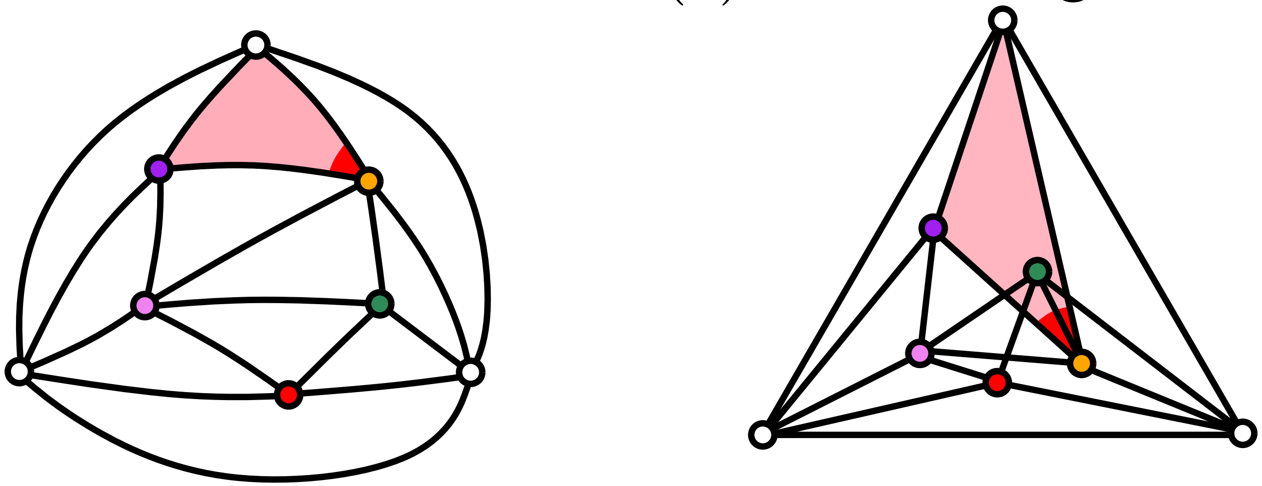

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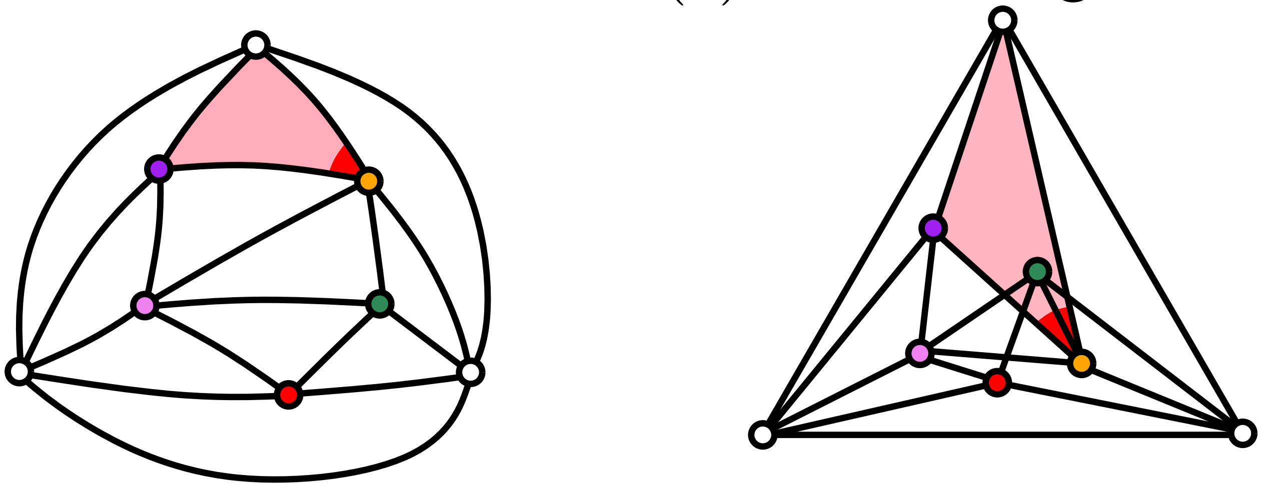

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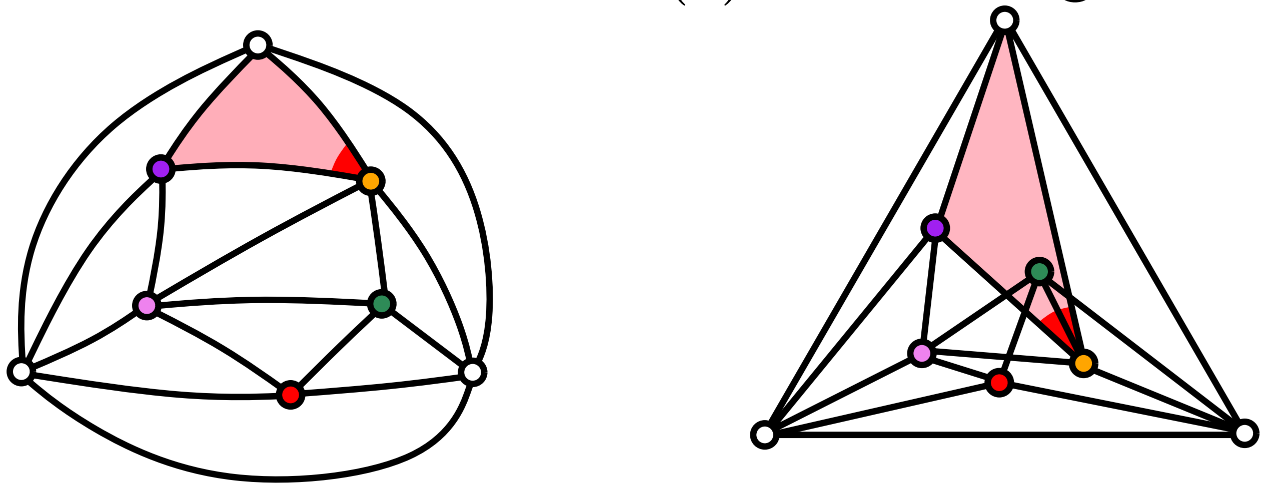

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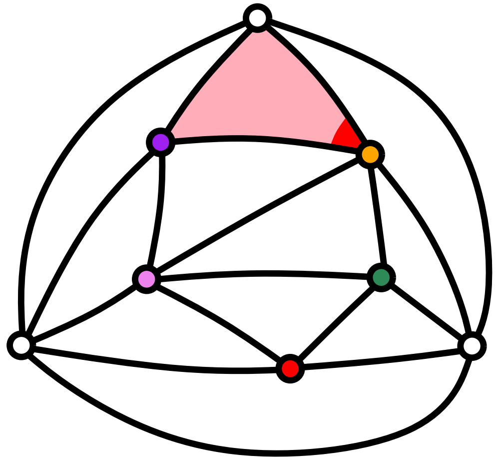

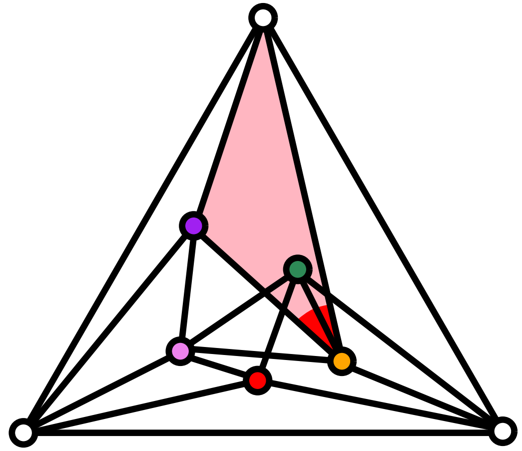

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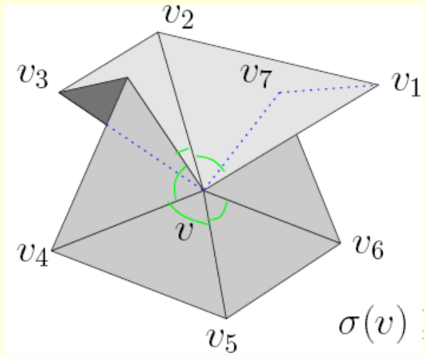
Hence locally planar at each vertex

(no "folding" of triangles at a vertex)

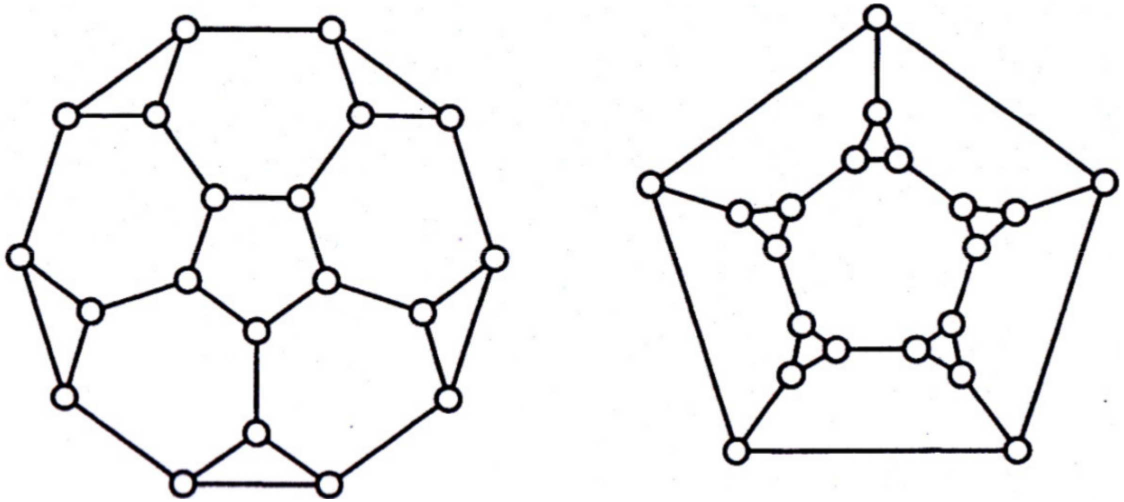
$\Rightarrow$ the drawing is planar



Tutte's barycentric method

- Outer vertices $v_1, \ldots, v_d$ are fixed at fixed positions (nailed)
- Each inner vertex is at the **barycenter of its neighbours**

$$x_i = \frac{1}{\Delta_i} \sum_{j \sim i} x_j \qquad y_i = \frac{1}{\Delta_i} \sum_{j \sim i} y_j \qquad \text{for } i \geq 4$$



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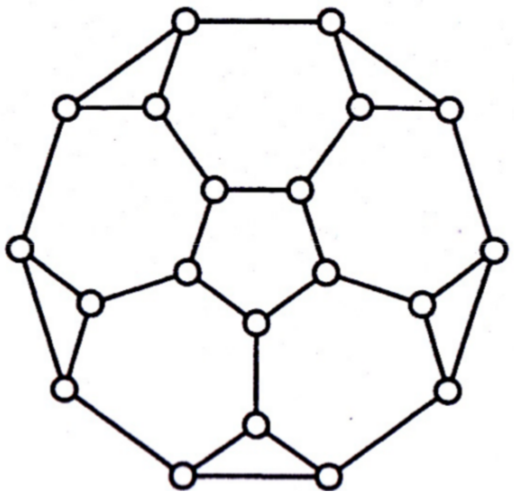
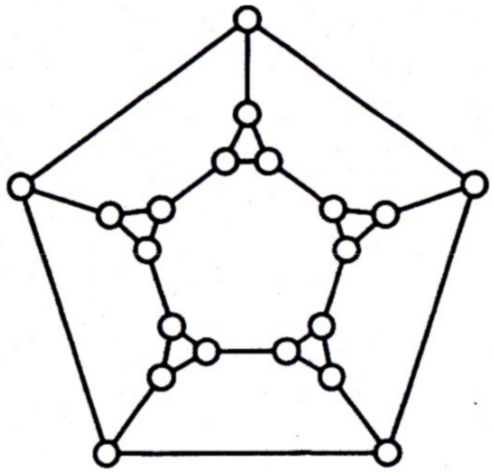

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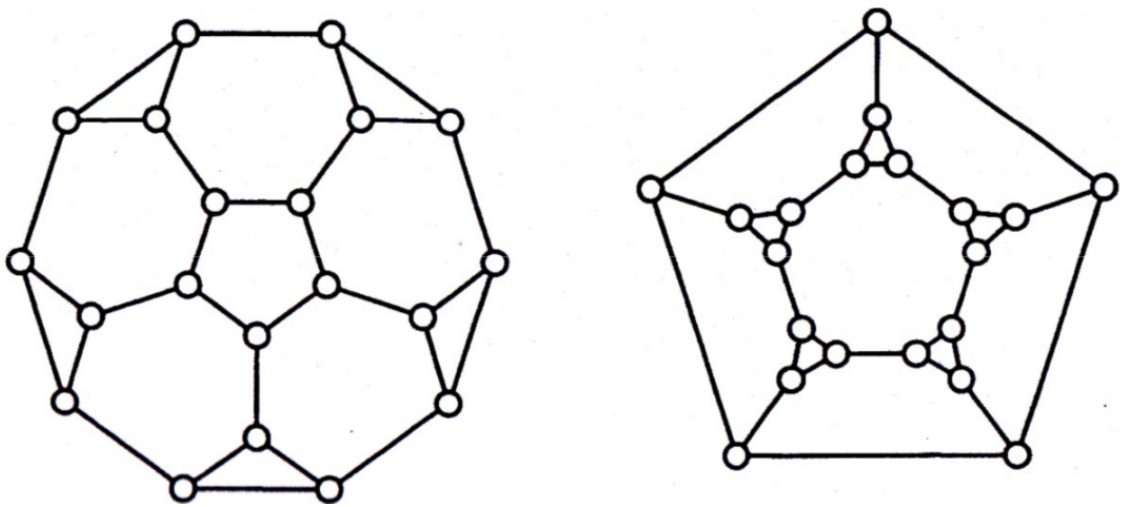

- This drawing **exists and is unique**. It minimizes the energy

$$\mathcal{P} = \sum_e \ell(e)^2 = \sum_{\{i,j\} \in T} (x_i - x_j)^2 + (y_i - y_j)^2$$

under the constraint of fixed $x_1, \ldots, x_d, y_1, \ldots, y_d$

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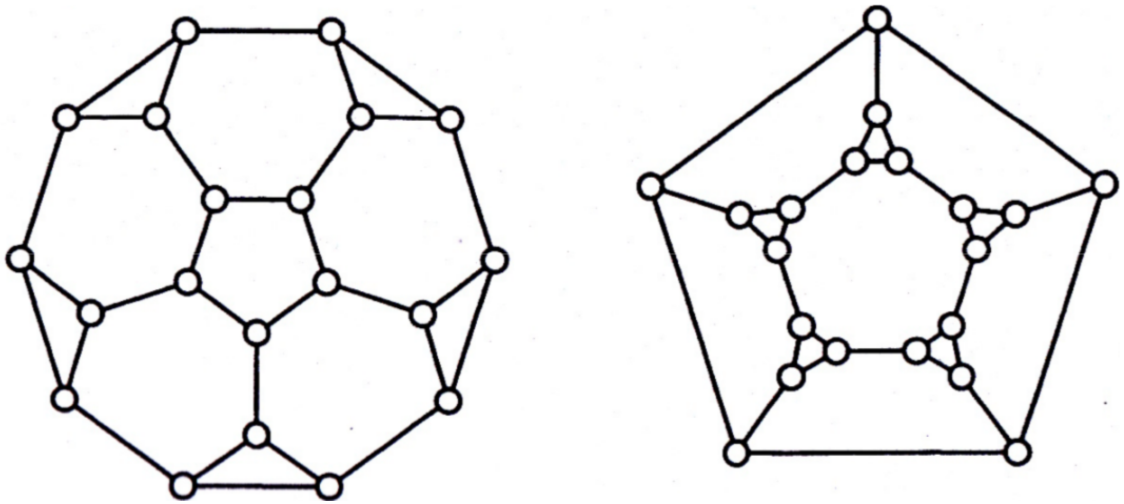

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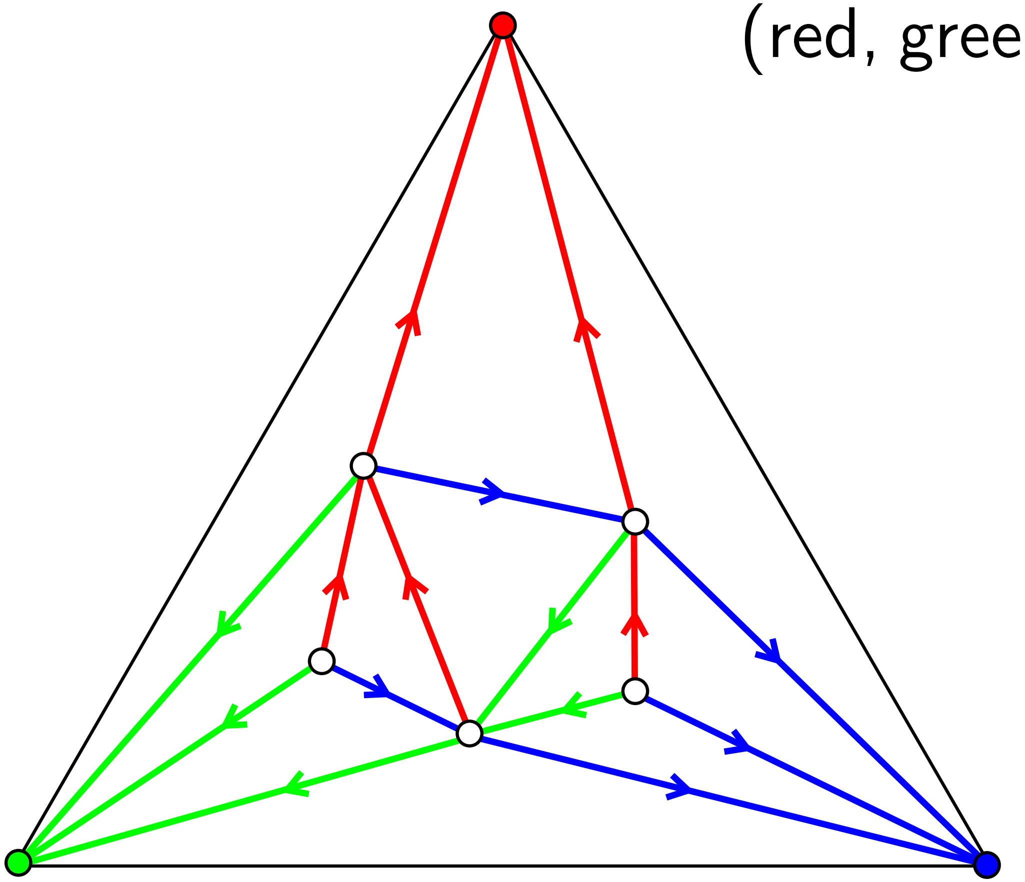
- also called **spring embedding** (each edge is a spring of energy $\ell(e)^2$)

Schnyder woods on triangulations

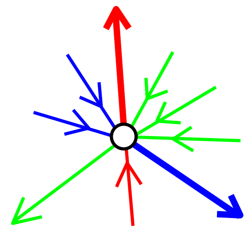
[Schnyder'89]



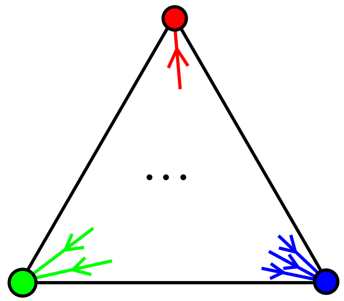
Schnyder wood = choice of a direction and color (red, green, or blue) for each inner edge, such that:

Local conditions:

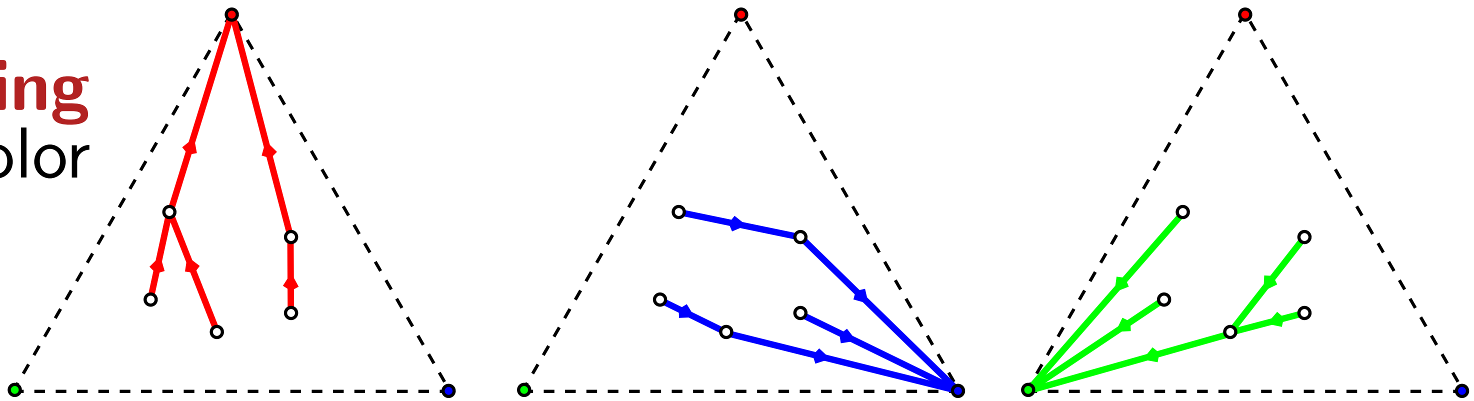
at each inner vertex



at the outer vertices



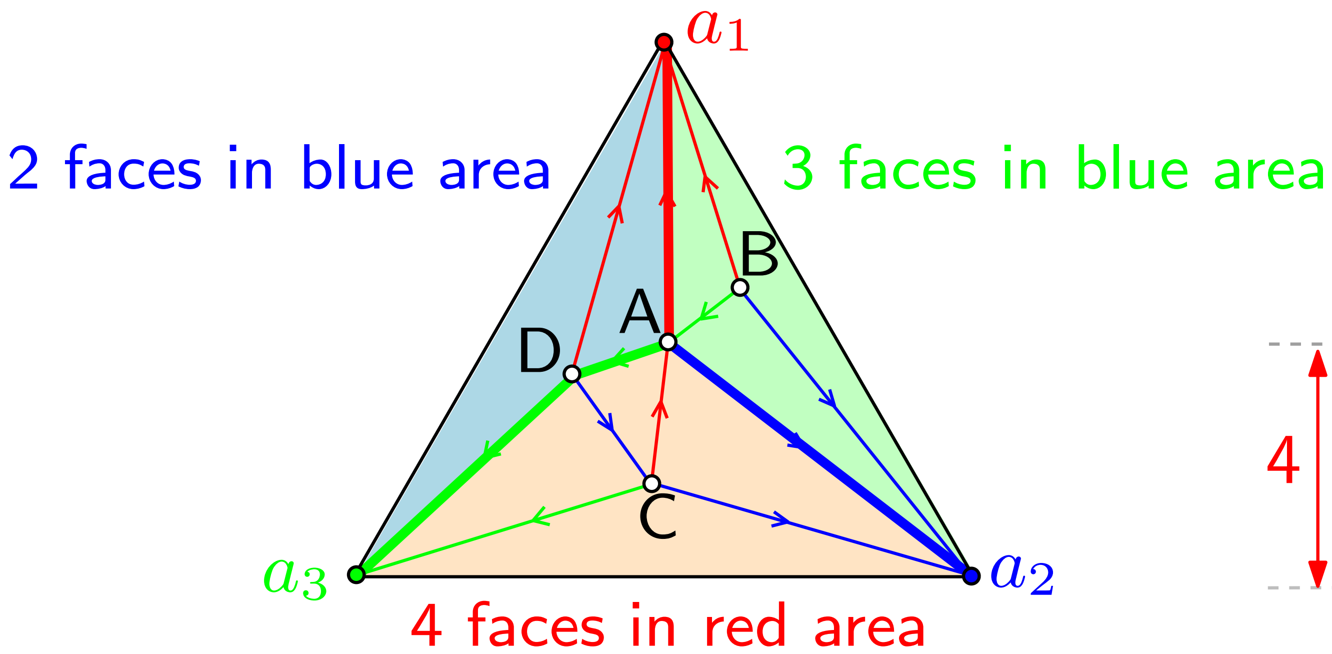
yields a **spanning tree** in each color

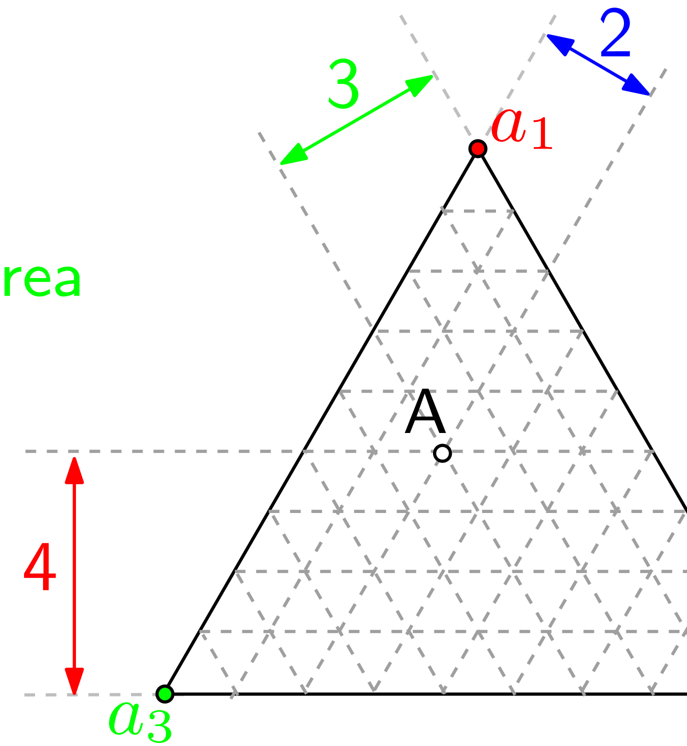


Schnyder's face-counting algorithm [Schnyder'90]

Outer vertices: equilateral triangle

Inner vertices: barycentric placement





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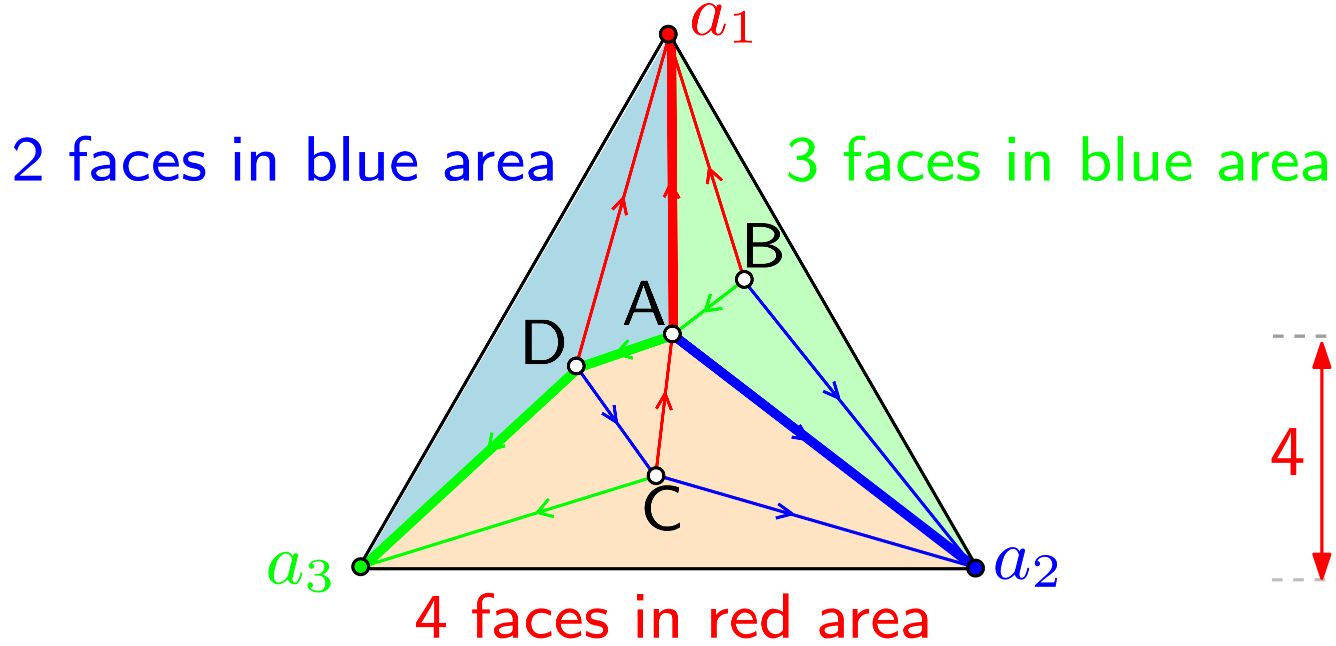

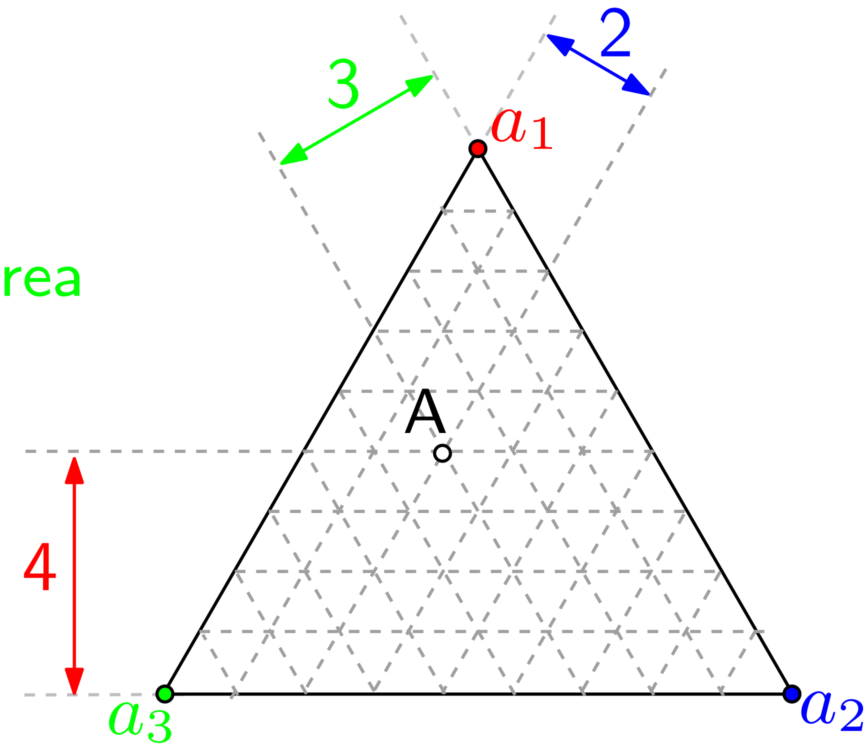

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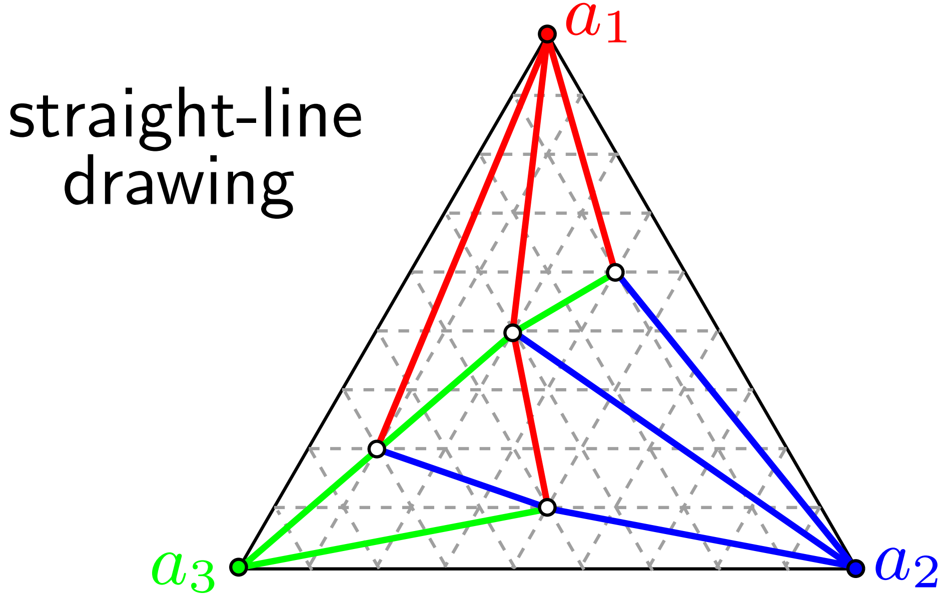

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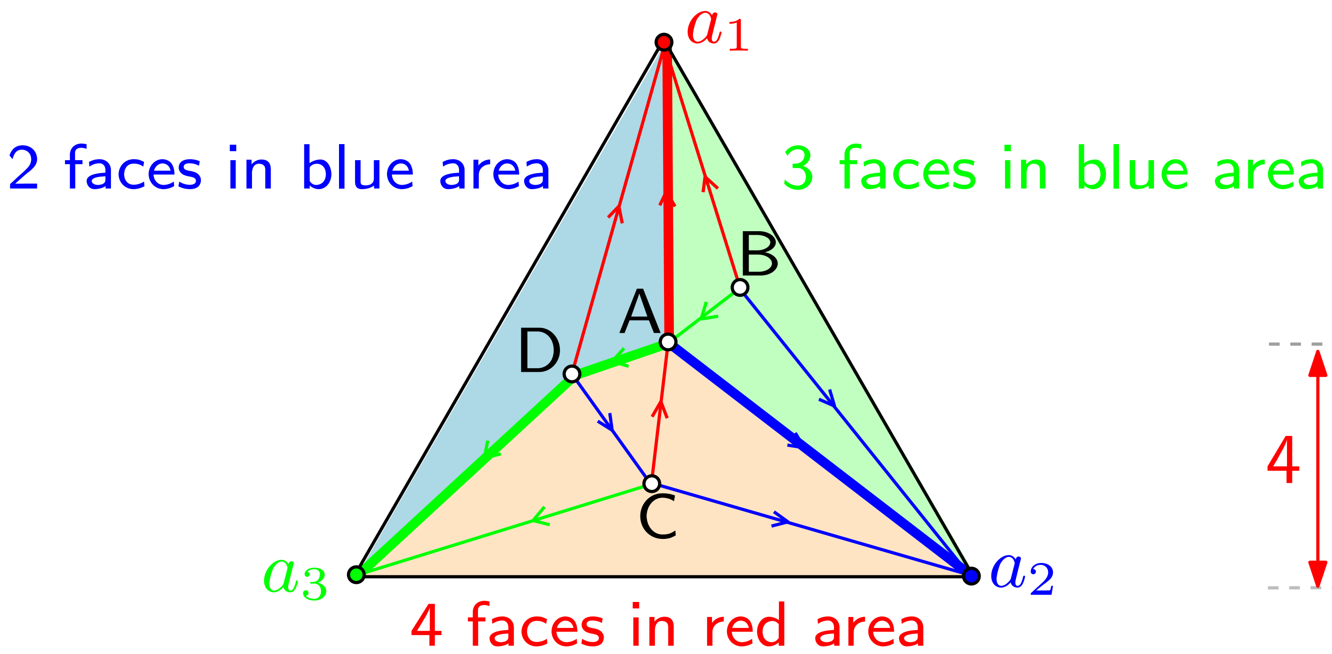
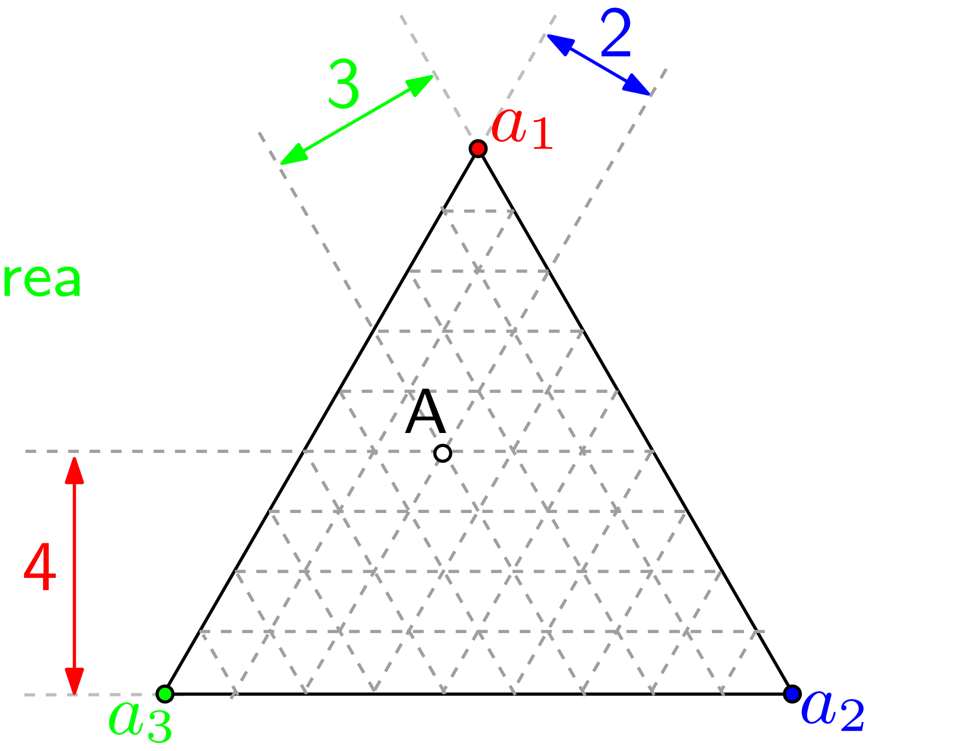
Outer vertices: equilateral triangle

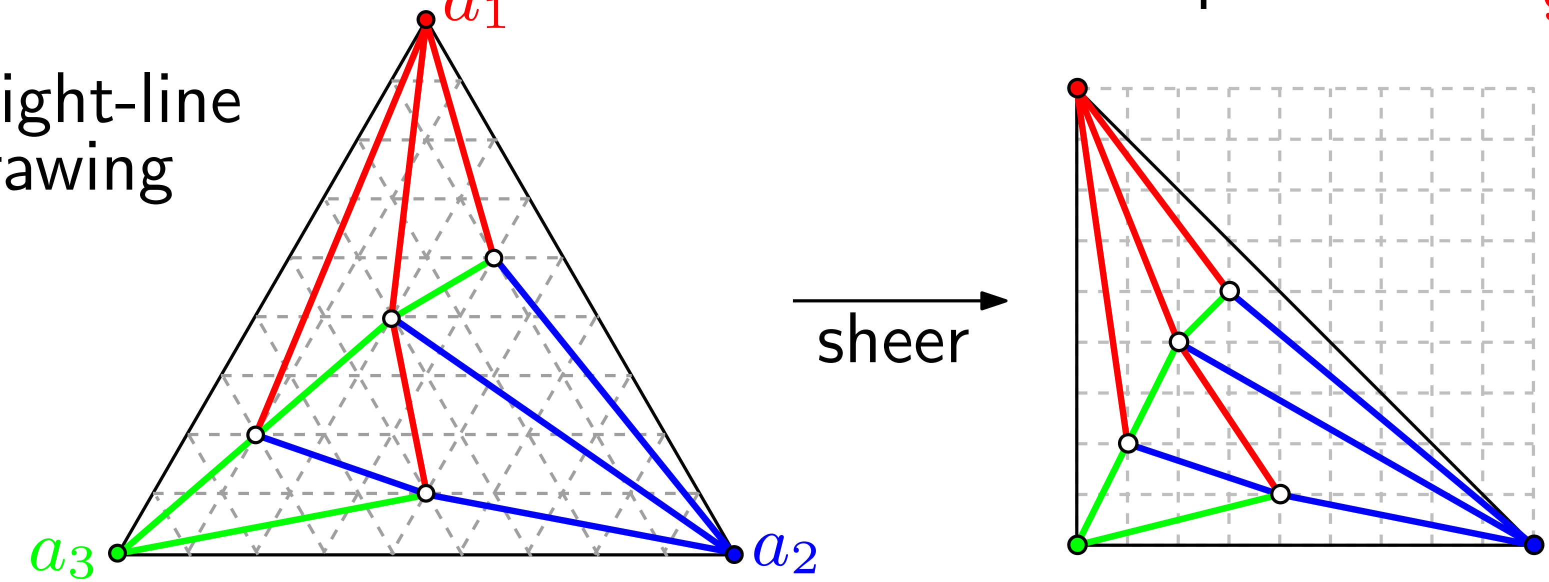
Inner vertices: barycentric placement





place A at $\frac{4}{9}a_1 + \frac{2}{9}a_2 + \frac{3}{9}a_3$

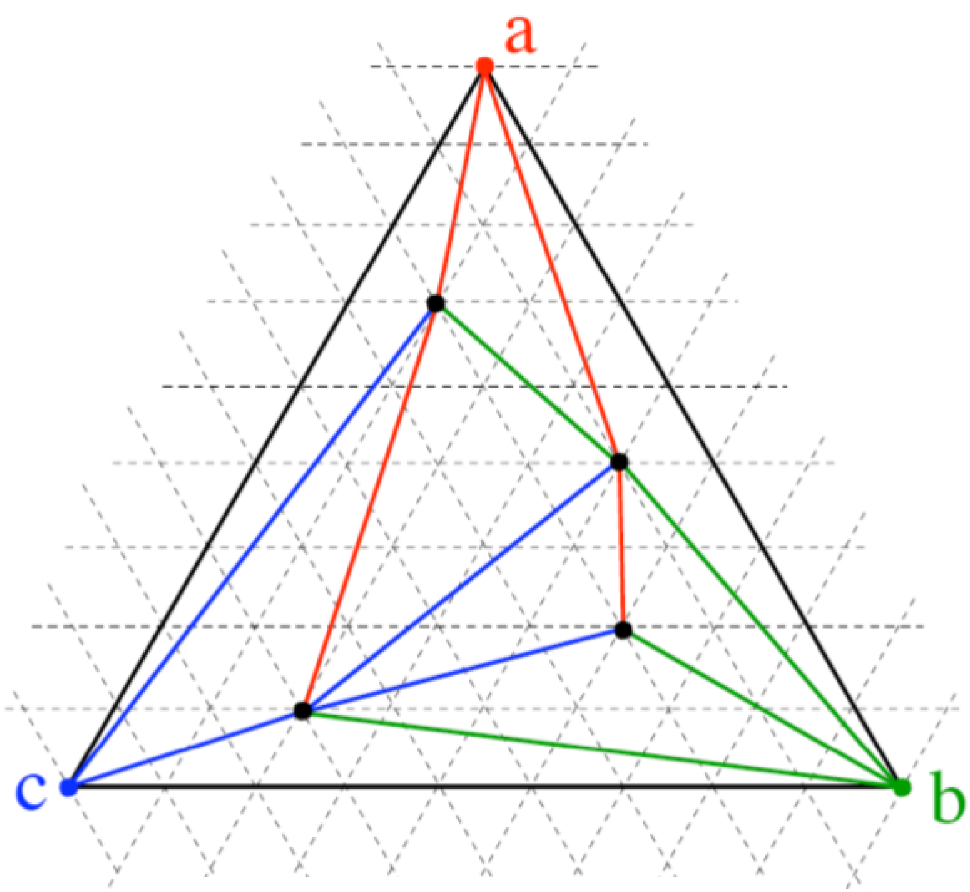
straight-line drawing



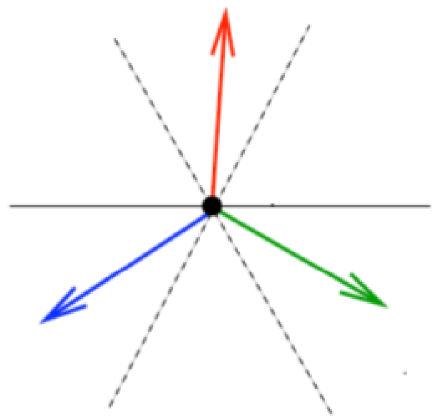
n vertices

grid $(2n-5) \times (2n-5)$

Proof of planarity



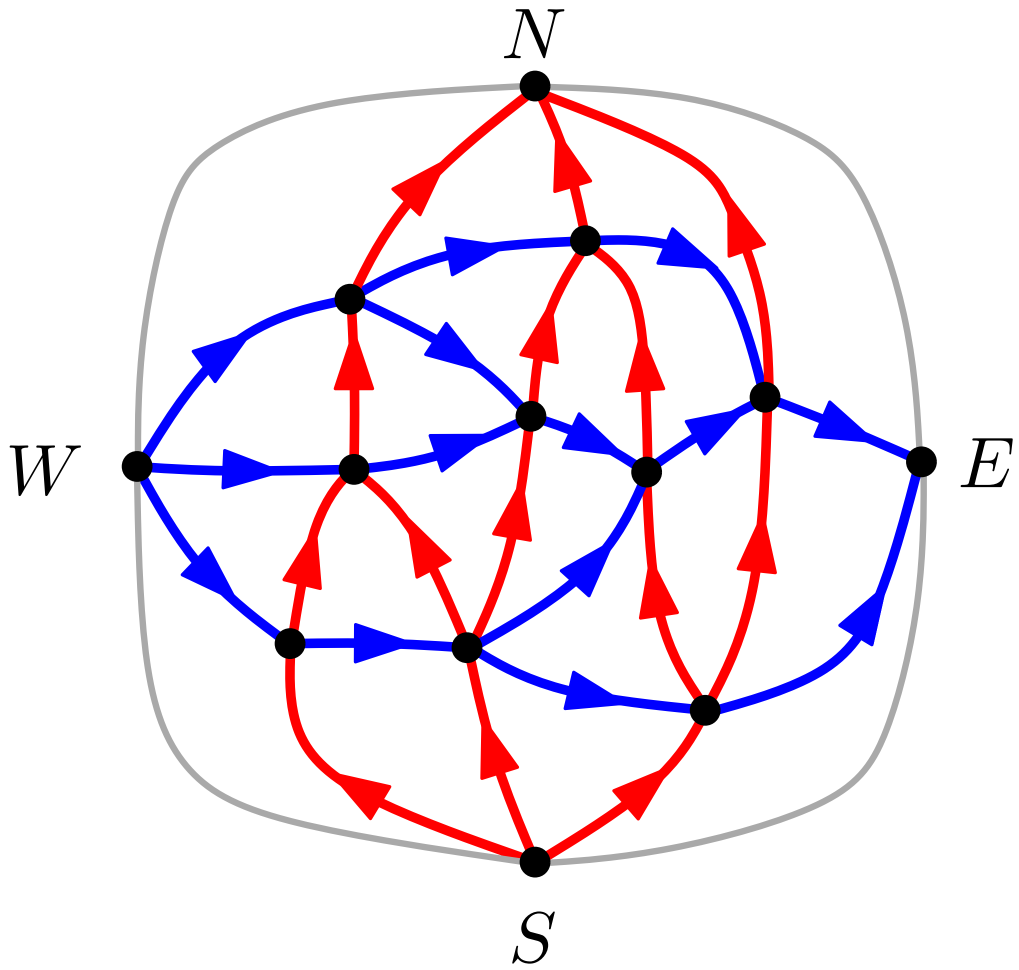
at each inner vertex:



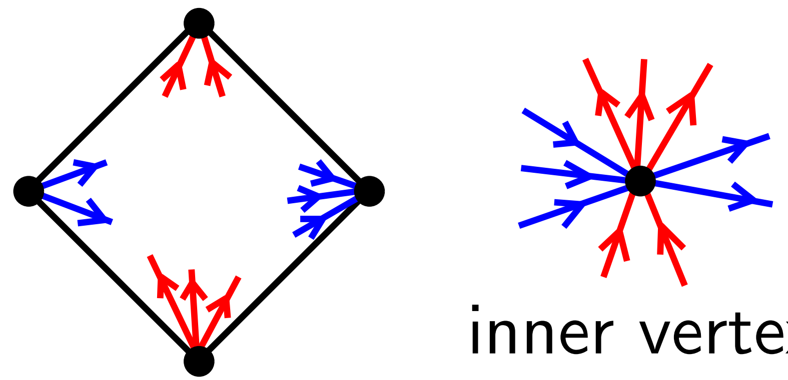
(hence inside the convex hull of neighbours)

Transversal structures

For T a triangulation of the 4-gon, a transversal structure is a partition of the inner edges into 2 transversal Hasse diagrams

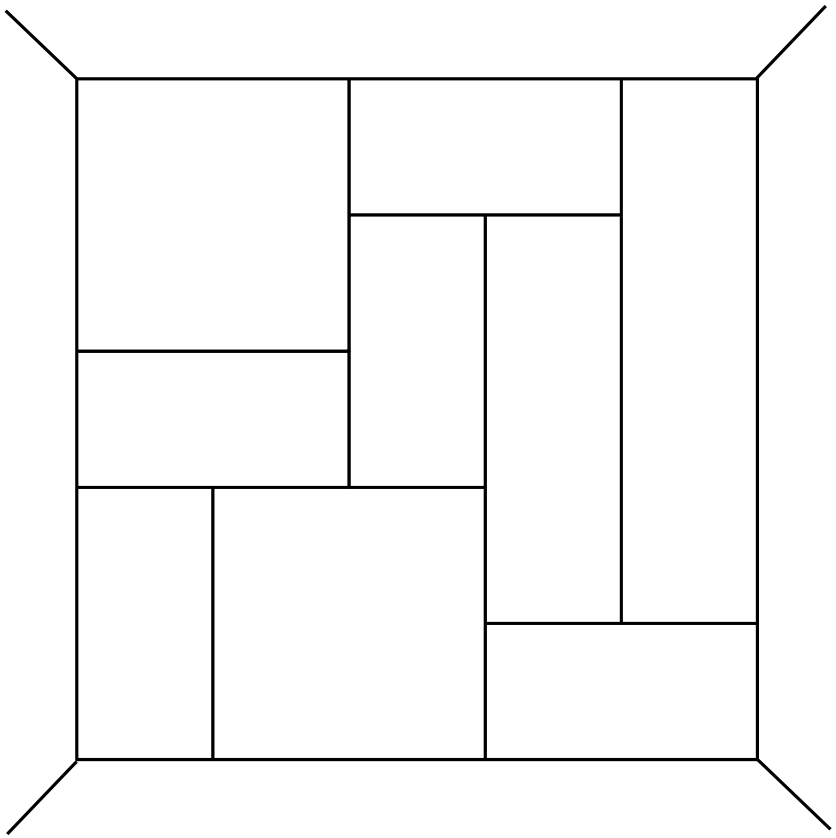


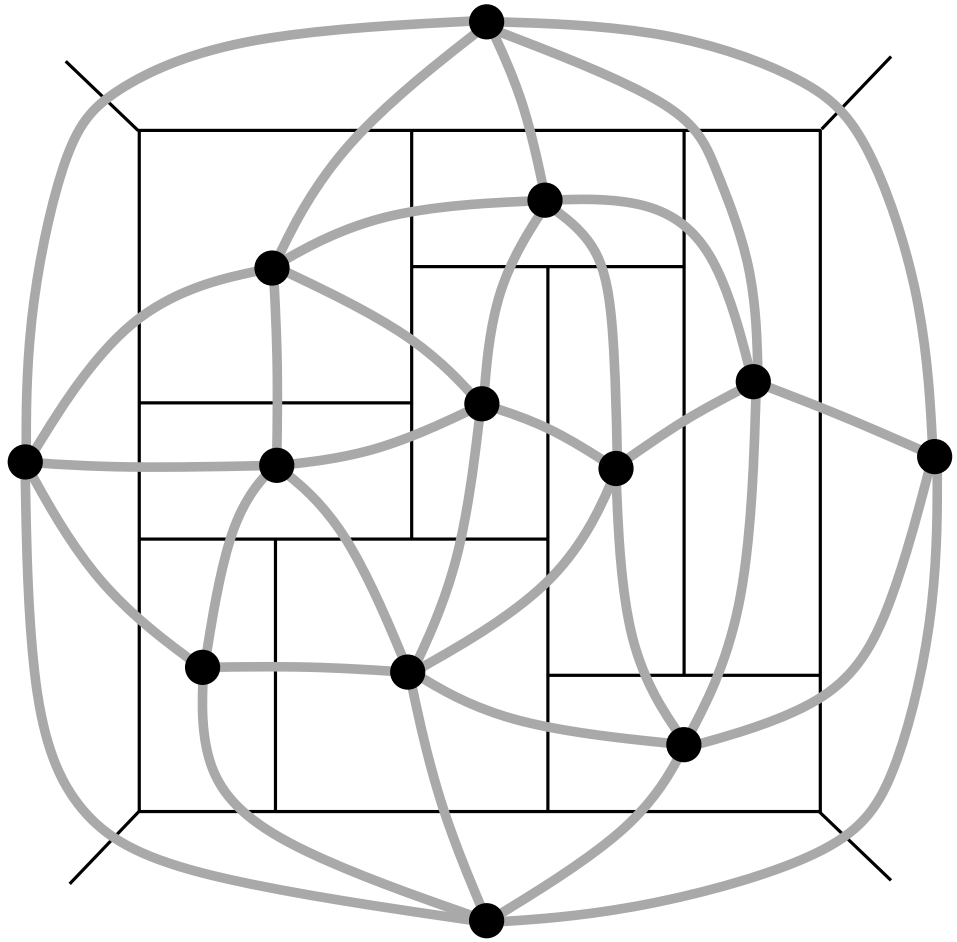
characterized by local conditions:



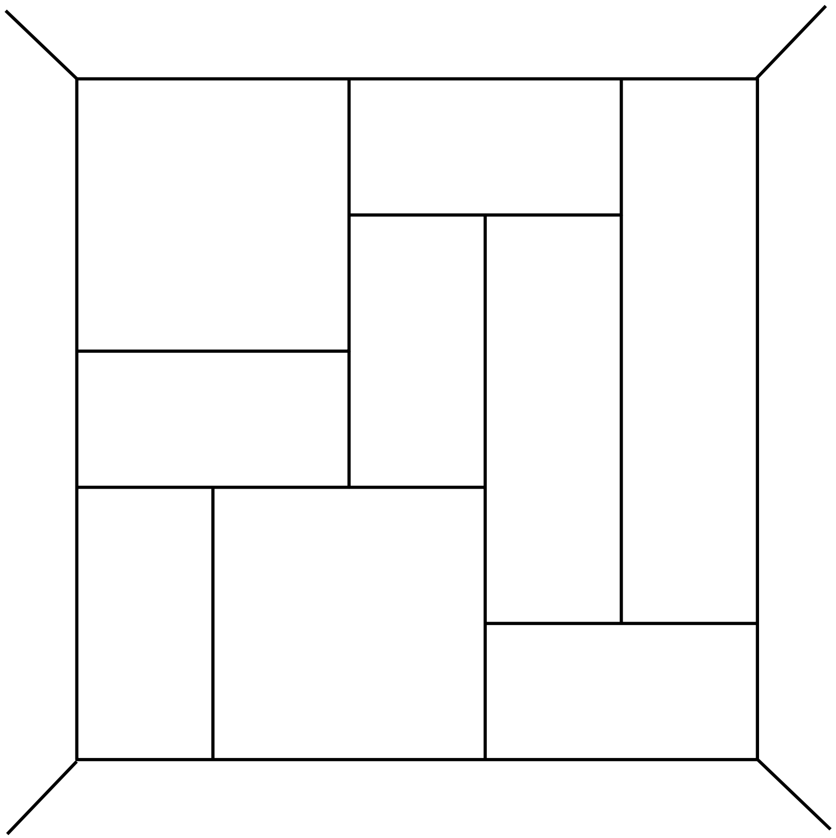
T admits a transversal structure iff every 3-cycle is facial

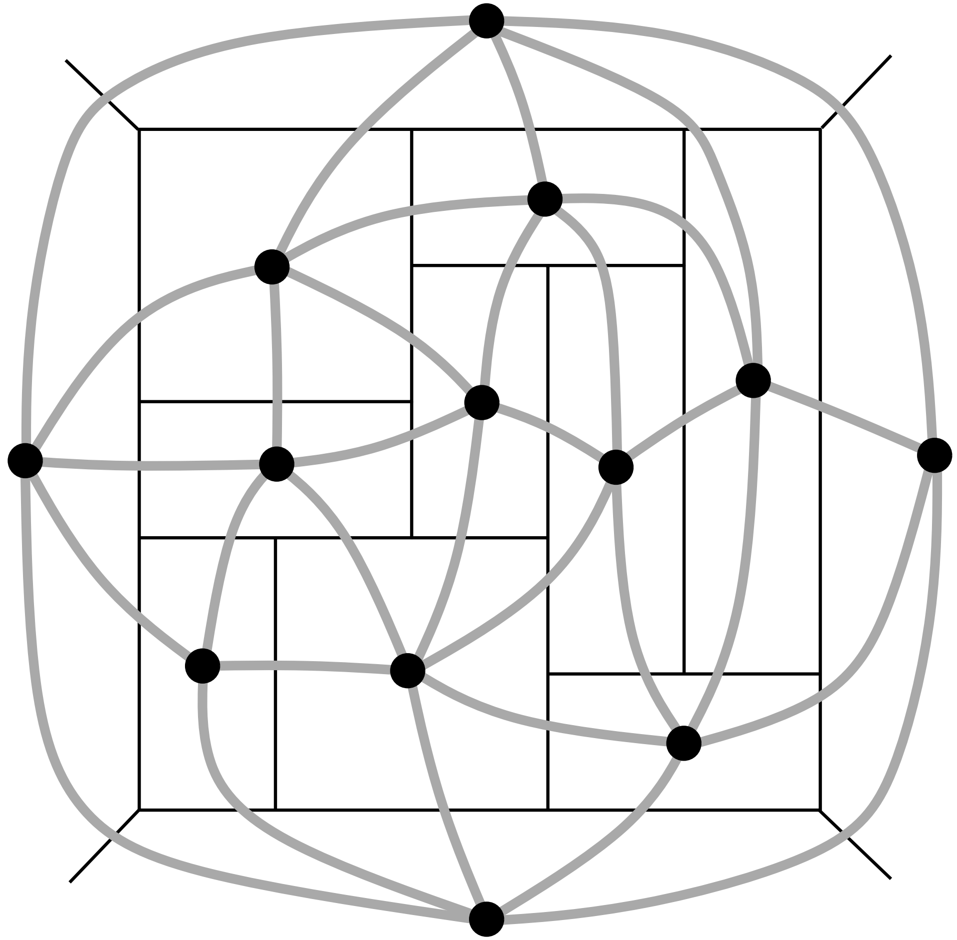
Rectangle tilings and dual triangulation





Rectangle tilings and dual triangulation



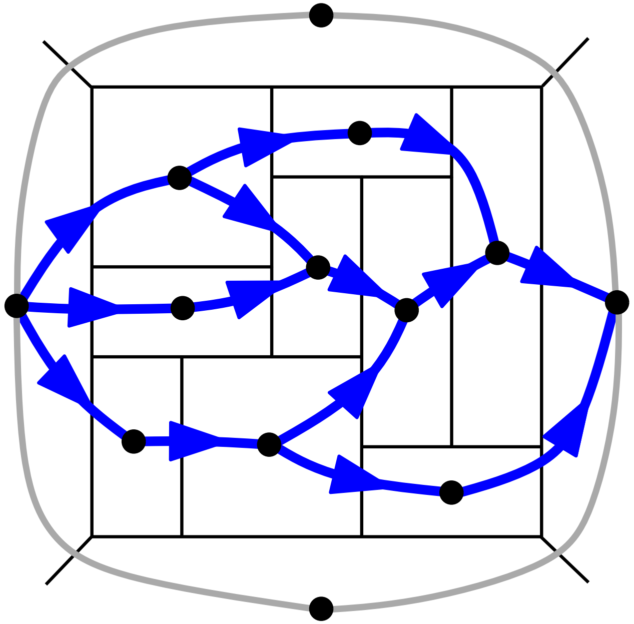


The dual map is a triangulation of the 4-gon, where every 3-cycle is facial

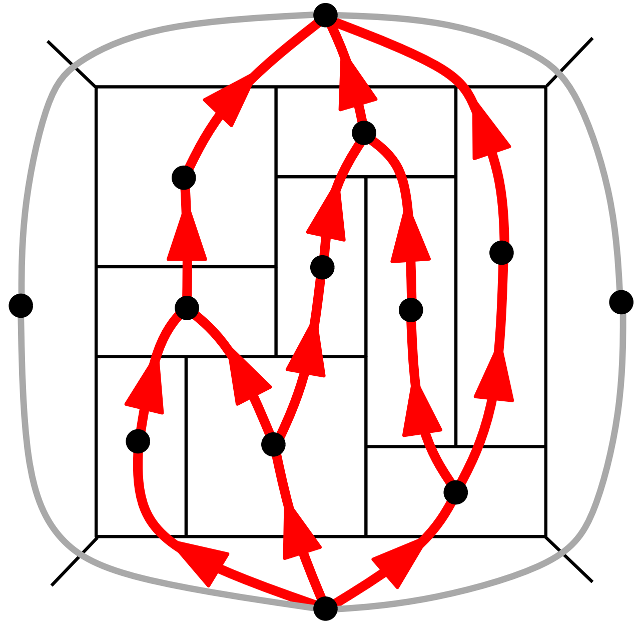
Rectangle tilings and dual triangulation

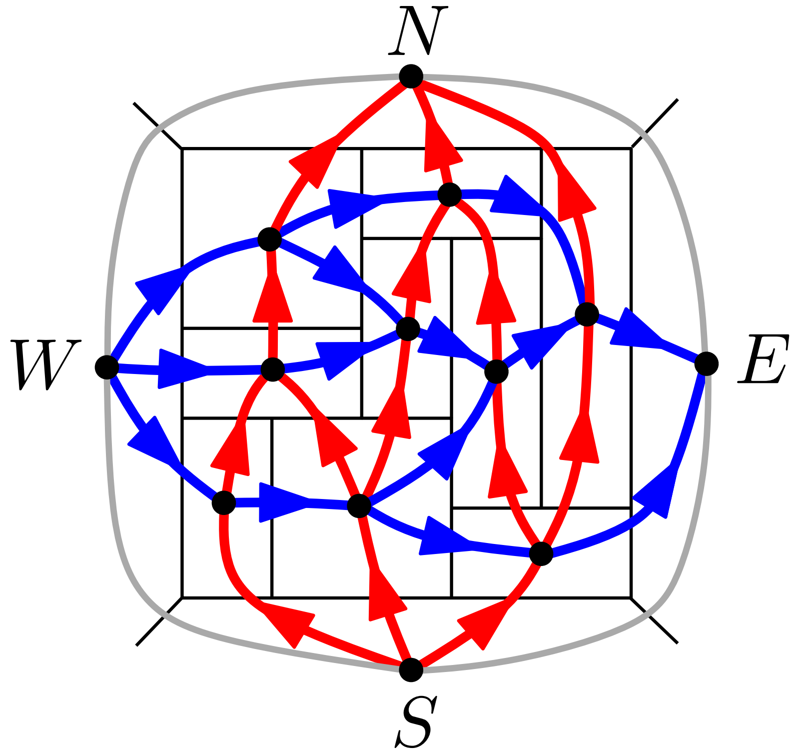
The dual is naturally endowed with a transversal structure

dual for vertical edges



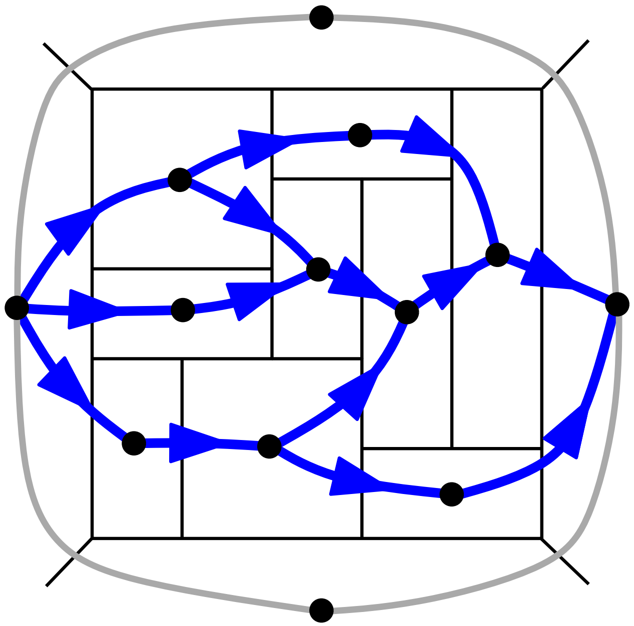
dual for horizontal edges





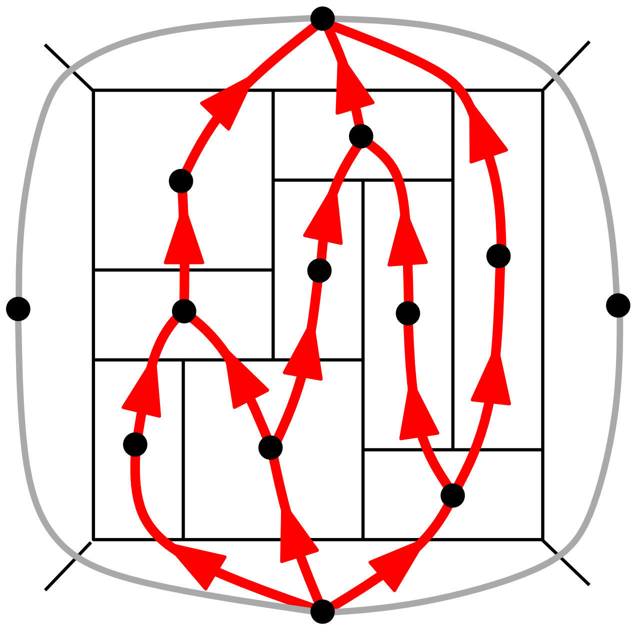
Face-labelling of the two Hasse diagrams

dual for vertical edges



a horizontal segment in each face

dual for horizontal edges



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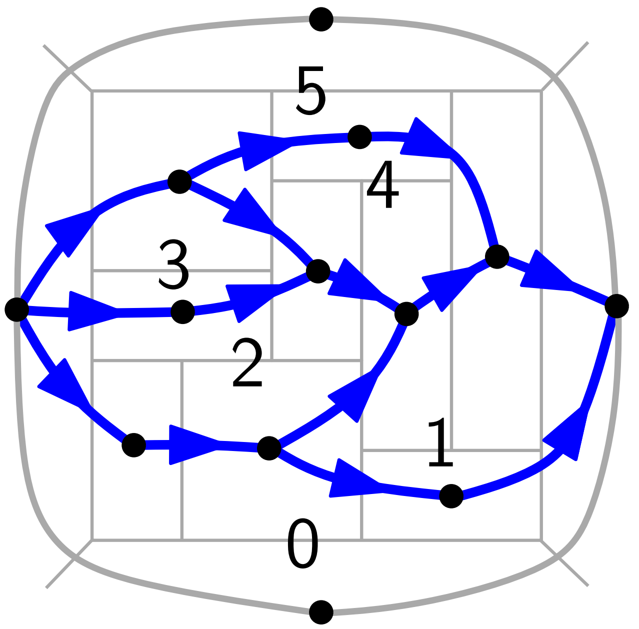
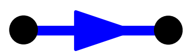


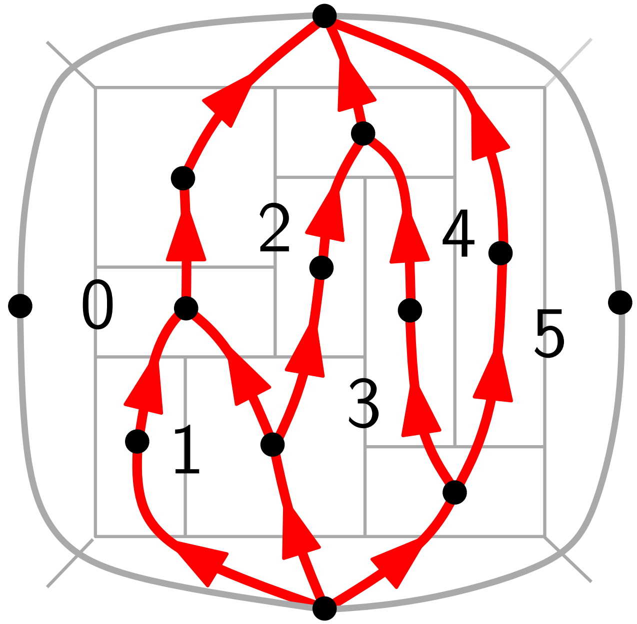
a horizontal segment in each face

label the face by the y-coordinate of segment



$j > i$

dual for horizontal edges



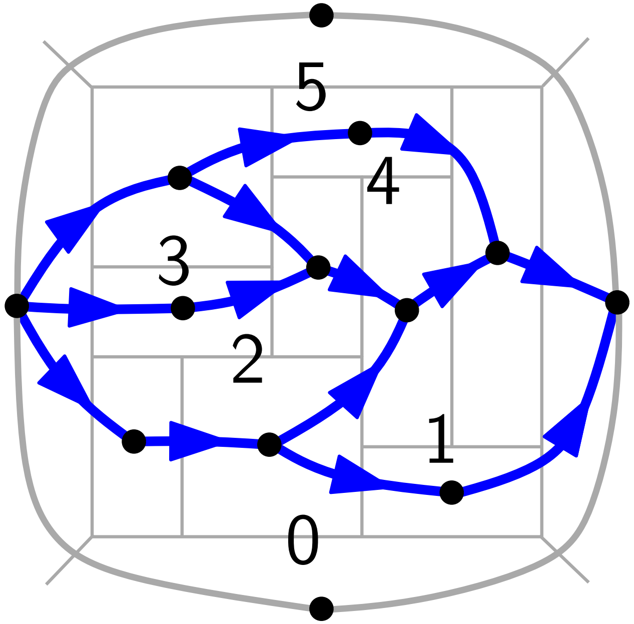
a vertical segment in each face

label the face by the x-coordinate of segment

$\ell > k$

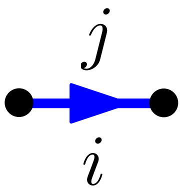
Face-labelling of the two Hasse diagrams

dual for vertical edges

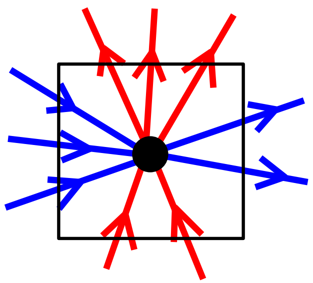


a horizontal segment in each face

label the face by the y-coordinate of segment

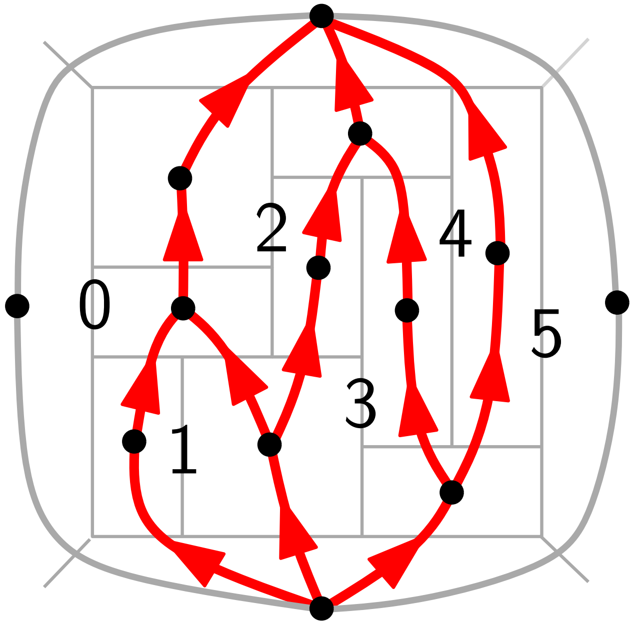


$j > i$



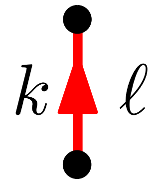
vertex $v \leftrightarrow$ rectangle $R(v)$

dual for horizontal edges



a vertical segment in each face

label the face by the x-coordinate of segment



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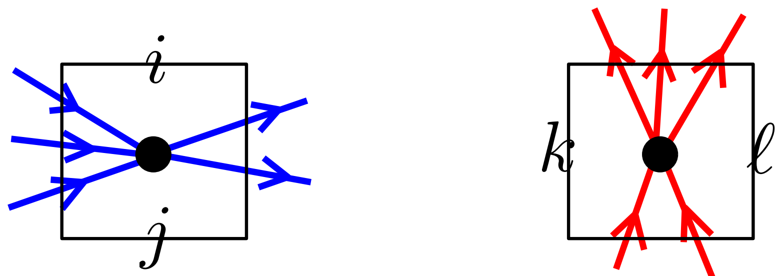
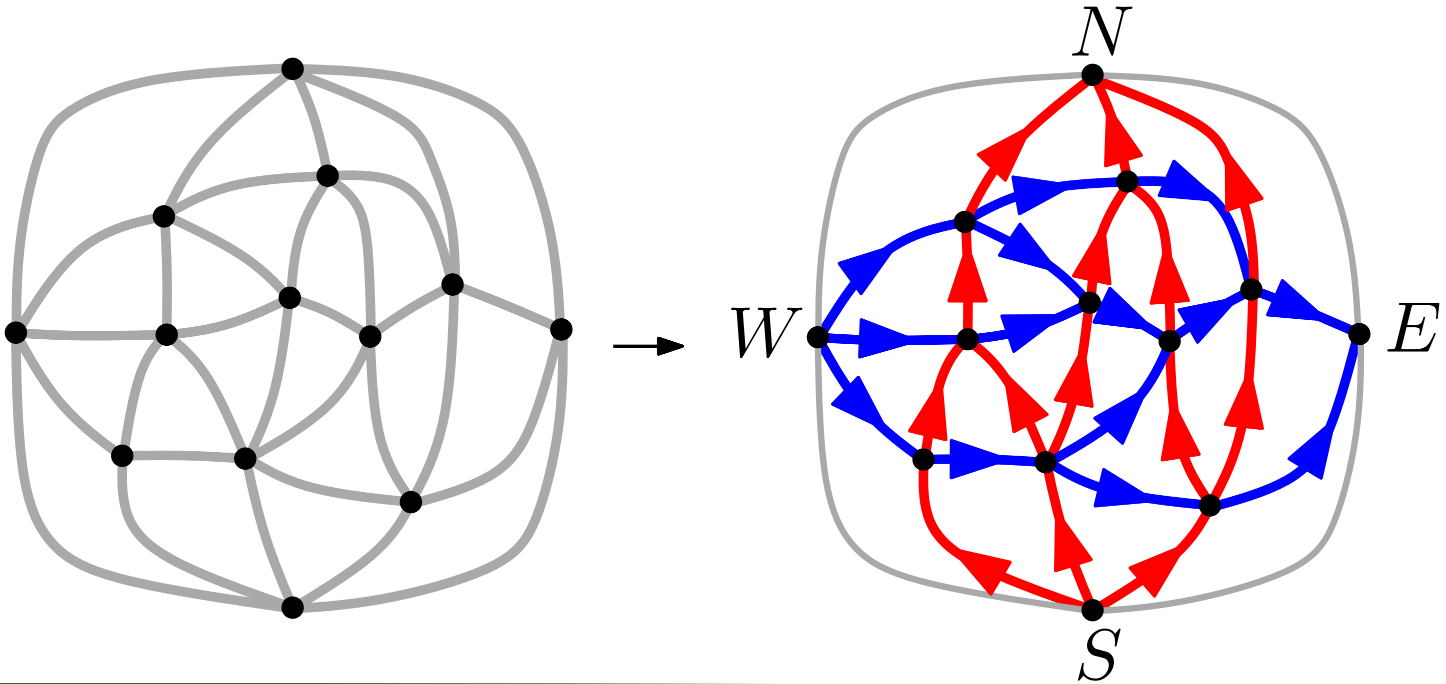


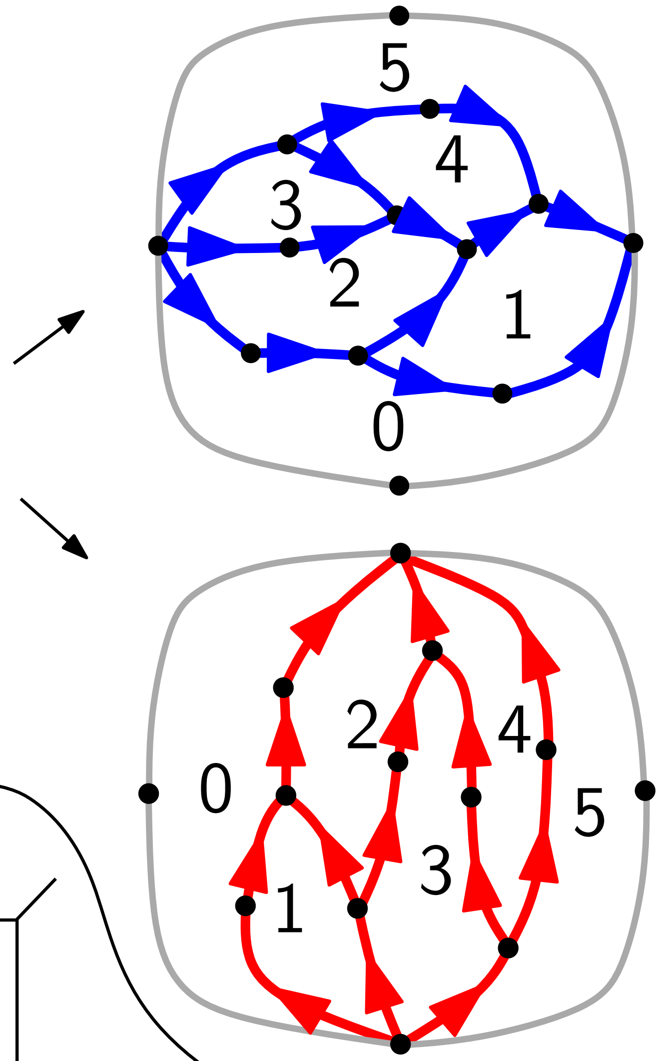
bounding x, y-coordinates given by labels

Algorithm by reverse-engineering

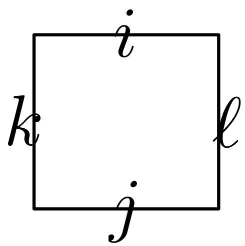
[Kant, He'92]

For T a triangulation of the 4-gon without separating 3-cycle

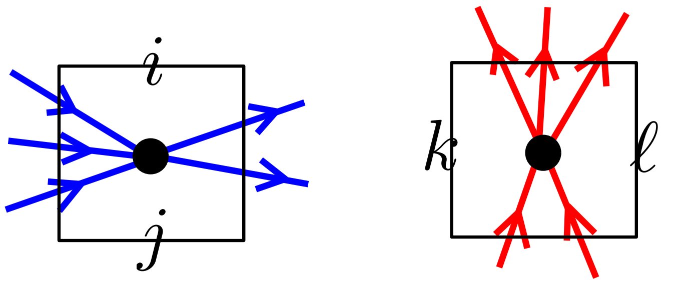


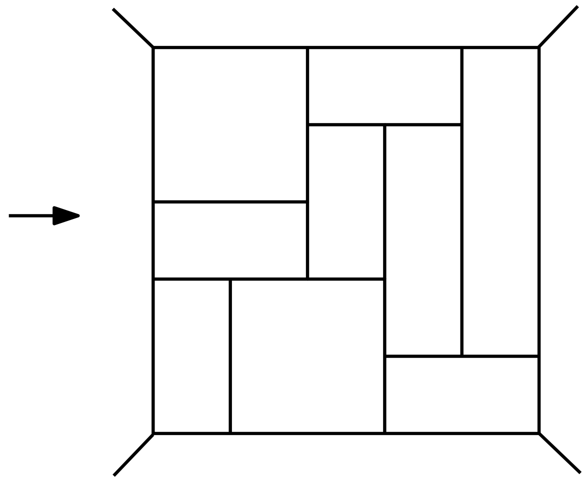


each vertex → box



where





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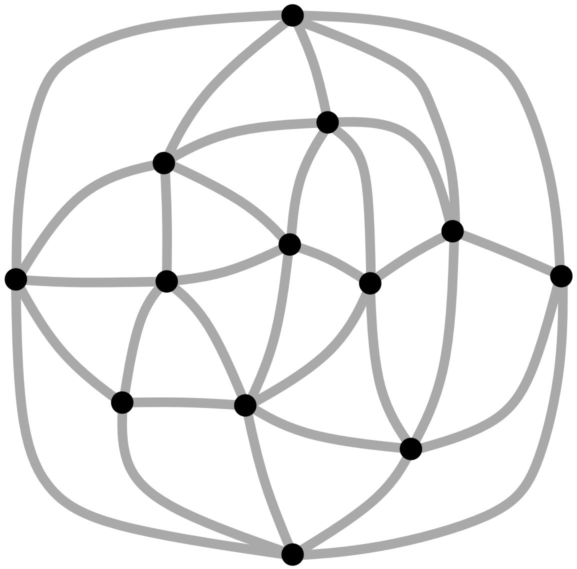

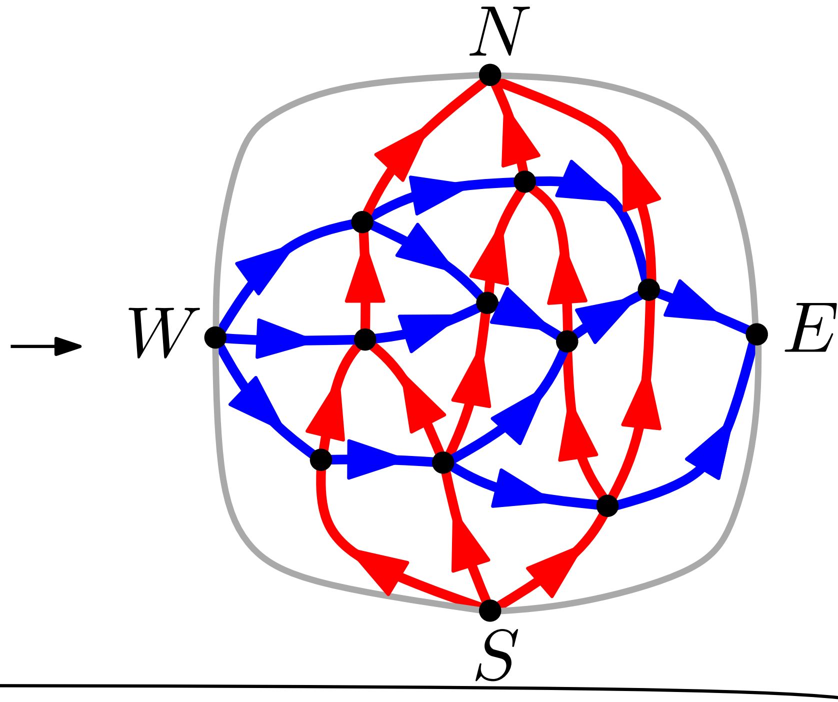

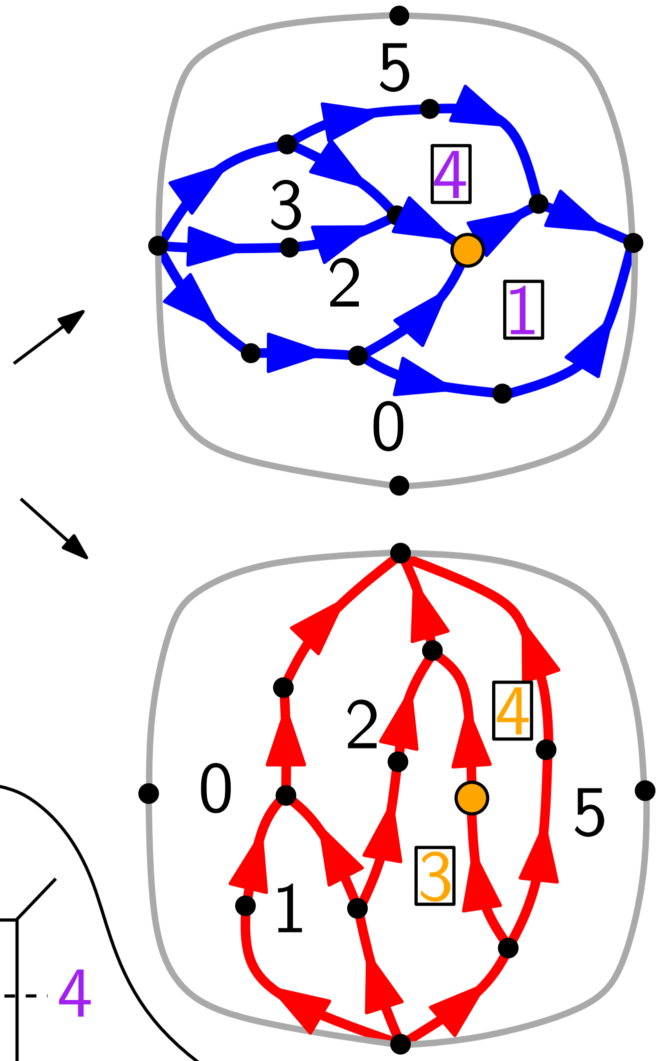

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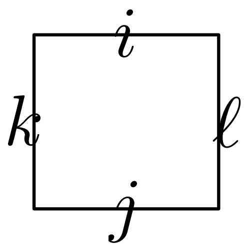
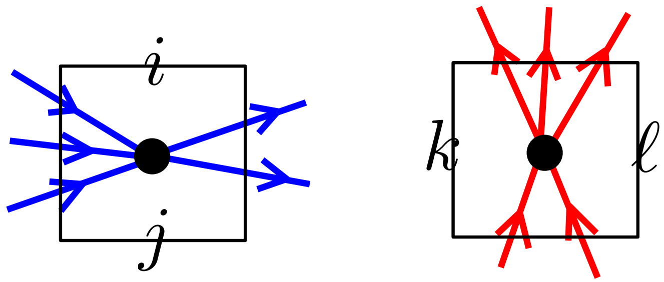
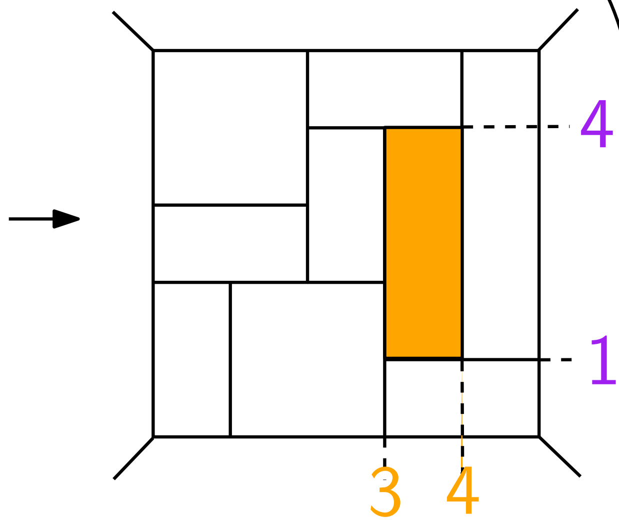


where

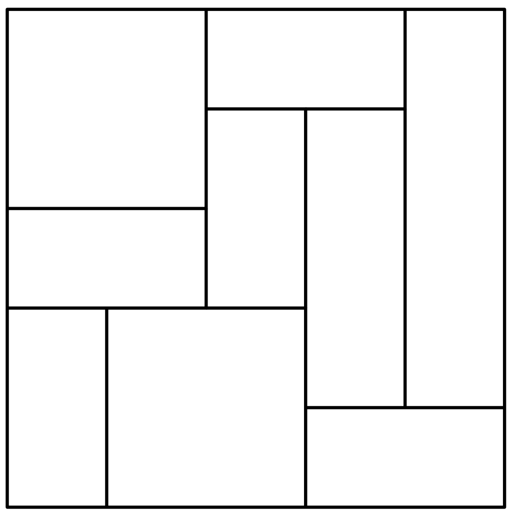


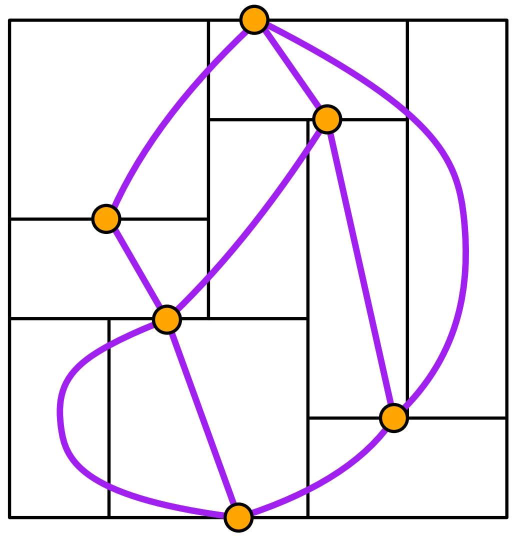


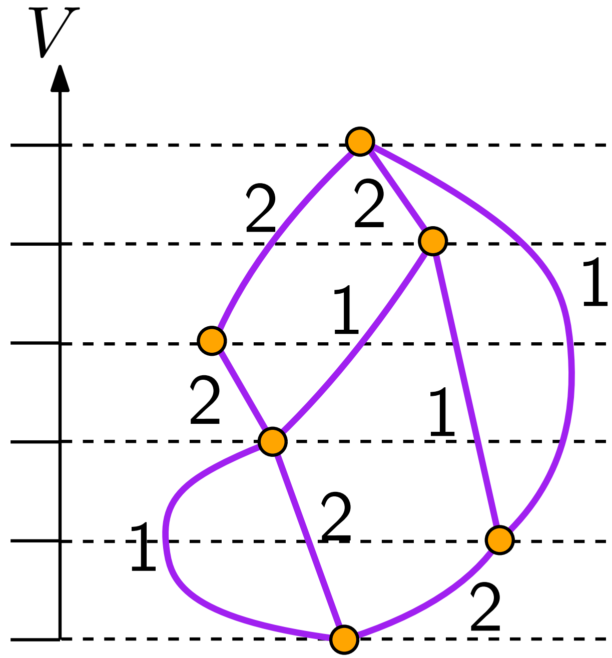
Rectangle tilings and electrical networks

other way of associating a planar map to a rectangle tiling

nice way to visualize Kirchhoff's laws







Rk: aspect ratio of a rectangle $\leftrightarrow$ resistance of corresponding link in the network

Rectangle tilings and electrical networks

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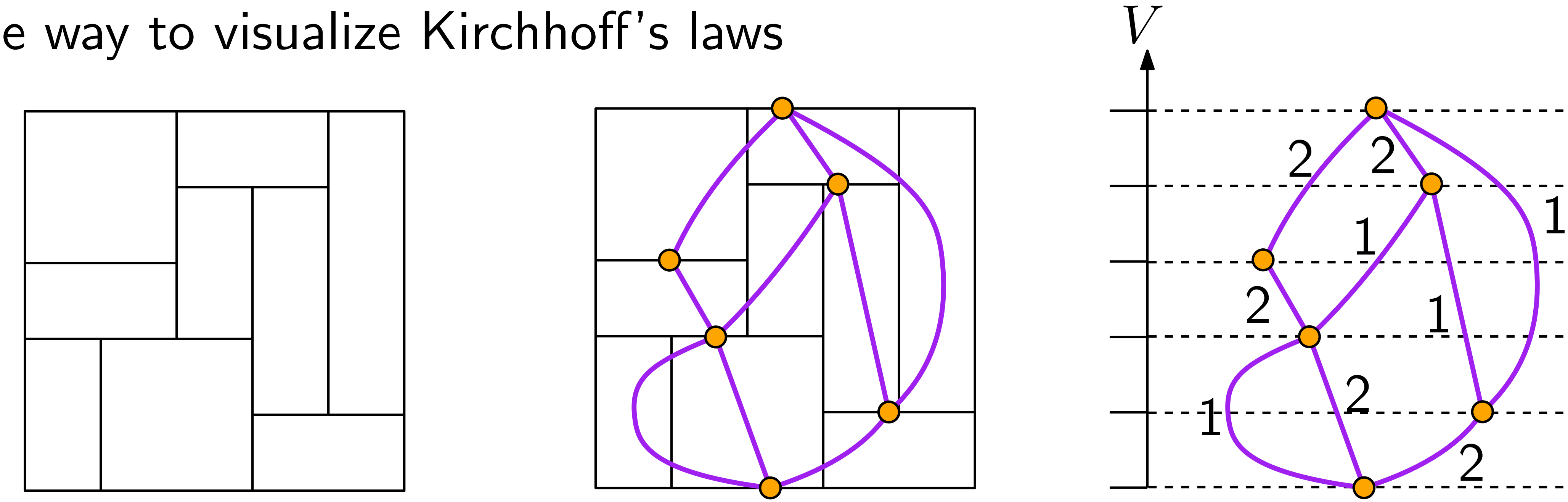
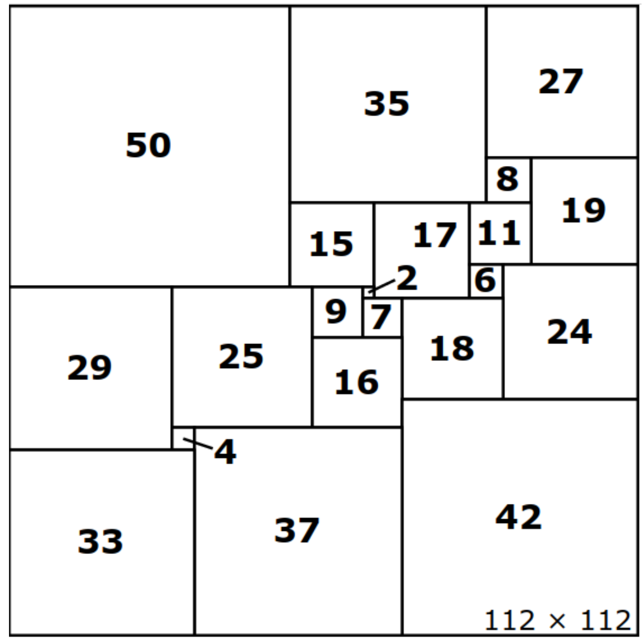


Rk: aspect ratio of a rectangle $\leftrightarrow$ resistance of corresponding link in the network

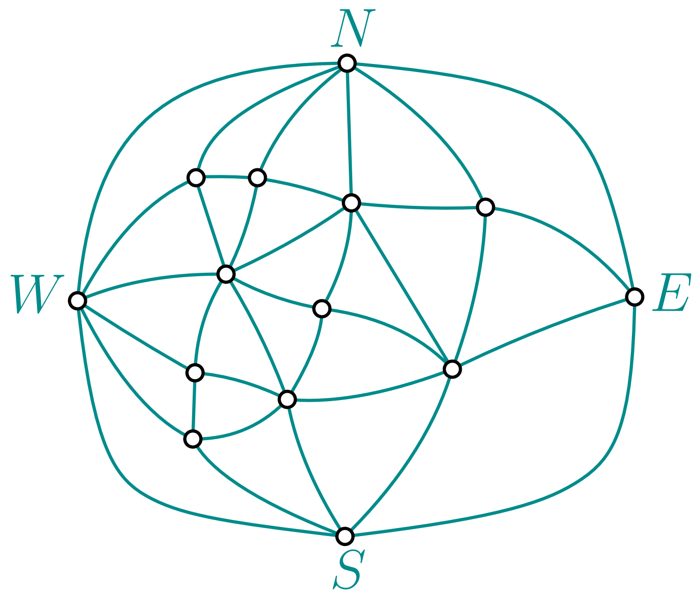
Given a network with resistances $= 1$
one gets a square tiling representation
by solving the Kirchhoff's laws

cf 'squaring the square'



Square tilings dual to triangulations [Schramm'93]

Question: Given T a triangulation of the 4-gon, does there always exist a square tiling whose dual is T?



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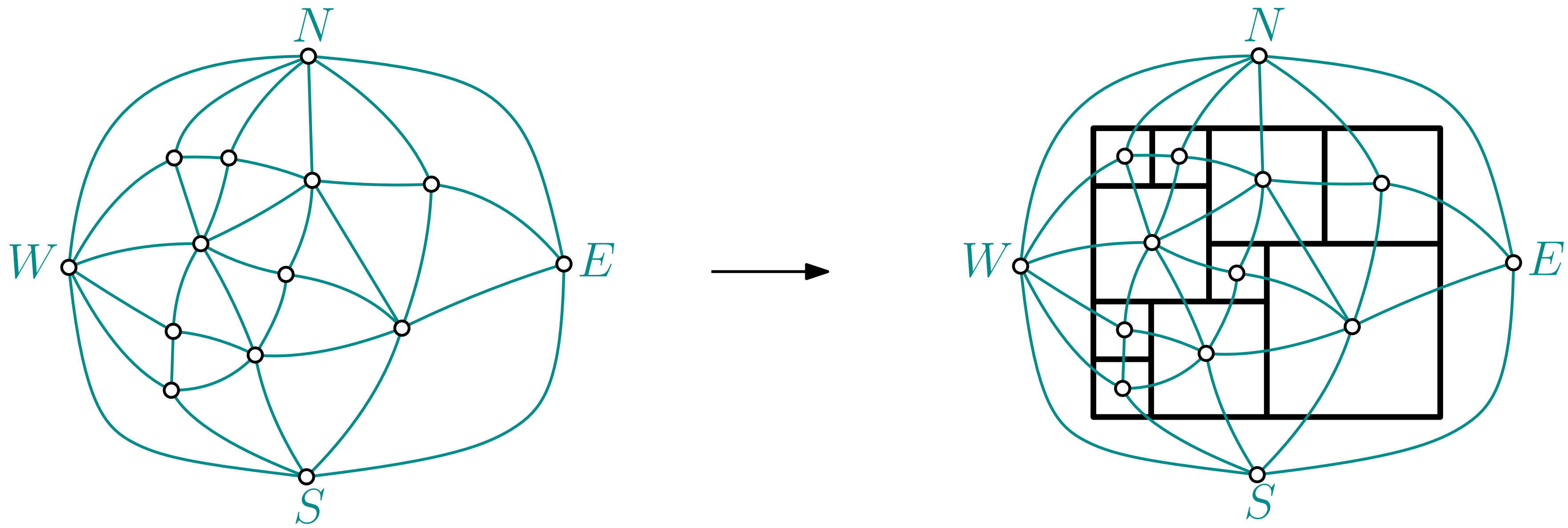


Yes ! up to allowing for degeneracies (empty squares)

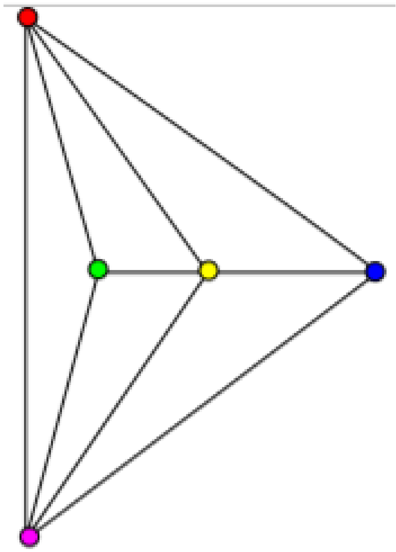
solution via computing the 'optimal metric' of T

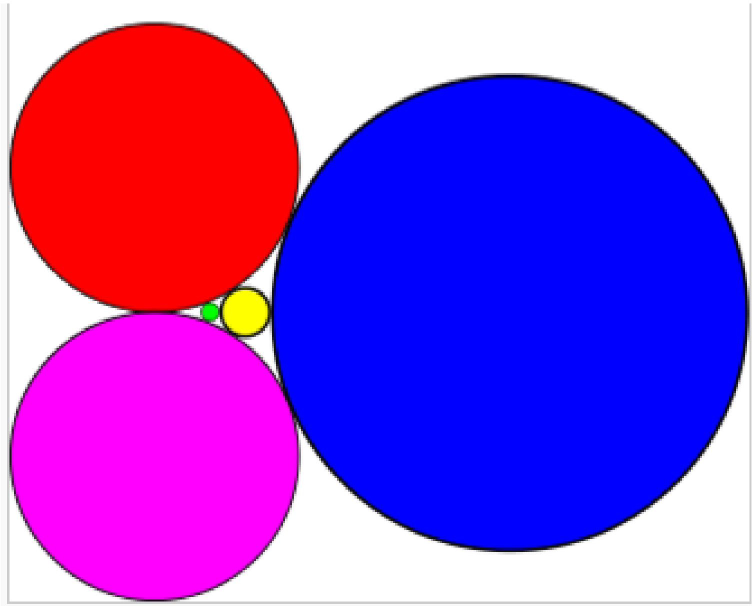
(no known algorithm by solving linear equation systems)

Circle packing

[Koebe'36, Andreev'70, Thurston'85]: every planar triangulation admits a contact representation by disks

The representation is unique if the 3 outer disks have prescribed radius





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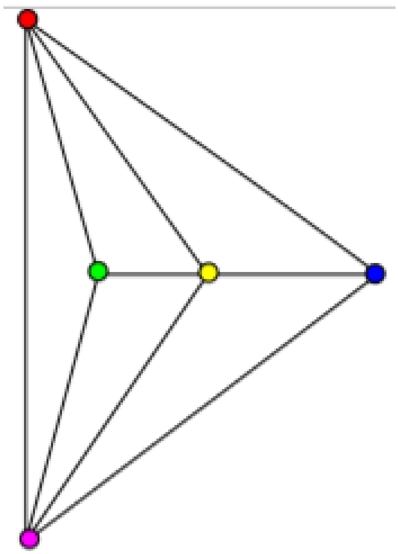
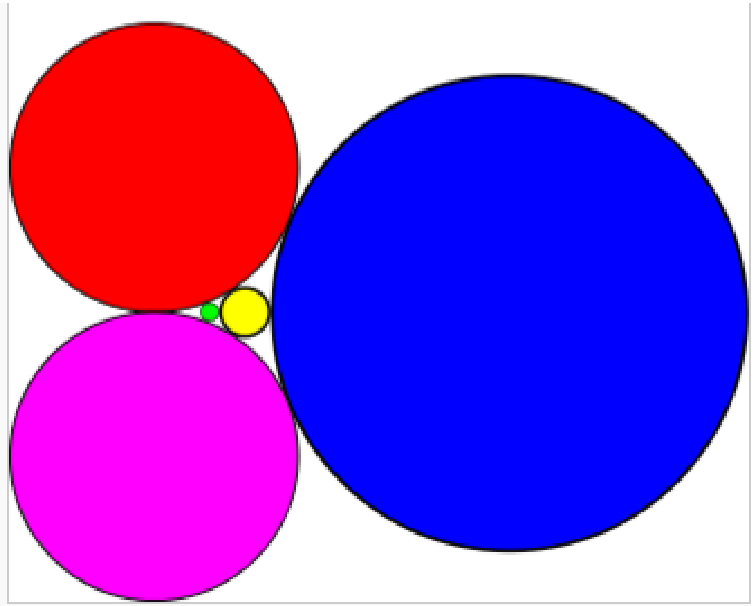
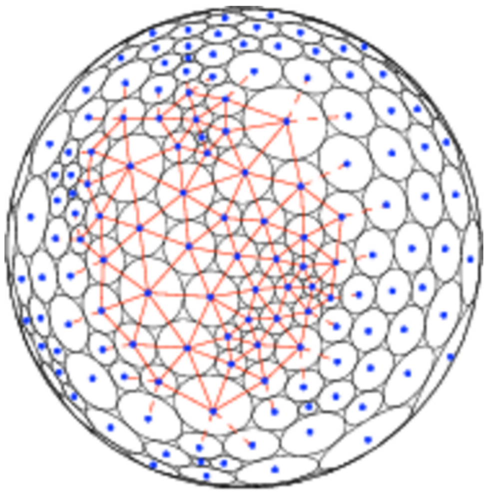




Exercise: the stereographic projection maps circles to circles (considering lines as circle of radius $+\infty$).



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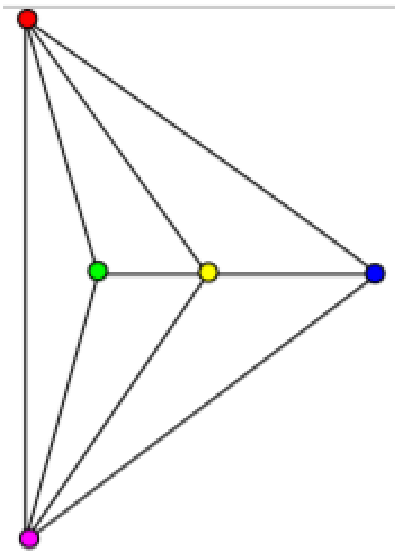
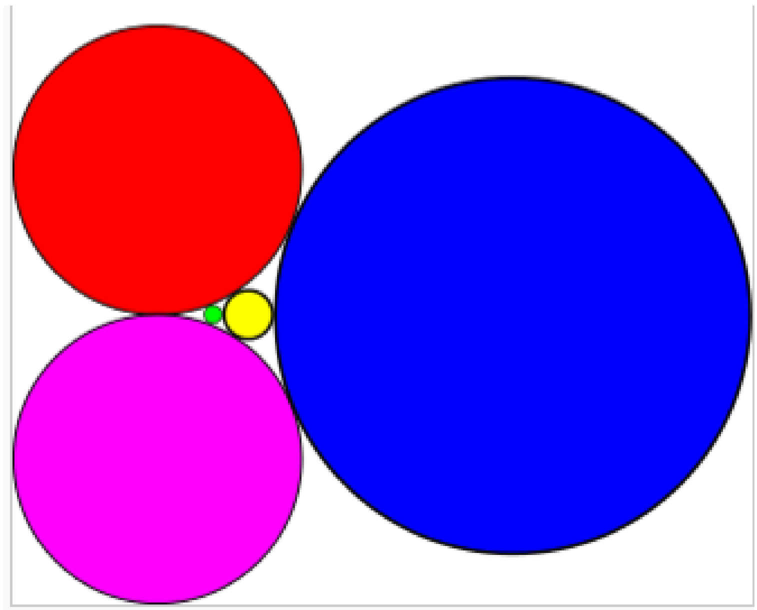
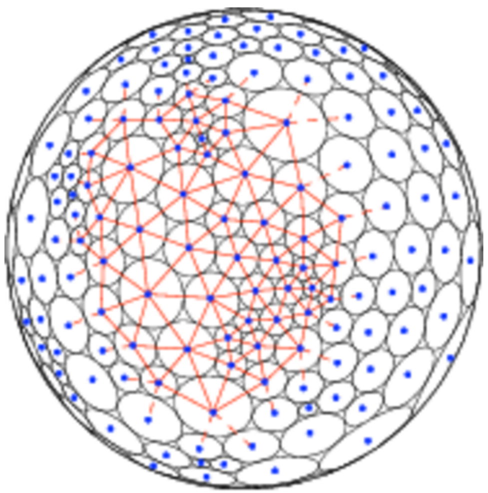




Exercise: the stereographic projection maps circles to circles (considering lines as circle of radius $+\infty$).

Hence one can lift to a circle packing on the sphere



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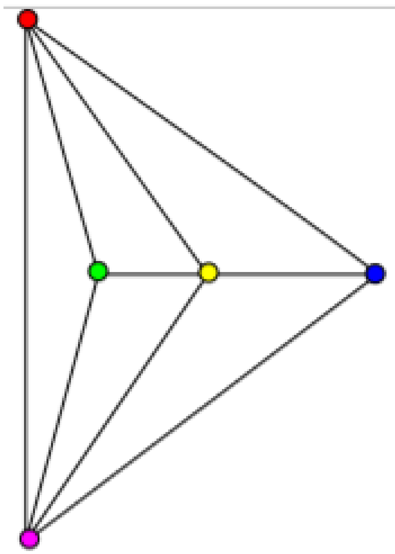

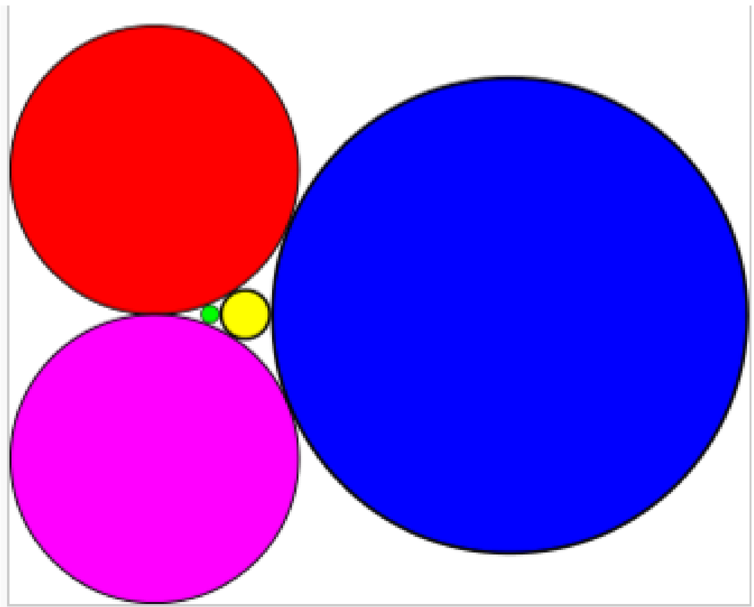

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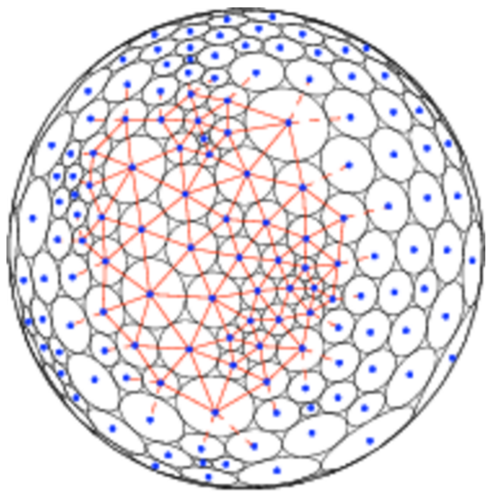
Hence one can lift to a circle packing on the sphere

There is a unique representation where the centre of the sphere is the barycenter of the contact points



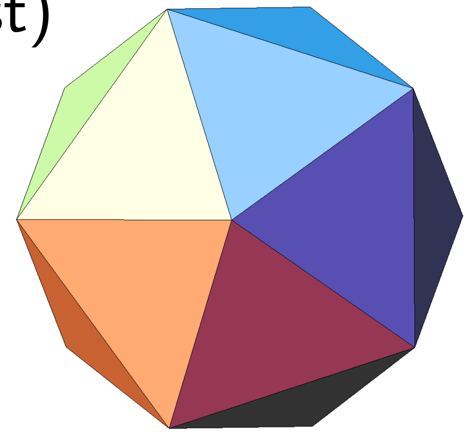
Contact representations with prescribed shapes

Generalized statement: **[Schramm's PHD 1990]**

for any triangulation T and a prescribed convex shape for each vertex
there exists a contact representation of T

(possibility of degeneracies if shapes are not smooth)

Example (Eppstein's blog post)



isocahedron