Maximizing the sum of the distances between four points on the hemisphere

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Outline

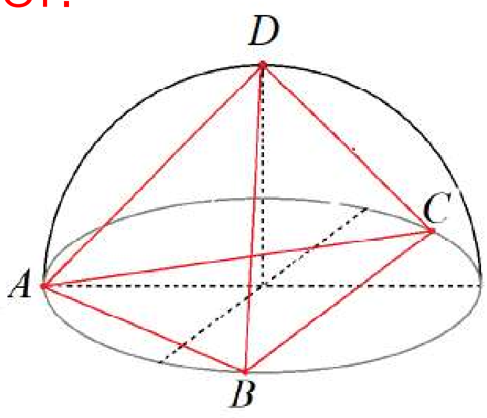
- The geometric problem
- Main Result
- Lemmas
- The automated deduction process
- Open questions

A geometric optimization problem

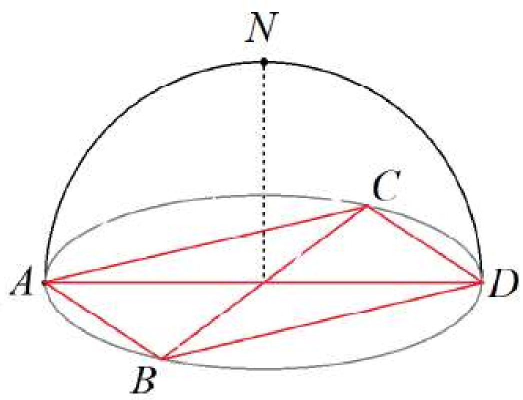
- Put n (n=4) points A, B, C, D on the hemisphere of radius 1, so that the sum of their mutual distances is maximal.
- Two possible solutions:
- (a): D at the North Pole.
- (b): A,B,C,D on the Equator.

$$3\sqrt{3}+3\sqrt{2} = 9.4387\cdots$$

$$4+2\sqrt{2} = 9.6568\cdots$$

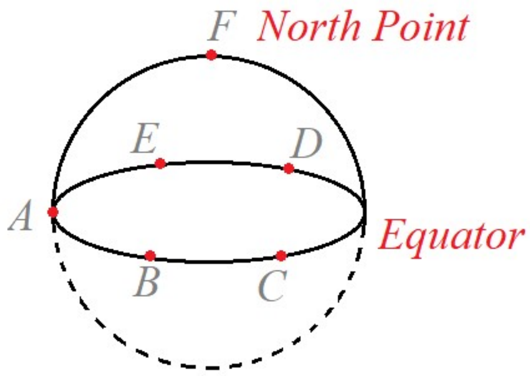


(a)

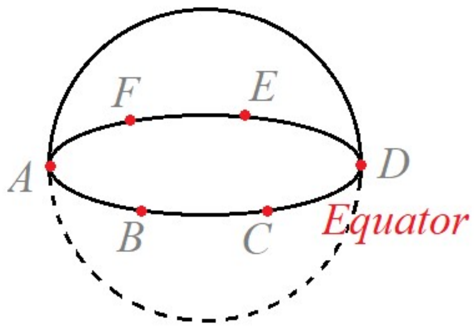


(b)

Numerical Searching (for n=4,5,⋯,12)



(a)



(b)

Table 1. Comparison of distance sums between n points on the hemisphere

(Configuration marked by ★ is better than the other one.)

n points	(a) the regular pyramid with apex on the North Pole	(b) the regular polygon inscribed to the equator
$n=4$	$3\sqrt{3}+3\sqrt{2}=9.4387\cdots$	$4+4\sqrt{2}=9.6568\cdots$ ★
$n=5$	$4+8\sqrt{2}=15.3137\cdots$	$15.3884\cdots$ ★
$n=6$	$22.4594\cdots$ ★	$12+6\sqrt{3}=22.3923\cdots$
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Main Result

For any four points *A, B, C, D* on the hemisphere of radius 1, the sum of their distances is not great that 4+4*sqrt(2): i.e.,

$$AB + AC + AD + BC + BD + CD \leqslant 4 + 4\sqrt{2} = 9.5658\cdots,$$

and, up to congruent, the optimal configuration is formed by

$$A = (0, -1, 0),\ B = (\frac{\sqrt{3}}{2},\ \frac{1}{2}, 0),\ C = (-\frac{\sqrt{3}}{2},\ \frac{1}{2}, 0),$$

$$D = (0, 0, 1).$$

Lemmas

- Lemma 1 (by human proof): If {A, B, C, D} is an optimal configuration for the maximal sum, then the center of the hemisphere must be contained in the interior (or surfaces) of the convex hull of A, B, C, D, and therefore, at least three of the four points lie on the equator of the hemisphere.

- Lemma 2 (easy): The following four points:

  A=(0, -1, 0), B=(sqrt(3)/2, 1/2, 0), C=(-sqrt(3)/2, 1/2, 0), D=(0, 0, -1)

  form a local maximal configuration of the original problem.

Lemmas

- Lemma 3 (automated deduction): For the following neighborhoods *U, V, W* and any points *B in U∩S^2, C in V ∩S^2, and D in W ∩S^2,* we have

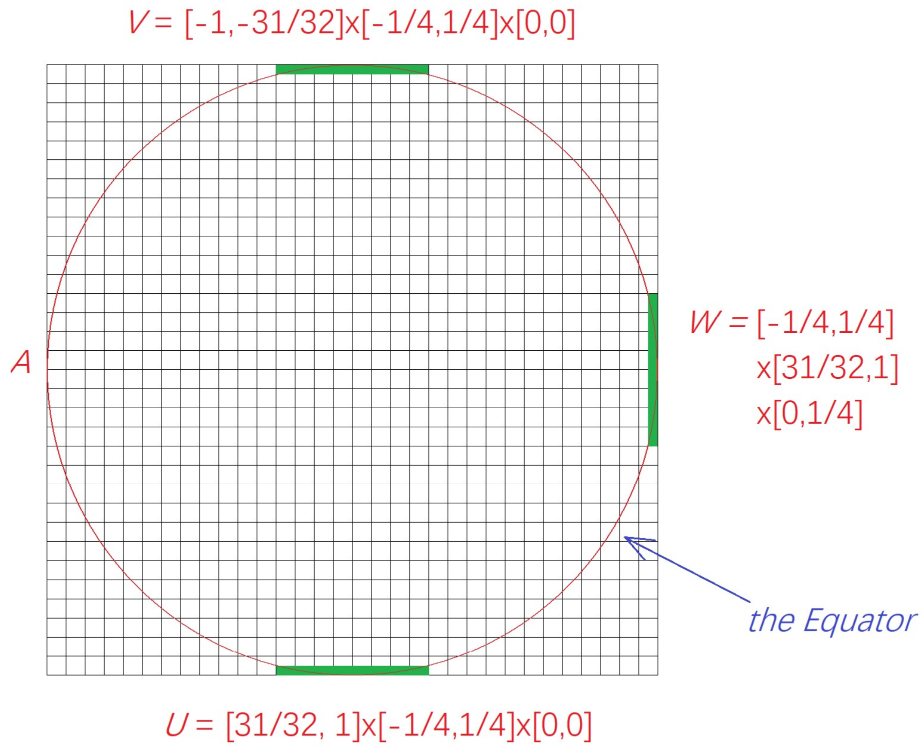
$$AB + AC + AD + BC + BD + CD \leqslant 4 + 4\sqrt{2} = 9.5658\cdots,$$

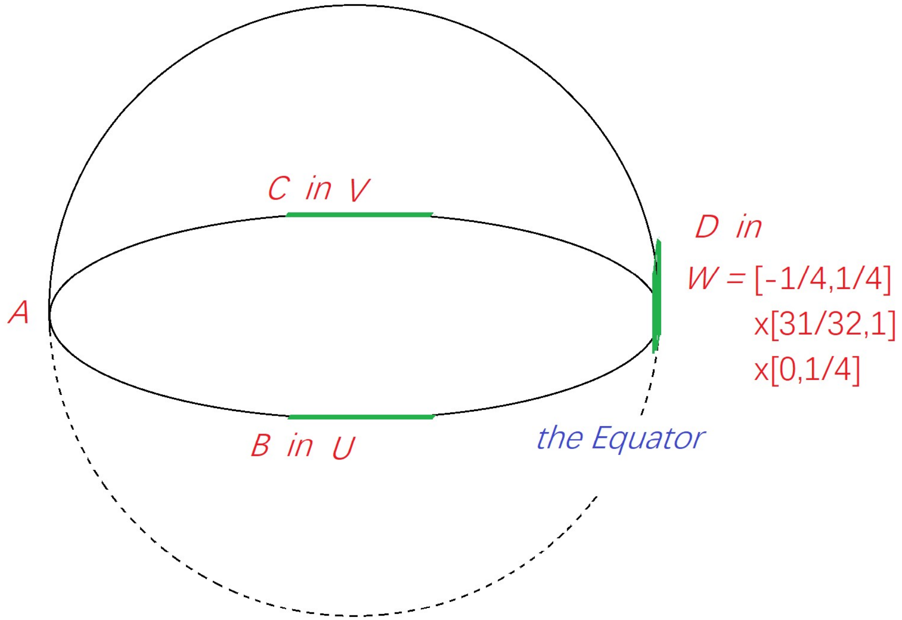
$where \quad A = (0, -1, 0), \quad and$

$$U := \{(x, y, 0) | 1 - \delta_1 < x \leq 1, \ -\delta_2 < y < \delta_2\}$$
$$V := \{(x, y, 0) | -1 \leq x < -1 + \delta_1, \ -\delta_2 < y < \delta_2\}$$
$$W := \{(x, y, z) | -\delta_2 < x < \delta_2, \ 1 - \delta_1 < y \leq 1, \ 0 \leq z < \delta_2\}$$

$$\delta_1 = 1/32, \delta_2 = 1/4.$$

Three neighborhoods in Lemma 3





U, *V* are 2D squares in the equator plane, *W* is a 3D cube.

Lemmas

- Lemma 4 (machine proof by numerical search): For A=(0,-1,0) and any B,C,D on the hemisphere that satisfy (B,C,D) in S^2\(U x V x W), the inequality

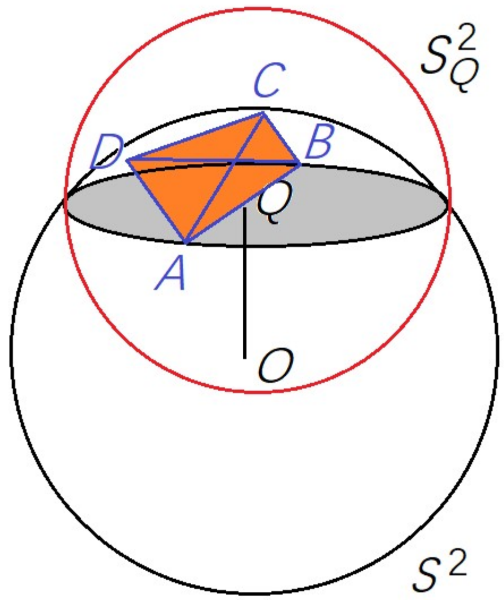
$$AB + AC + AD + BC + BD + CD \leqslant 4 + 4\sqrt{2} = 9.5658\cdots,$$

is also valid.

Sketch of the Proof of Lemma 1.

- Lemma 1 (by human proof): If {A, B, C, D} is an optimal configuration for the maximal sum, then the center of the hemisphere must be contained in the interior (or surfaces) of the convex hull of A, B, C, D, and therefore, at least three of the four points lie on the equator of the hemisphere.

- Step1: Assume that {A,B,C,D} is an optimal configuration, K is the convex hull of {A, B, C, D}, O is the center of the hemisphere, and O\not\in K. Let Q \in K be the point so that d(O,Q)=min{d(O,P), P\in K}. Draw a plane perpendicular to OQ through point Q. Then this plane divides the sphere into two parts, one part is smaller and the other part is larger, and K is contained in the smaller part.
- Step2: Draw the smallest sphere (red one in fig) that contains the small spherical cap.



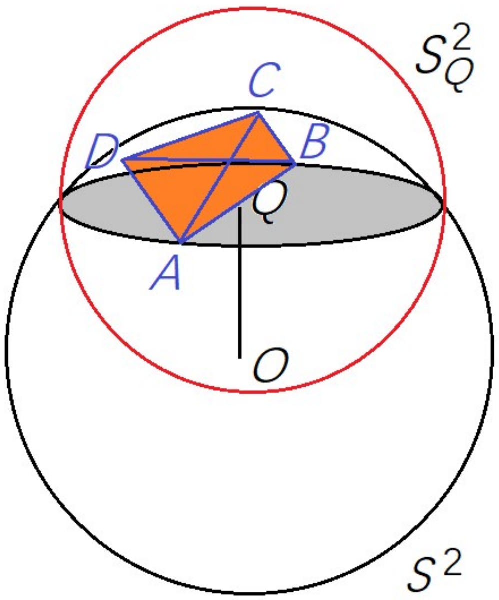
Sketch of the Proof of Lemma 1.

- Step2: Draw the smallest sphere (red one in fig) that contains the small spherical cap. The radius of this new sphere is less than 1.

- Now K is contained the smaller hemisphere, formed by the perpendicular plane and the smaller sphere.

- Let A',B',C',D' be the intersection points of QA,QB,QC,QD with the new sphere.

- A'B'>AB, A'C'>AC, ···, C'D'>CD.



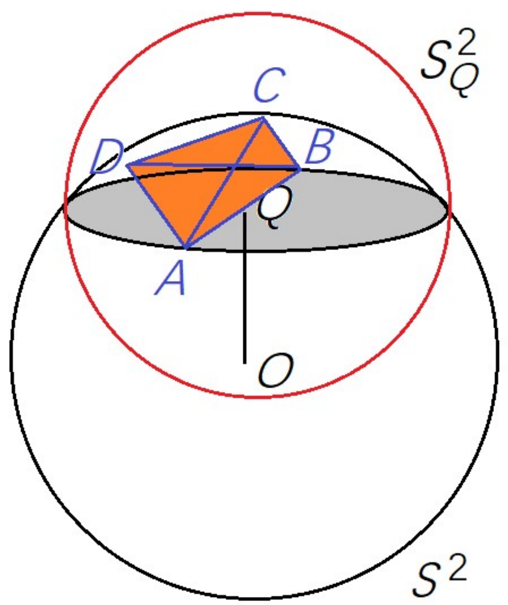
Sketch of the Proof of Lemma 1.

- A'B'>AB, A'C'>AC, ···, C'D'>CD.

- We got 4 points A', B', C', D' on a smaller hemisphere with
- A'B'+A'C'+···+C'D'>AB+AC+···+CD,

- This is contradicts to the assumption that {A,B,C,D} is optimal.

- This proves that O\in K.



Lemma 3

- Lemma 3 (automated deduction): For the following neighborhoods *U, V, W* and any points *B in U∩S^2, C in V ∩S^2, and D in W ∩S^2,* we have

$$AB + AC + AD + BC + BD + CD \leqslant 4 + 4\sqrt{2} = 9.5658\cdots,$$

where $A = (0, -1, 0)$, *and*

$$U := \{(x, y, 0) | 1 - \delta_1 < x \leq 1,\ -\delta_2 < y < \delta_2\}$$
$$V := \{(x, y, 0) | -1 \leq x < -1 + \delta_1,\ -\delta_2 < y < \delta_2\}$$
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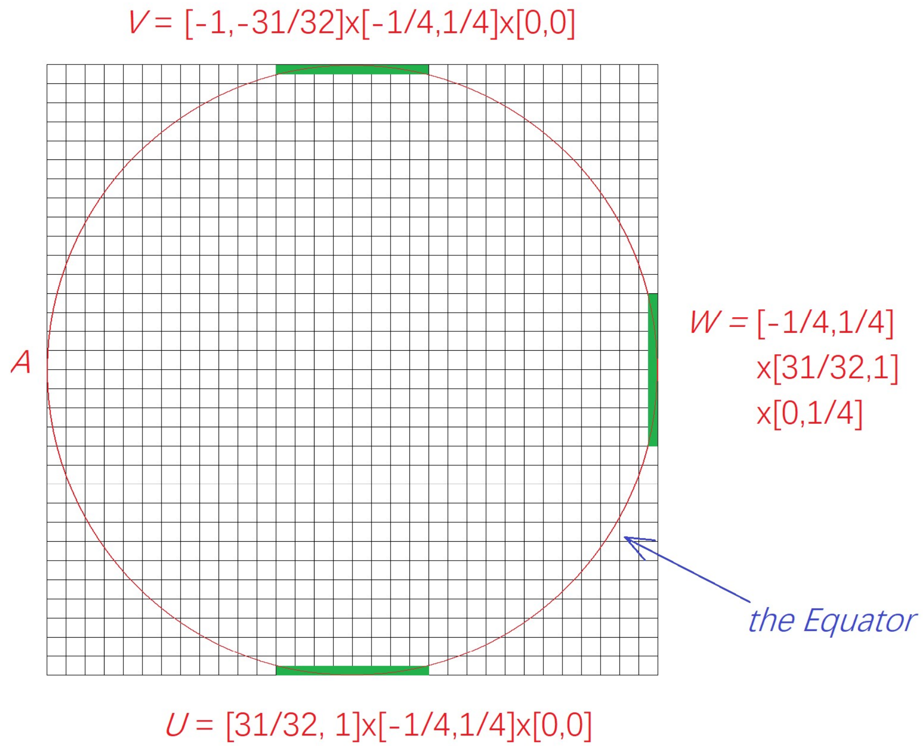
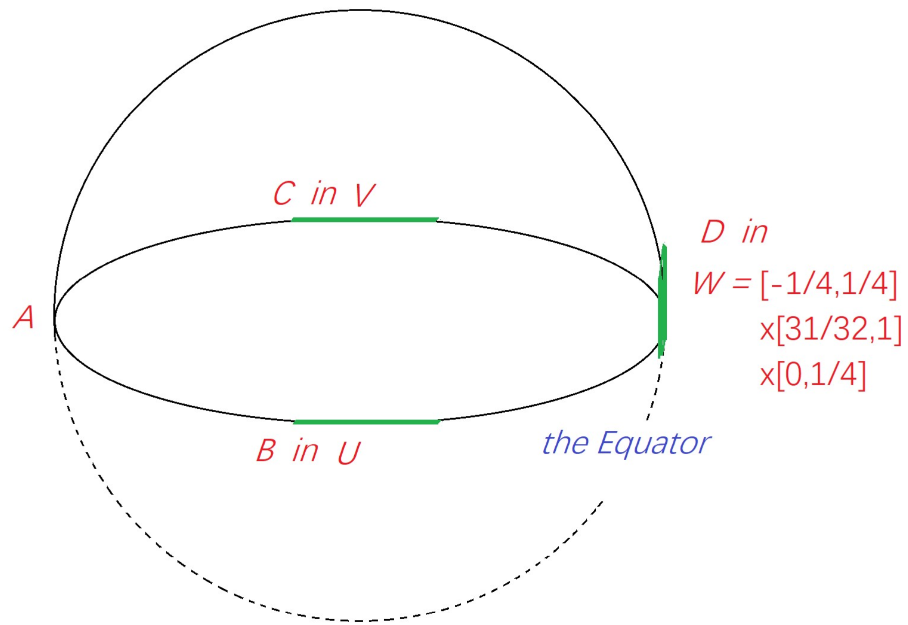
$$\delta_1 = 1/32, \delta_2 = 1/4.$$

Proof of Lemma 3





$$A = (0, -1, 0), \quad B = (\frac{1-s^2}{1+s^2}, \frac{2s}{1+s^2}, 0), \quad C = (-\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}, 0), \quad \text{and} \quad D = (\frac{2u}{1+u^2}, \frac{1-v^2}{1+v^2}, z_3) \in S^2_{\geq 0},$$

Lemma3

(when P or Q lies on the equator)

For any two points $P = (x, y, z), Q = (x', y', z') \in S^2$ with $z \cdot z' = 0$, we have

$$PQ = d(P, Q) = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} = \sqrt{2 \cdot (1 - x\,x' - y\,y')}.$$

$$f(A, B, C, D) = d(A,B)+d(B,C)+d(C,A)+d(D,A)+d(D,B)+d(D,C)$$
$$= AB + BC + CA + DA + DB + DC = R(s,\ t,\ u,\ v)$$

each distance is a square-root of a polynomial of s,t,u,v.

How to change the sum of square-roots to a rational function?

Lemma 3: estimate the square-root

Lemma. *For* $-1 \leq x \leq 1$,

$$\sqrt{1-x} \leq 1 - \frac{1}{2}\,x - \frac{1}{8}\,x^2 - \frac{1}{16}\,x^3.$$

Proof.

$$\left(1 - \frac{1}{2}\,x - \frac{1}{8}\,x^2 - \frac{1}{16}\,x^3\right)^2 - (1-x) = \frac{x^4\,(x^2 + 4\,x + 20)}{256} > 0.$$

□

Lemma 3:

$$f(A,B,C,D) \le 4\sqrt{2} + 4 + \frac{1}{Q(s,t,u,v)}\left(X \cdot H \cdot X^T + \text{h.o.t.}\right) = 4 + 4\sqrt{2} + \frac{J(s,t,u,v)}{8\,(t^2+1)^3\,(s^2+1)^3\,(v^2+1)^3\,(u^2+1)^3},$$

$X=(s,t,u,v)$

high order terms $H_3, H_4, \dots, H_{24}$

H_2 **Hessian matrix**

$$H_2 = -8(\sqrt{2}+1)s^2 - 16st - 8(\sqrt{2}+1)t^2 + 8\sqrt{2}su - 8\sqrt{2}tu - 8\sqrt{2}u^2 - 8v^2,$$

$$\begin{aligned}
H_3 =& -4\sqrt{2}\left(su - tu + 3u^2 - 4v^2\right)(s+t)\\
H_4 =& -18v^4 - 8\sqrt{2}s^4 - 8\sqrt{2}t^4 - 8\sqrt{2}u^4 - 48\sqrt{2}s^2t^2 - 32\sqrt{2}s^2u^2 - 8\sqrt{2}s^2v^2\\
&+ 16\sqrt{2}su^3 - 32\sqrt{2}t^2u^2 - 8\sqrt{2}t^2v^2 - 16\sqrt{2}tu^3 - 24\sqrt{2}u^2v^2 - 44s^2t^2\\
&- 24s^2u^2 - 48s^2v^2 - 24t^2u^2 - 48t^2v^2 - 24u^2v^2 - 40s^3t - 40st^3\\
&- 48stu^2 - 48stv^2 - 24\sqrt{2}s^2tu + 24\sqrt{2}st^2u + 8\sqrt{2}suv^2 - 8\sqrt{2}tuv^2 - 18s^4 - 18t^4.
\end{aligned}$$

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& + 16\sqrt{2}su^3 - 32\sqrt{2}t^2u^2 - 8\sqrt{2}t^2v^2 - 16\sqrt{2}tu^3 - 24\sqrt{2}u^2v^2 - 44s^2t^2 \\
& - 24s^2u^2 - 48s^2v^2 - 24t^2u^2 - 48t^2v^2 - 24u^2v^2 - 40s^3t - 40st^3 \\
& - 48stu^2 - 48stv^2 - 24\sqrt{2}s^2tu + 24\sqrt{2}st^2u + 8\sqrt{2}suv^2 \\
& - 8\sqrt{2}tuv^2 - 18s^4 - 18t^4.
\end{aligned}$$

$\cdots \cdots$

$$H_{23} = 16\sqrt{2}s^6t^5u^6v^6 + 16\sqrt{2}s^5t^6u^6v^6,$$

$$H_{24} = -11\,s^6t^6u^6v^6.$$

The number of monomials in $H_5, H_6, \cdots, H_{24}$ are listed as follows:

$$20, 59, 44, 101, 70, 134, 88, 145, 90, 133, 74, 100, 50, 59, 26, 29, 10, 10, 2, 1,$$

Lemma 3: estimate the higher order terms

$$J(s,t,u,v) \leq H_2 + H_3 + H_4 + \left(\frac{5}{4}s^2 + \frac{5}{4}t^2 + u^2 + \frac{5}{4}v^2\right).$$

(*under assumption -1 ⩽ s, t, u, v ⩽ 1/7*).

Lemma . *If $x_1, x_2, \cdots, x_n$ are positive real numbers such that $x_1, x_2, \cdots, x_n \leq r$, then*

$$x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} \leq \frac{r^{N-2}}{N}\left(x_1^2 \cdot d_1 + x_2^2 \cdot d_2 + \cdots + x_n^2 \cdot d_n\right),$$

where $N = d_1 + d_2 + \cdots + d_n$.

Lemma 3: estimate the higher order terms

under the assumption $-1/7 < s,t,u < 1/7$ and $s+t>0$,

$$J(s,t,u,v) \leq H_2 + \left(\frac{122\sqrt{2}}{147} + \frac{453}{196}\right)s^2 + \left(\frac{122\sqrt{2}}{147} + \frac{453}{196}\right)t^2 + \left(\frac{164\sqrt{2}}{147} + \frac{73}{49}\right)u^2$$

$$\leq \frac{1}{2}(s,\ t,\ u)\begin{bmatrix} -\frac{2108\sqrt{2}}{147} - \frac{1115}{98} & -16 & 8\sqrt{2} \\ -16 & -\frac{2108\sqrt{2}}{147} - \frac{1115}{98} & -8\sqrt{2} \\ 8\sqrt{2} & -8\sqrt{2} & -\frac{2024\sqrt{2}}{147} + \frac{146}{49} \end{bmatrix}\begin{pmatrix} s \\ t \\ u \end{pmatrix}.$$

this is a negative semi-definite matrix

Lemma 3

- Therefore, we proved that when $-1<s,t,u,v<1/7$, and $s+t>0$, the inequality

  f(A,B,C,D)=AB+BC+CA+AD+BD+CD <= 4+4*sqrt(2)

  is valid.

(the condition $s+t>0$ is a technical assumption, which is always true for optimal configuration).

This result implies that if A=(0,-1,0) and B in U, C in V, D in W, then

$$AB+BC+CD+AD+BD+CD \leqslant 4+4\sqrt{2}$$

Lemmas

- Lemma 4 (machine proof by numerical search): For A=(0,-1,0) and any B,C,D on the hemisphere that satisfy (B,C,D) in S^2\(U x V x W), the inequality

$$AB + AC + AD + BC + BD + CD \leqslant 4 + 4\sqrt{2} = 9.5658\cdots,$$

is also valid.

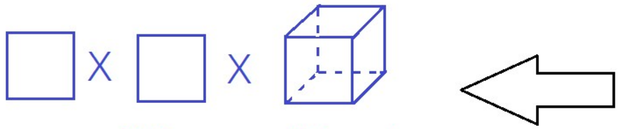
Sketch of the Proof of Lemma 4

- We need to verify that:

For $A=(0,-1,0)$ and all $(B,C,D) \in M = S^1 x S^1 x S^2_{\geq 0} \setminus (U x V x W)$,

$$f(A,B,C,D)=AB+BC+CA+AD+BD+CD<4+4\sqrt{2}.$$

- Method: Construct a finite number of super-cubes in R^7, to cover the set M,



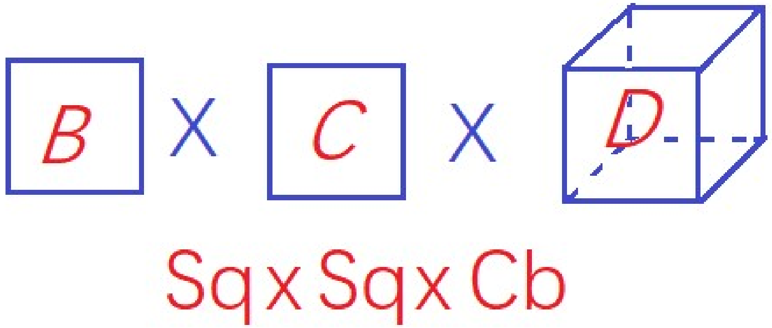
2D square 2D sq. 3D cube

a super cube in R^7

- The edge of the 7D cube is 1/8.

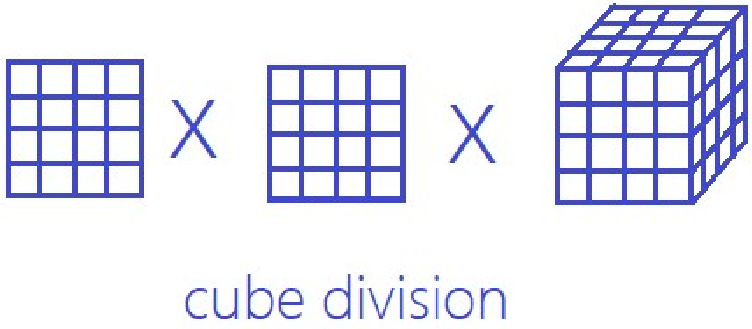
Sketch of the Proof of Lemma 4

- For each cube Sq x Sq x Cb, estimate the upper bound value of the function f(A,B,C,D) on this super cube.



- If the upper bound is less than
- 4+4*sqrt(2), then drop this cube;

- Otherwise, divide the 7D cube to 2^(14)=16,384 cubes of edge 1/32, drop those cubes which has no intersection with M, and estimate f(A,B,C,D) on the remaining cubes.



Sketch of the Proof of Lemma 4

- Do the above divide-and-conquer computation recursively, until on all small super-cubes the upper bound of f(A,B,C,D)<4+4sqrt(2) is verified.

round 1: cubes of edge 1/8 // total number: 806,400//verified rate: 99.467% //remaining 4,300;

15.922 seconds

round 2: cubes of edge 1/32 // total number: 1,105,782//verified rate: 98.263% //remaining 19,206;

1,170.266 seconds

round 3: (depth-first-search) 1/128>>>1/512: 19,007 passed on cubes of edge 1/128, 199 passed on 1/512

8,777.250 seconds

Unsolved Problems

(1) Prove or disprove that for n=5, the optimal configuration is the regular pentagon inscribed to the equator.

(2) Prove or disprove that for n=6,⋯,12, the optimal configuration is the regular pyramid with apex at the north pole.

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Thank You!