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Parametric Root Finding to support discovering geometric inequalities in GeoGebra

(ADG 2021, Hagenberg, September 16)

Inequality Exploration in planar Euclidean geom.

Create a triangle construction in a DGS and assume that a USER want to explore an INEQUALITY, that is, all the possible ratio between

$m = g_1/g_2$, where g_j can be perimeter, area, circumradius, sum of the medians, etc.,
for all nondegenerate triangles from a certain class.
We want to support this symbolically! Previous talk:
algebraically as 1st order real quantifier elimination problem (RQE).
We use tarski's (Qepcad) and Mathematica's RQE implementations

Observations

Because the algebraization in GGBA is coordinate-based, we have several variables in the semialgebraic representations.

When the #vars is more than 6, the RQE problem cannot be solved often in a reasonable time (~5sec)

However, the semialgebraic system for certain classes has only FINITELY MANY SOLUTIONS for a fixed *m*, if *wlog we fix a triangle side*.

For instance, for Isosceles Triangle (IT) or the Right Triangle (RT) classes.⟹

Maybe other methods, approaches can help here to avoid general (full dimensional) RQE and to reduce practical computational time.

Observations [2]

In[ ]:=
```
{Resolve[Exists[{a, b, c},
    a + b > c ∧ a + c > b ∧ b + c > a ∧ c == b ∧ (a^2 + b^2 + c^2) == m (a b + b c + c a)], Reals],
  Resolve[Exists[{b}, 2 b > 1 ∧ (1 + 2 b^2) == m (b^2 + 2 b)], Reals]}
```

Out[ ]= $\{1 \le m < 2, 1 \le m < 2\}$

In[ ]:=
```
(a^2 + b^2 + c^2) / (a b + b c + c a) /.
  {{a → 2, b → 4, c → 4}, {a → 1, b → 2, c → 2}, {a → 1/2, b → 1, c → 1}}
```

Out[ ]= $\left\{\frac{9}{8}, \frac{9}{8}, \frac{9}{8}\right\}$

In[ ]:=
```
Solve[(1 + 2 b^2) == m (b^2 + 2 b) /. m -> 9/8]
```

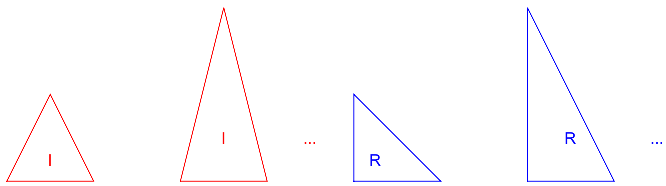
Out[ ]= $\left\{\left\{b \to \frac{4}{7}\right\}, \{b \to 2\}\right\}$

Description of the problem

From elementary planar Euclidean geometry:

Consider Inequality Exploration problems from the class of nondegenerate isosceles triangles or the class of right triangles

Out[ ]=



Description of the problem [2]

From the algebraic/logical point of view: The EXPLORATION PROBLEM for IT/RT is

not a decision problem, not a SAT/UNSAT problem, but it is very close to that,
one free variable *m* and *n* existentially bound variables,
the nonlinear real algebraic model has Hilbert dimension 1.

We can reduce the RQE problem to finitely many SAT problems, in fact to real root counting (RRC).

Description of the (new) PRF method

Detect via Gröbner basis computations the “wrong/critical” points of the m-parameter space (where the #sol of the real SAS may change): $O_{\text{crit}} \cup O_{\text{in}} \cup O_{\text{inf}}$ (Computation of MDV via reduction to Elimination)

Decompose the m-space into finitely many cells, generate sample for open cells

Solve the Real Root Counting// Real SAT problem and generate a qfree formula based on this.
Ref.: [Lazard 2007][Moroz 2006, 2011], [Xia, Hou 2002]

Description of the (new) PRF method [2]

Existing Implementation : [Maple PRF Package, Maple RegChains]
(not only for 1pm)

Problems/difficulties:
disjunctions, non-strict inequalities, well-behaved systems, orderings.

In fact the IEP for general triangles lead to 2pms problems, but maybe a recursive classification of the 2D pm space helps!

A very simple example

For an isosceles triangle, denote the length of the three sides AB , BC, CA, by a=1,b,c=b (wlog a=1)

What is the (range of the) ratio of the sum of the squares of the sides $\left(AB^2 + BC^2 + CA^2 = 2\,b^2 + 1\right)$ and the sum of the products of the sides $\left(AB\,BC + AB\,CA + BC\,CA = b\,b + b + b = b^2 + 2\,b\right)$?

A very simple example [2]

What is the (range of the) ratio of the sum of the squares of the sides $\left(AB^2 + BC^2 + CA^2 = 2\,b^2 + 1\right)$ and the sum of the products of the sides $\left(AB\,BC + AB\,CA + BC\,CA = b\,b + b + b = b^2 + 2\,b\right)$?

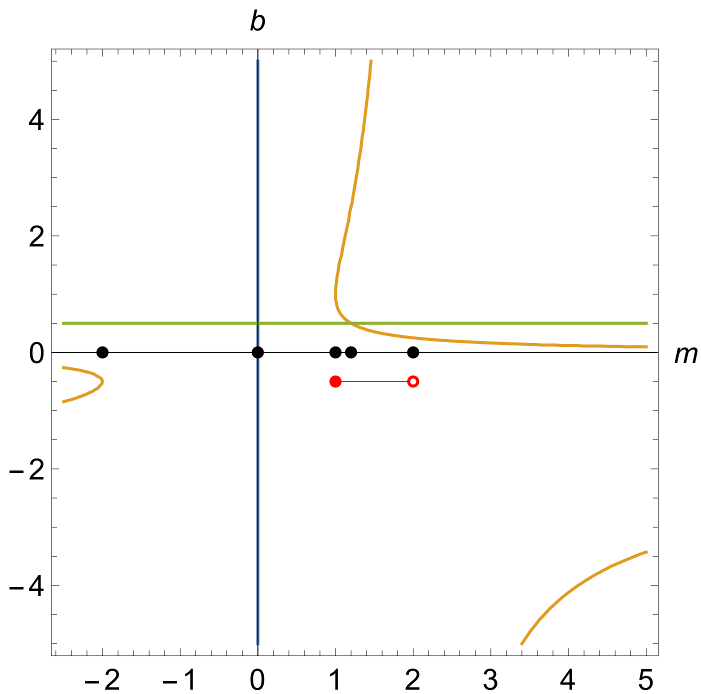
As an RQE (NONSAT) problem with one free and one (existentially bound) variable:

```
Resolve[∃_b (2 b - 1 > 0 ∧ 2 b^2 + 1 == m (b^2 + 2 b)), Reals]
```

Out[ ]= $1 \le m < 2$

A very simple example [3]

Out[ ]=



A very simple example [4]

Reduction via Groebner Basis:

Moroz: 1D Ocrit ∪ Oinequs ∪ Oinfinity ⟹Induces a complete 1D CAD (open intervals and points)

One typical computation for Ocrit via the partial Jacobian, that is, for detecting the value m=1 (first call)

In[ ]:=
```
Flatten[{Factor[GroebnerBasis[
    {(2 b^2 + 1) - m (b^2 + 2 b), D[(2 b^2 + 1) - m (b^2 + 2 b), b], m (2 b - 1) t + 1}, {m}, {t, b}]],
  Factor[GroebnerBasis[{(2 b^2 + 1) - m (b^2 + 2 b), m (2 b - 1) - u, t u - 1}, {m, u}, {t, b}] /.
    u → 0], Factor[GroebnerBasis[
     {(2 b^2 + u^2) - m (b^2 + 2 b u), t u m (2 b - u) - 1, b - 1}, {m, u}, {t, b}] /. u → 0]}]
```

Out[ ]= $\{(-1+m)(2+m),\ m^2(-6+5m),\ -2+m\}$

Out[ ]= $\left\{\{1\},\ \left\{\frac{6}{5}\right\},\ \{2\}\right\}$

A very simple example [5]

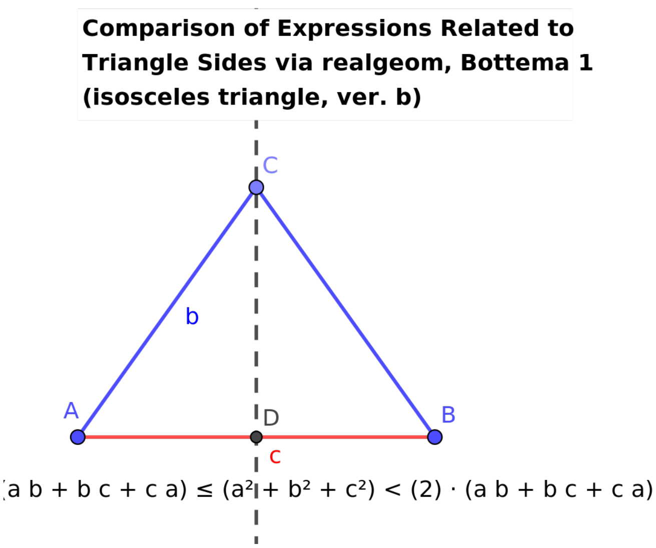
Final Solution (1≤m<2) the points and intervals for which the value True assigned

```
In[ ]:= AbsoluteTiming[Table[Resolve[Exists[{b}, 2 b - 1 > 0 ∧ (2 b ^ 2 + 1) == m (b^2 + 2 b)], Reals],
          {m, {1 / 2, 1, 11 / 10, 6 / 5, 3 / 2, 2, 3}}]]

Out[ ]= {0.013465 , {False , True , True , True , True , False , False}}

In[ ]:= Reduce[m == 1 ∨ 1 < m < 6 / 5 ∨ m == 6 / 5 ∨ 6 / 5 < m < 2, x, Reals]

Out[ ]= 1 ≤ m < 2
```

A very simple example [6]

```
Reduce[
 Resolve[Exists[{v10, v11, v13}, (m > 0) ∧ (v11 > 0) ∧ (v13 > 0) ∧ (-4 * v10 ^ 2 + 4 * v11 ^ 2 - 1 == 0) ∧
    (-4 * v10 ^ 2 + 4 * v13 ^ 2 - 1 == 0) ∧ (-m * v11 * v13 - m * v11 - m * v13 + v11 ^ 2 + v13 ^ 2 + 1 == 0)],
  Reals], Reals](*GGBA CB version*)
```

Out[∘]= 1 ≤ m < 2



Motivation

It is a practical work, intuitively it hopes to profit from the reduction of the number of variables (number of CAD-cells) (and from theoretical comp. results) .

STAT: For IT/RT, 2-12 (bound) variables.
We have a pre-computed RQE Benchmark sets (>100 test cases)
Can we reach with the PRF method the same or even better results than with RQE?

Findings

We worked with maple's packages and and our prototype implementation in Mathematica based on a selected BOTTEMA-problem collection (BM 1.1, 1.19, 4.2, 5.1, 5.3, 6.1, 8.1)

All the GGBA generated CB-based IT/RT problems could be treated with PRF but some refinements in our implementation are needed, ongoing work...

In[ ]:=
```
0.862 ... ≤ m < 1 // ToRadicals
```

Out[ ]=

$$-\frac{1}{2}+\sqrt{5}-\sqrt{3-\sqrt{5}}\leq m<1$$

```
Inpsas94RBM81pb =
 {{(4 * v16 ^ 2 - v8 ^ 2 - 1), (4 * v17 ^ 2 - 4 * v8 ^ 2 - 1), v18 ^ 2 - v8 ^ 2 - 1, (v19 ^ 2 - v8 ^ 2),
   4 * v20 ^ 2 - v8 ^ 2 - 4, (-m * v18 - m * v19 - m + v16 + v17 + v20)}, {m, v16, v17, v18, v19, v20}}

auxd2d[{Inpsas94 , m}]
```

$$\{\{v16, v17, v18, v19, v20, v8\}, True, \{(81 - 72\,m - 232\,m^2 - 32\,m^3 + 16\,m^4)$$
$$(81 + 72\,m - 232\,m^2 + 32\,m^3 + 16\,m^4)\,(-81 + 324\,m^2 + 36\,m^3 + 144\,m^4 + 416\,m^5 - 64\,m^6 + 64\,m^7)$$
$$(81 - 324\,m^2 + 36\,m^3 - 144\,m^4 + 416\,m^5 + 64\,m^6 + 64\,m^7)\},$$
$$\{4\,(-1 + m)\,m^{10}\,(1 + m)\,(-1 + 2\,m)^2\,(1 + 2\,m)^2\,(-3 + 4\,m^2)^8\,(3 + 4\,m^2)^8$$
$$(81 + 324\,m - 324\,m^2 - 1296\,m^3 + 2304\,m^4 - 2304\,m^5 + 1728\,m^6 + 256\,m^8)^2$$
$$(81 - 324\,m - 324\,m^2 + 1296\,m^3 + 2304\,m^4 + 2304\,m^5 + 1728\,m^6 + 256\,m^8)^2\},$$
$$\{\{-u^2 + 4\,v16^2 - v8^2\}, \{-u^2 + 4\,v17^2 - 4\,v8^2\}, \{-u^2 + v18^2 - v8^2\}, \{v19^2 - v8^2\},$$
$$\{-4\,u^2 + 4\,v20^2 - v8^2\}, \{-m\,u + v16 + v17 - m\,v18 - m\,v19 + v20\}\},$$
$$\{\{m\}, \{v16\}, \{v17\}, \{v18\}, \{v19\}, \{v20\}\}, \{\{-67\,108\,864\,(-1 + m)\,m^4\,(1 + m)\,(-1 + 2\,m)^2\,(1 + 2\,m)^2\},$$
$$\{-1\,048\,576\,(-1 + m)\,m^4\,(1 + m)\,(-1 + 2\,m)^2\,(1 + 2\,m)^2\}, \{-16\,384\,(-1 + m)\,m^4\,(1 + m)\,(-1 + 2\,m)^2\,(1 + 2\,m)^2\},$$
$$\{16\,384\,(-1 + m)\,m^4\,(1 + m)\,(-1 + 2\,m)^2\,(1 + 2\,m)^2\}, \{-1\,048\,576\,(-1 + m)\,m^4\,(1 + m)\,(-1 + 2\,m)^2\,(1 + 2\,m)^2\},$$
$$\{268\,435\,456\,(-1 + m)^2\,m^8\,(1 + m)^2\,(-1 + 2\,m)^4\,(1 + 2\,m)^4\}\}, \{0.447\ldots, 0.448\ldots,$$
$$\frac{1}{2}, 0.547\ldots, 0.801\ldots, 0.862\ldots, \frac{\sqrt{3}}{2}, 1, 2.61\ldots, 5.02\ldots\},$$
$$\{\{\{m \to \frac{27}{226}\}\}, \{\{m \to \frac{96\,153}{214\,879}\}\}, \{\{m \to \frac{893}{1936}\}\}, \{\{m \to \frac{1092}{2131}\}\}, \{\{m \to \frac{35}{57}\}\},$$
$$\{\{m \to \frac{452}{553}\}\}, \{\{m \to \frac{21\,851}{25\,317}\}\}, \{\{m \to \frac{679}{754}\}\}, \{\{m \to \frac{10}{7}\}\}, \{\{m \to \frac{68}{21}\}\}, \{\{m \to 32\}\}\},$$
$$\{False, False, False, False, False, False, True, True, False, False, False\},$$
$$\{False, False, False, False, False, False, 0.862\ldots < m < \frac{\sqrt{3}}{2},$$
$$\frac{\sqrt{3}}{2} < m < 1, False, False, False\}, \{False, False, False, False, False,$$
$$m == 0.862\ldots, m == \frac{\sqrt{3}}{2}, False, False, False\}, 0.862\ldots \le m < 1\}$$

Conclusion

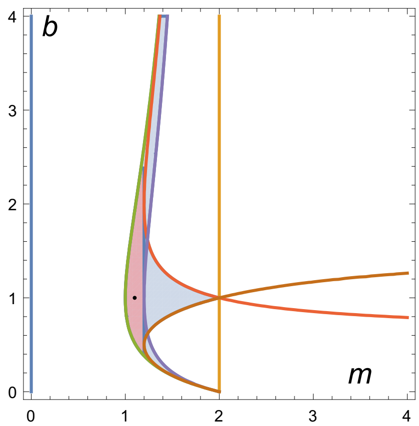
If GB and (nonlinear) real SAT or RRC are implemented and they are fast, it COULD be a viable approach instead of the general RQE.

Educational applications all the sub-algorithms should be implemented in a free software (GB ⟶ Giac, SAT ⟶ tarski, SMT-RAT,..., WS?)

The exploration problems for a GENERAL triangles are not 1D problems. MDV in a 2D space: 2D generic CAD, also recursive analysis of curves?

Discussion: Any suggestion? SEE GT for $m = \left(AB^2 + BC^2 + CA^2\right)/(AB + BC + CA)$

Out[ ]=



References

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[Liang-Gerhard-Jeffrey-Moroz 2009] A package for solving parametric polynomial systems, ACM Communications in Computer Algebra 169(43), 2009.

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