On automating triangle constructions in absolute and hyperbolic geometry

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Solving ruler and compass construction problems

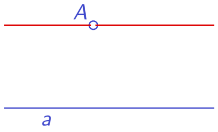
- One of the most studied problems in mathematical education
- Task: to describe a construction of geometrical figure which satisfies given set of constraints
  " construct $\triangle ABC$ given α, β and $|AB|$"
- Constructions are procedures
- Some instances are unsolvable (e.g. angle trisection)

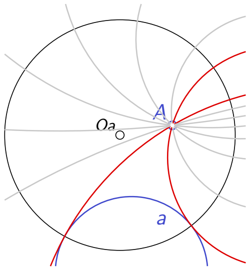
Different geometries

- Many different geometries exist
- Absolute geometry is based on four groups of axioms: incidence, order, congruence, and continuity
- By adding the appropriate axiom of parallelism, we get either Euclidean geometry or hyperbolic geometry

Different geometries (2)

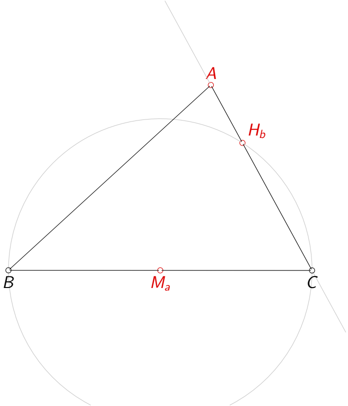
- Euclidean geometry: a unique line parallel to a given line a through a point A not on the line
- Hyperbolic geometry: infinitely many parallels to a given line a through a point A not on the line

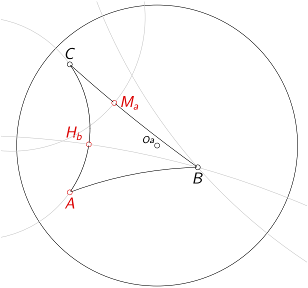




Goal

- Many ruler and compass constructions are valid only in Euclidean geometry
- We want to automatically find constructions that are valid in absolute geometry
- We want to automatically find constructions that are valid in hyperbolic geometry





Constructions using straightedge and compass

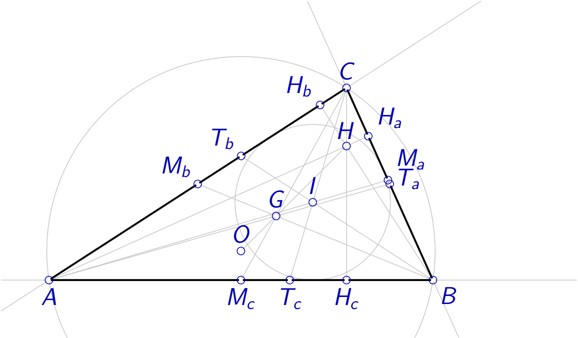
- Tools: straightedge and compass
- Elementary steps:
  - construction of an arbitrary point
  - construction of a line through two given points
  - construction of a circle centered at given point passing through another point
  - construction of an intersection of two circles, two lines, or a line and a circle
- We usually use compound construction steps

Automating triangle constructions

- System for automated solving of location construction problems from the given corpus (ArgoTriCS, authors: V. Marinković, P. Janičić)
- Initially focused solely on Euclidean geometry
- Export textual descriptions of constructions, and formal procedures in GCLC format
- The main problem in solving:
  combinatorial explosion – huge search space
- Adjusting the system for usage in education is the subject of current work (e.g., next-step guidance feature)

Corpora of construction problems

- Wernick's corpus (1982)
- Task: construct triangle ABC if locations of three significant points in the triangle are given



Corpora of construction problems (2)

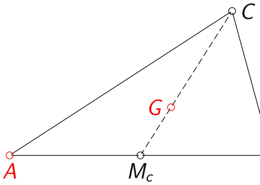
Wernick's corpus: in total $\binom{16}{3} = 560$ instances, 139 non-trivial, significantly different problems; 3 redundant (R); 23 locus dependent (L); 74 solvable (S); 39 unsolvable (U)

1.	A, B, O	L
2.	A, B, M_a	S
3.	A, B, M_c	R
4.	A, B, G	S
5.	A, B, H_a	L
6.	A, B, H_c	L
7.	A, B, H	S
8.	A, B, T_a	S
9.	A, B, T_c	S

25. A, …		
26. A, M_a, …		
27. A, M_a, I	S	[9]
28. A, M_b, M_c	S	
…, T_b, I	S	
55. A, H, T_a	S	
56. A, H, T_b	U	[9]
57. A, H, I	S	[9]
… A, T_a, T_b	S	[9]
… T_a, I	L	
… T_b, T_c	S	
… I	S	
… M_b	S	
… G	S	
… H_a	L	
… H_b	S	
…	S	
…	L	
…	U	[9]
…	S	
…	S	
…	R	
…	U	[9]
…	U	[9]
… H_b	U	[9]
… H	S	
… T_a	S	
… H_a, T_b		
… H_a, I		
… O, H, T_a	U	[9]
80. O, H, I	U	[9]
81. O, T_a, T_b		
82. O, T_a, I	S	[9]
83. M_a, M_b, M_c	S	
84. M_a, M_b, G	S	
85. M_a, M_b, H_a	S	
86. M_a, M_b, H_c	S	
87. M_a, M_b, H	S	[9]
88. M_a, M_b, T_a	U	[9]
89. M_a, M_b, T_c	U	[9]
90. M_a, M_b, I	U	[10]
91. M_a, G, H_a	L	
92. M_a, G, H_b	S	
93. M_a, G, H	S	
94. M_a, G, T_a	S	
95. M_a, G, T_b	U	[9]
96. M_a, G, I	S	[9]
97. M_a, H_a, H_b	S	
98. M_a, H_a, H	L	
99. M_a, H_a, T_a	L	
100. M_a, H_a, T_b	U	[9]
101. M_a, H_a, I	S	
102. M_a, H_b, H_c	L	
103. M_a, H_b, H	S	
104. M_a, H_b, T_a	S	
105. M_a, H_b, T_b	S	
106. M_a, H_b, T_c	U	[9]
107. M_a, H_b, I	U	[9]
108. M_a, H, T_a	U	[9]
109. M_a, H, T_b	U	[10]
110. M_a, H, I	U	[10]
111. M_a, T_a, T_b	U	[10]
112. M_a, T_a, I	S	
113. M_a, T_b, T_c		
114. M_a, T_b, I	U	[9]
115. G, H_a, H_b	U	[9]
116. G, H_a, H	S	
117. G, H_a, T_a	S	
118. G, H_a, T_b		
119. G, H_a, I		
120. G, H, T_a	U	[9]
121. G, H, I	U	[9]
122. G, T_a, T_b		
123. G, T_a, I		
124. H_a, H_b, H_c	S	
125. H_a, H_b, H	S	
126. H_a, H_b, T_a	S	
127. H_a, H_b, T_c		
128. H_a, H_b, I		
129. H_a, H, T_a	L	
130. H_a, H, T_b	U	[9]
131. H_a, H, I	S	[9]
132. H_a, T_a, T_b		
133. H_a, T_a, I	S	
134. H_a, T_b, T_c		
135. H_a, T_b, I		
136. H, T_a, T_b		
137. H, T_a, I		
138. T_a, T_b, T_c	U	[11]
139. T_a, T_b, I	S	

Knowledge representation

Problem: *Construct a triangle ABC given vertices A and B and its centroid G*
Solution: *Construct the midpoint M_c of the segment AB, and then construct a point C such that it holds $\overrightarrow{M_cC}/\overrightarrow{M_cG} = 3$*



Following knowledge is used:

- M_c is the midpoint of the segment AB (definition of the point M_c)
- G is the centroid of the triangle ABC (definition of the point G)
- it holds: $\overrightarrow{M_cG} = 1/3\overrightarrow{M_cC}$ (lemma)
- given points X and Y, construct the midpoint of the segment XY (primitive construction)
- given points X and Y, construct a point Z: $\overrightarrow{XZ}/\overrightarrow{XY} = m/n$ (primitive construction)

How to adapt ArgoTriCS for non-Euclidean geometries?

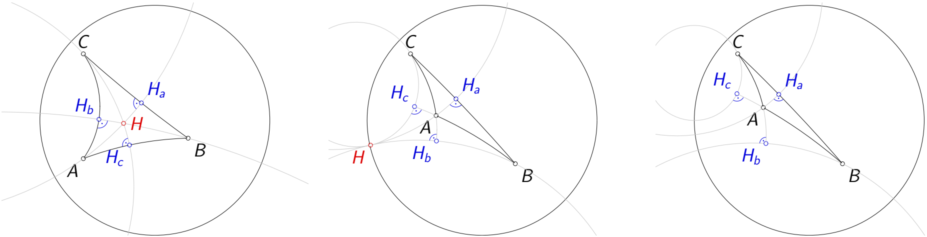
- Change definitions (when necessary)
- Change lemmas (when necessary)
- Change primitive construction steps (when necessary)
- The search algorithm remains the same
- Guiding heuristics might be adapted for better efficiency

Definitions and pseudo-elements

- In the Euclidean case many notions can be defined in equivalent ways
  For example,
  - a median is the segment that connect a triangle vertex with the midpoint of its opposite side
  - a median is a segment that divides the triangle area in two exact halves
- In hyperbolic case these need not coincide, so we define different objects
  For example, we distinguish:
  - median (definition 1) and
  - pseudo-median (definition 2)
- Some Euclidean theorems hold only for pseudo-elements (e.g., Euler line does not exist, but pseudo-Euler line exists)
- Unfortunately, some pseudo-elements are not ruler and compass constructible

Theorems of absolute geometry (weaker than in Euclidean geometry)

- The sum of internal angles of a triangle is less or equal to π
- The three medians of a triangle intersect in one point (the centroid G)
- The three internal angle bisectors of a triangle intersect in one point (the incenter I)
- The perpendicular bisectors of triangle sides belong to the same pencil of lines (the circumcenter need not exist)
- The altitudes a triangle belong to the same pencil of lines (the orthocenter need not exist)

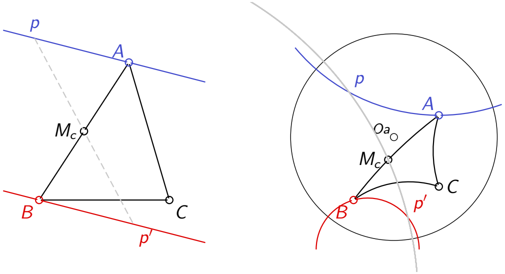


Euclidean lemmas that fail in hyperbolic geometry

- The centroid G does not divide the median in 2:1 ratio
- The inscribed angle subtended by a diameter need not be right
- Locus of points subtending a segment under a given angle is not a circular arc
- Equidistant curve is not a line
- ...

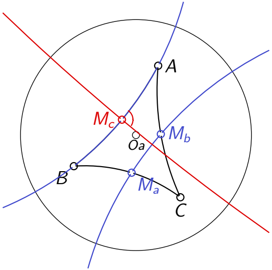
Lemmas added to the system

- If a vertex A of triangle ABC belongs to the line p, then a vertex B belongs to a line which is an image of line p under the reflection wrt. point M_c



Lemmas added to the system

- Lines M_aM_b and AB are hyperparallel and M_c is the foot of their common perpendicular (this one is specific for hyperbolic geometry)



Primitive constructions

Some primitive constructions fail in the hyperbolic case. For example:

- Given points X, Z, and W, and a rational number r one cannot construct a point Y for which holds: $\overrightarrow{XY}/\overrightarrow{ZW} = r$

However, special cases of those constructions can be done

- Given points X and Y construct the midpoint Z of the segment XY
- Given points X and Y construct the point Z symmetric to X wrt. point Y

Reflections were not primitive steps in Euclidean geometry solver, since they could be realized by other steps, but we needed to add them to the hyperbolic solver

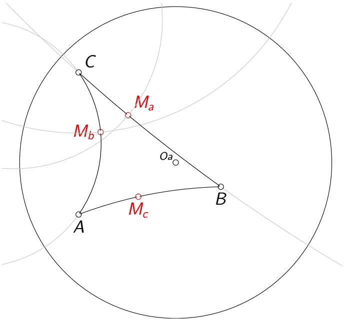
- Given a line m and a point P, construct its image under the reflection wrt. line m

Example

Problem: Construct the triangle ABC given three side midpoints M_a, M_b, and M_c

Solution:

1. Construct the line a that is hyperparallel to the line through points M_b and M_c with point M_a being the foot of their common perpendicular;
2. Construct the line b that is hyperparallel to the line through points M_a and M_c with point M_b being the foot of their common perpendicular;
3. Construct the intersection point C of the lines a and b;
4. Construct the point B symmetric to C wrt. point M_a;
5. Construct the point A symmetric to C wrt. point M_b.



Results

- From 139 significantly different problems, 31 determined solvable (and solved), 1 redundant and 11 locus dependent
- Compendium of solutions in hyperbolic geometry available here: http://poincare.matf.bg.ac.rs/~vesnap/animations_hyp/compendium_wernick_hyperbolic.html

Conclusions

- We have identified definitions, lemmas and primitive constructions relevant for absolute and hyperbolic geometry
- We have adapted ArgoTriCS for solving constructions in absolute and hyperbolic geometry
- Ruler and compass constructions are much harder in absolute and hyperbolic geometry (we believe that many problems are not RC-constructible)