GeoGebra Discovery in context

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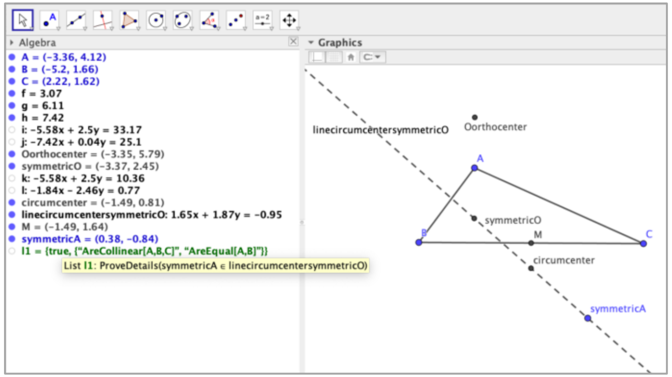
Our main aim…

From

GeoGebra Automated reasoning tools in Geometry

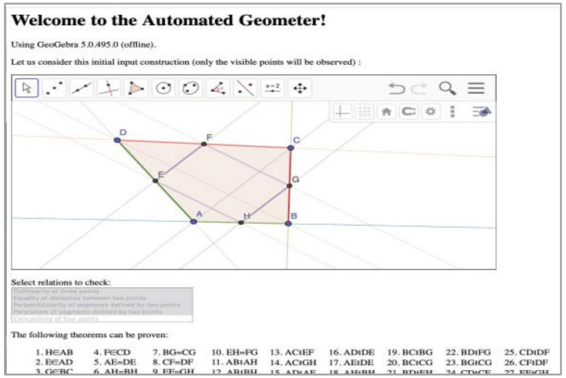
Exploration, discovery or verification of some guessed or conjectured property in a figure



to

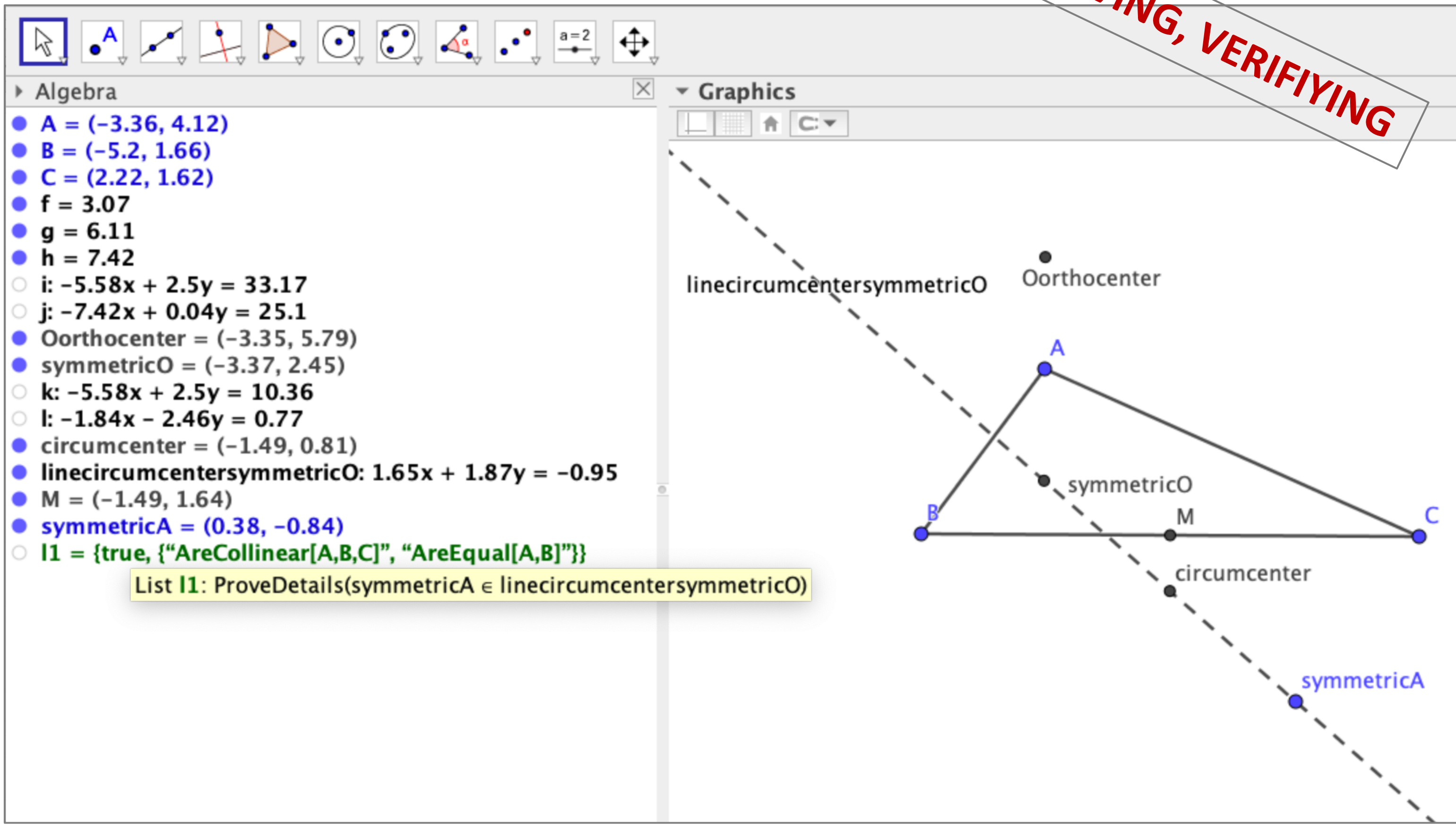
Mechanical geometer program

Finding a large collection of properties in a figure without previously guessing any property



Going from …

PROVING, VERIFIYING



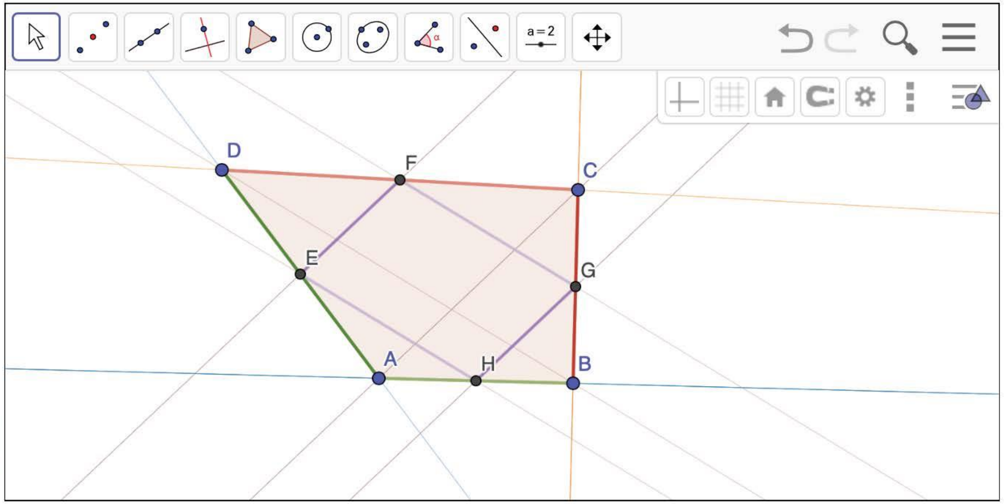
to...

Welcome to the Automated Geometer!

DISCOVERY

Using GeoGebra 5.0.495.0 (offline).

Let us consider this initial input construction (only the visible points will be observed) :



Select relations to check:

Collinearity of three points
Equality of distances between two points
Perpendicularity of segments defined by two points
Parallelism of segments defined by two points
Concyclicity of four points

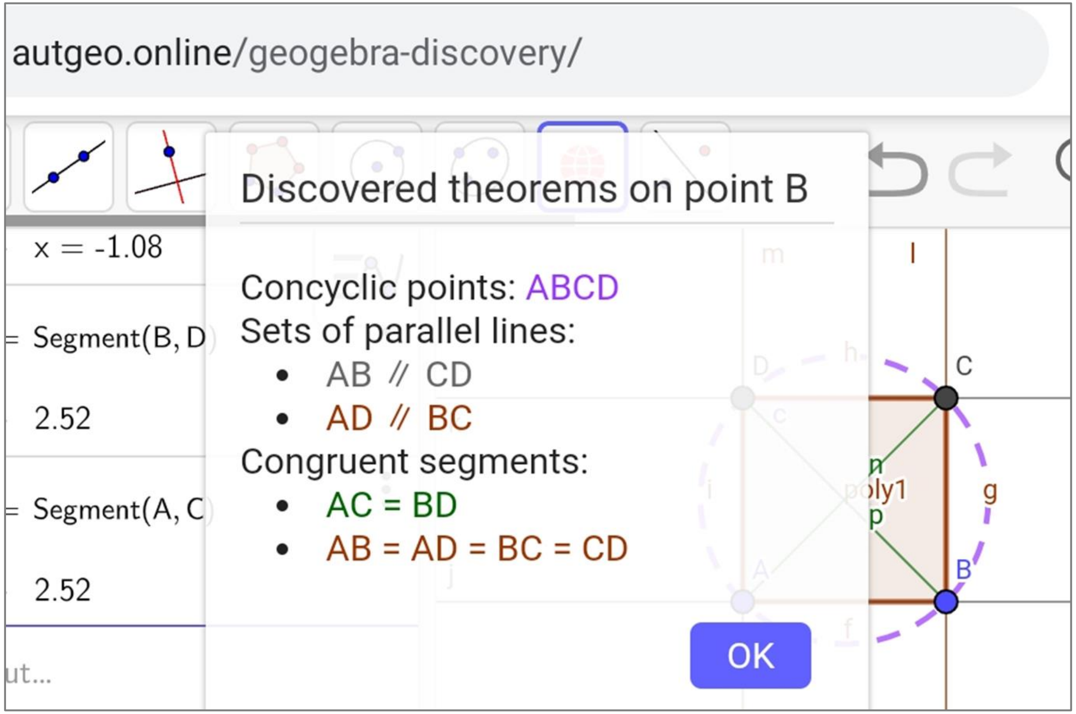
The following theorems can be proven:

1. H∈AB
2. E∈AD
3. G∈BC
4. F∈CD
5. AE=DE
6. AH=BH
7. BG=CG
8. CF=DF
9. EF=GH
10. EH=FG
11. AB∥AH
12. AB∥BH
13. AC∥EF
14. AC∥GH
15. AD∥AE
16. AD∥DE
17. AE∥DE
18. AH∥BH
19. BC∥BG
20. BC∥CG
21. BD∥EH
22. BD∥FG
23. BG∥CG
24. CD∥CF
25. CD∥DF
26. CF∥DF
27. EF∥GH
28. EH∥FG

Finished, found 28 theorems among 1190 possible statements.
Elapsed time: 0h 0m 5s

Automated Geometer: http://www.autgeo.online/ag/automated-geometer.html?offline=1

Through improvements in GeoGebra ART



➢ **GeoGebra Discovery**

What’s GeoGebra Discovery?

GeoGebra Discovery is an experimental version of GeoGebra.

- ✓ some new GeoGebra features to conjecture, discover and prove statements based on complex and real algebraic geometry
- ✓ under development
- ✓ not yet included in the official GeoGebra version

Online version (GeoGebra 6)
http://autgeo.online/geogebra-discovery/

Desktop version (GeoGebra 5)
https://github.com/kovzol/geogebra/releases

GeoGebra Discovery's website
https://github.com/kovzol/geogebra-discovery#geogebra-discovery

GeoGebra Discovery tools and commands

- **`Prove`** and **`ProveDetails`**: proving the truth or failure of a given statement (improved, inequalities).
- **`LocusEquation`**: discovering how to modify a given figure so that a wrong statement becomes true (improved).
- **`Envelope`**: computing the equation of a curve which is tangent to a family of objects while a certain parent of the family moves on a path (improved).
- **`Relation`**: discovering the relation holding among some concrete elements of the given figure (improved and new features, inequalities).
- **`Discover`**: discovering all statements holding true involving one element in the figure selected by the user (**new**, inequalities).
- **`Compare`**: comparison between segment lengths (**new**, inequalities).

ICMI Study "School Maths in the 90's"

Consider, for example, the following question (to other aspects of which we shall wish to refer later):

Two lines are drawn from one vertex of a square to the midpoints of the two non-adjacent sides. They divide the diagonal into three segments (see Figure 5.2).

(a) Are those three segments equal?

(b) Suggest several ways in which the problem can be generalised.

(c) Does your answer to (a) generalise?

(d) Can the argument you used in (a) be used in the more general cases?

(e) If your answer to (d) is 'No', can you find an argument which does generalise?

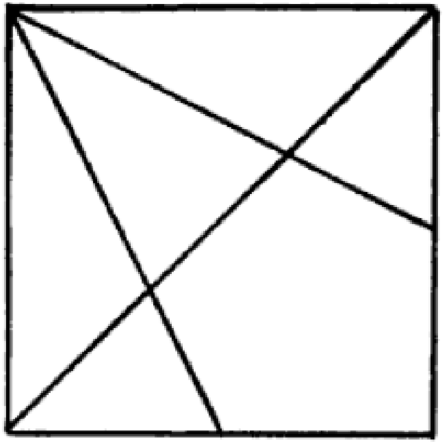


Fig. 5.2

Howson, G., Wilson, B. (1986) ICMI Study series: School mathematics in the 1990's. Cambridge University Press. Kuwait.

… verifying and discovering

`Relation(m,n)`

Relation

It is generally true that:

- **m** and **n** are parallel

under the condition:

- **A** and **B** are not equal

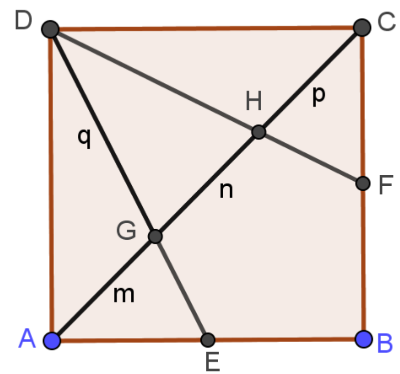
It is generally true that:

- **m** has the same length as **n**

under the condition:

- **A** and **B** are not equal

OK



`Relation(m,q)`

Relation

m does not have the same length as **q** (checked numerically)

More...

OK

Relation

It is generally true that:

- **m** $= (1/5 \cdot \sqrt{10}) \cdot$ **q**

under the condition:

- the construction is not degenerate

OK

Relation detects lenghts ratio

Dealing with inequalities

GEOMETRIC INEQUALITIES

BY

O. BOTTEMA
Delft, The Netherlands

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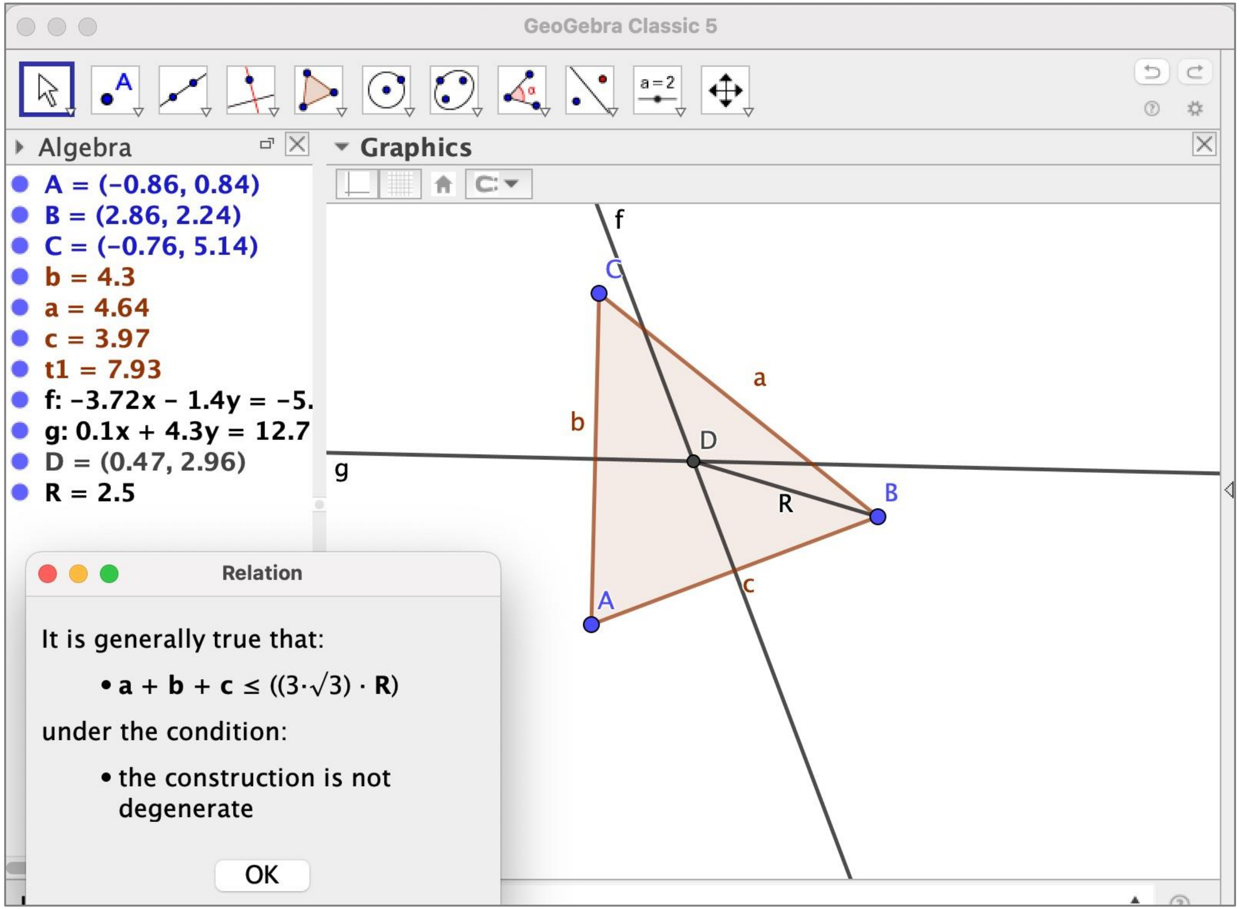
THE SIDES AND THE RADII OF A TRIANGLE

5.3 $a+b+c \leqslant 3R\sqrt{3}.$

Equality holds if and only if $a = b = c$.
S. Nakajima, Tôhoku Math. J. 25 (1925), 115–121.
A. Padoa, Period. Mat. (4) 5 (1925), 80–85.

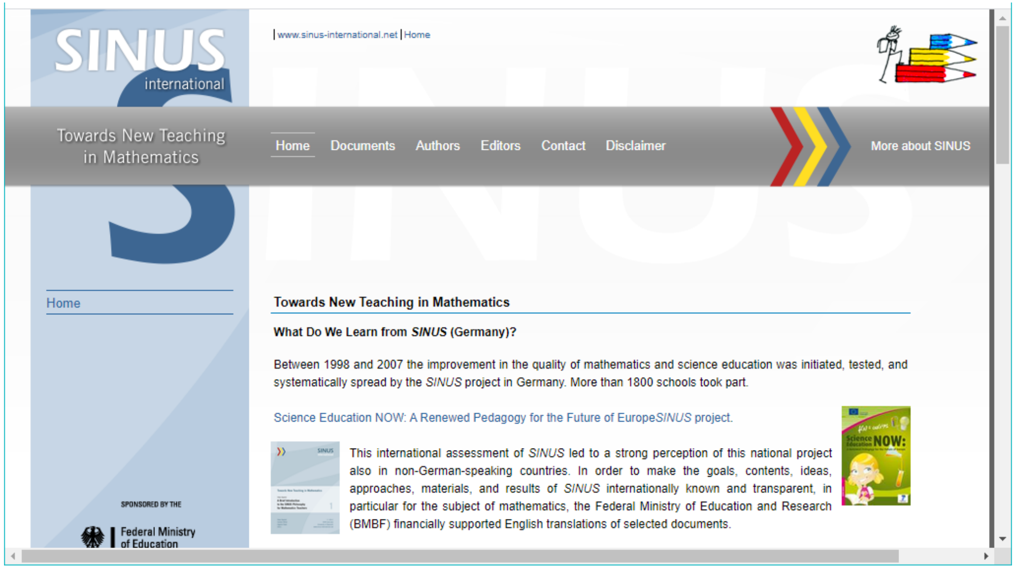
https://www.isinj.com/mt-usamo/Geometric%20Inequalities%20-%20Bottema,%20et.%20al.%20(1968).pdf

Relation gives also inequalities between lengths



Different *milieu* requires different tasks …

"**Open-ended tasks** are any tasks where students are asked to explore objects and to discover and investigate their mathematical properties"
V. Ulm (The SINUS Project 1998-2007)



Volker Ulm (2011) Teaching Mathematics – Opening up Individual Paths to Learning. In: *Towards New Teaching in Mathematics,* 3, SINUS International. http://sinus.uni-bayreuth.de/math/tnt_math_03.pdf

The Treasure Island Problem

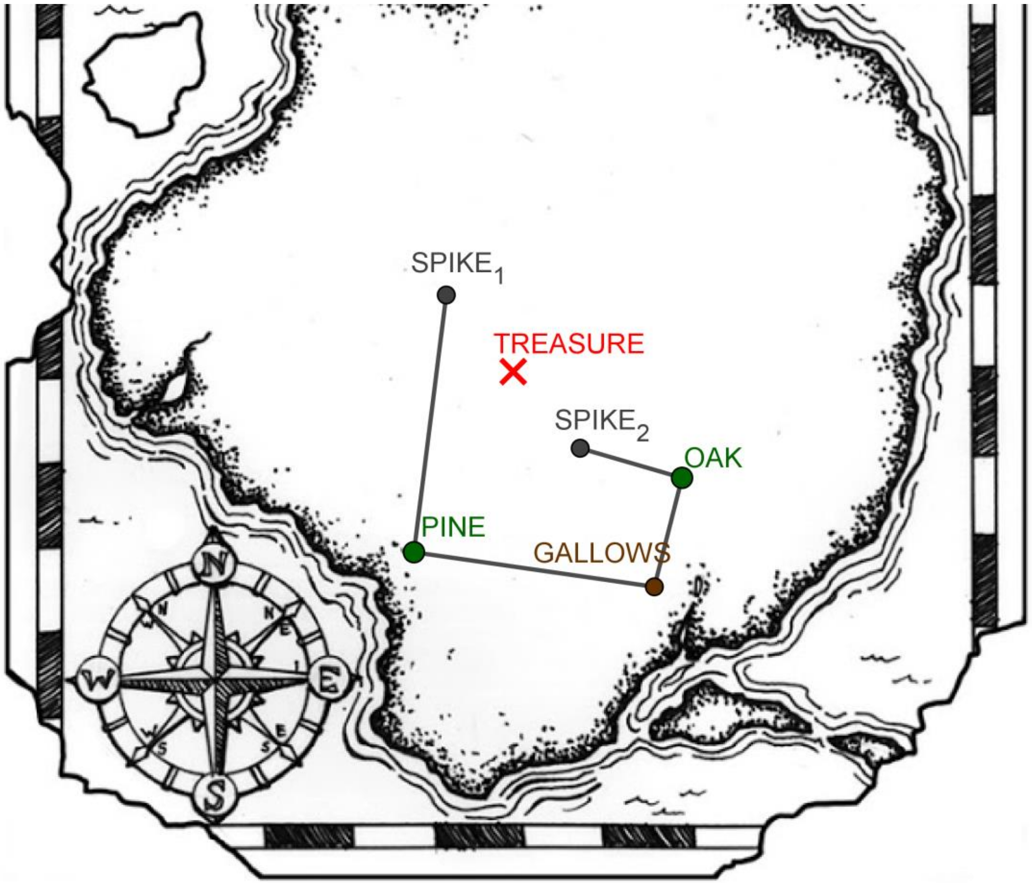
In http://jwilson.coe.uga.edu/EMT725/Treasure/Treasure.html is pointed out that *"In 1948, George Gamow wrote a book called 'One, Two, Three, . . . Infinity'. In it, he presents a problem suggested by a treasure map found in a grandfather's attic"*.

A young man was going through the attic of his grandfather's house and found a paper describing the location of a buried treasure on a particular island. The note said that on the island one would find a gallows, an oak tree, and a pine tree. To locate the treasure one would begin at the gallows, walk to the pine tree, turn right 90 degrees and walk the same number of paces away from the pine tree. A spike was to be driven at that point. Then return to the gallows, walk to the oak tree and turn left 90 degrees and walk the same number of paces away from the oak tree. Drive a second spike in the ground. The midpoint of a string drawn between the two spikes would locate the treasure.

The young man and his friends mounted an expedition to the island, found the oak tree and the pine tree but no gallows. It had been eliminated years ago without a trace. They returned home with the map above and no treasure.

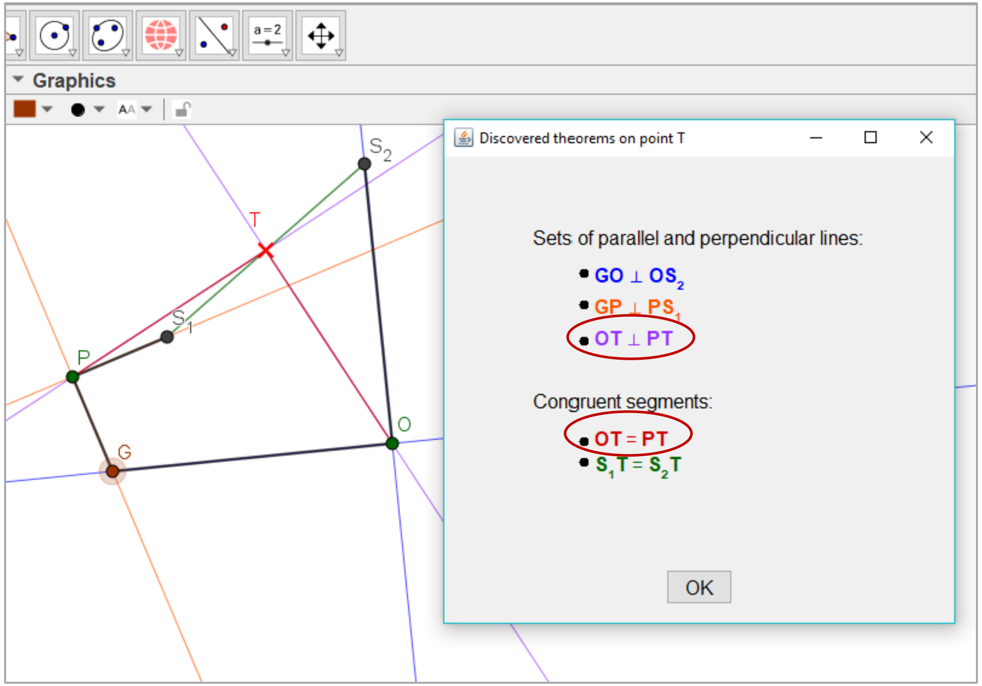
Show them where to look for the treasure.

The Treasure Island Problem-Draw in GeoGebra



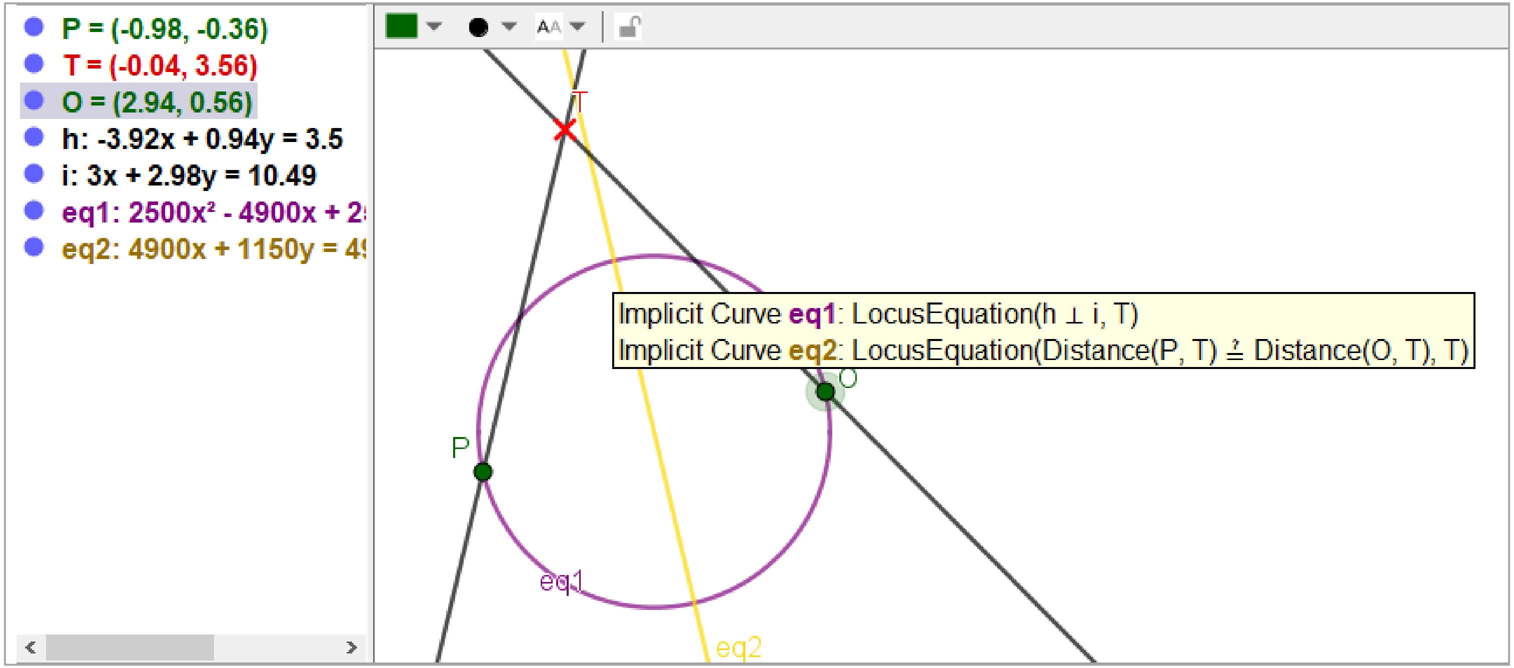
The Treasure Island Problem-Discovery

- OT ⊥ PT: the paths from the trees (P and O) to the treasure (T) are perpendicular
- OT = PT: the treasure (T) is at the same distance from both trees (P and O)



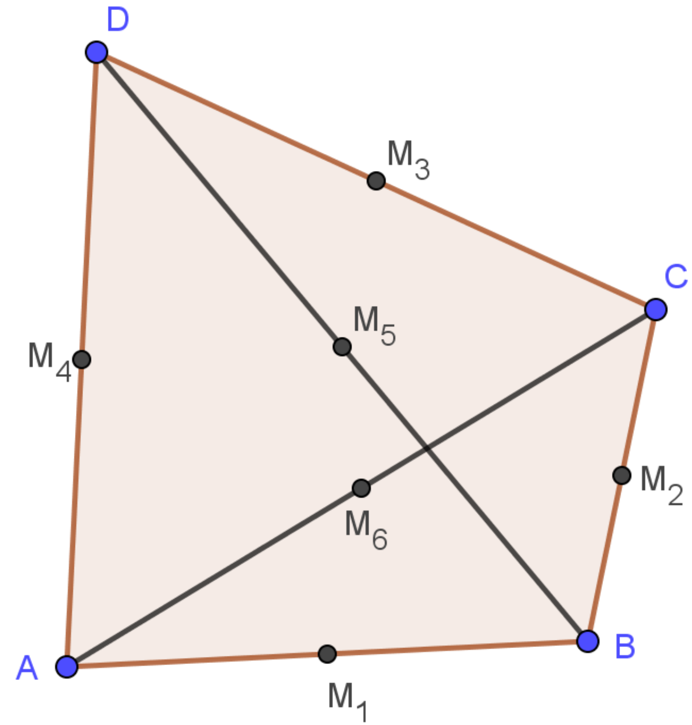
The Treasure Island Problem-Finding T

Ask GeoGebra Discovery where to put T such that OT ⊥ PT and OT =PT



Solving some “Problem Corner”

Problem 1. Let M1, M2, M3, M4, M5, M6 be the midpoints of the edges AB, BC, CD, DA, AC, BD. Prove that the segments M1M3, M2M4, M5M6 are concurrent in a point E that bisects them all.



Kovacs, Z., Recio, T. (2021) Alternative solutions and comments to the Problem Corner, October 2020 issue. The Electronic Journal of Mathematics and Technology.
https://php.radford.edu/~ejmt/ProblemCornerDocs/eJMT_Alternative_Solutions_to_Oct2020.pdf

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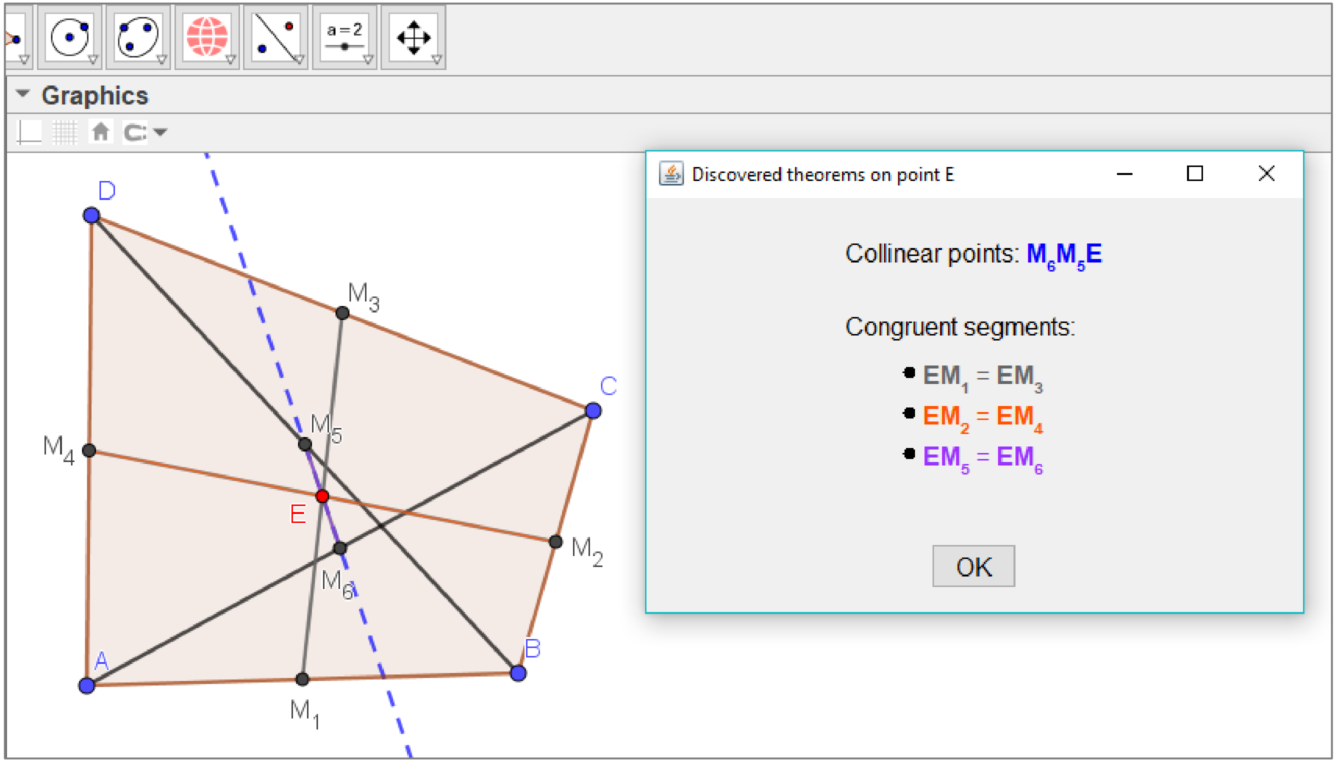
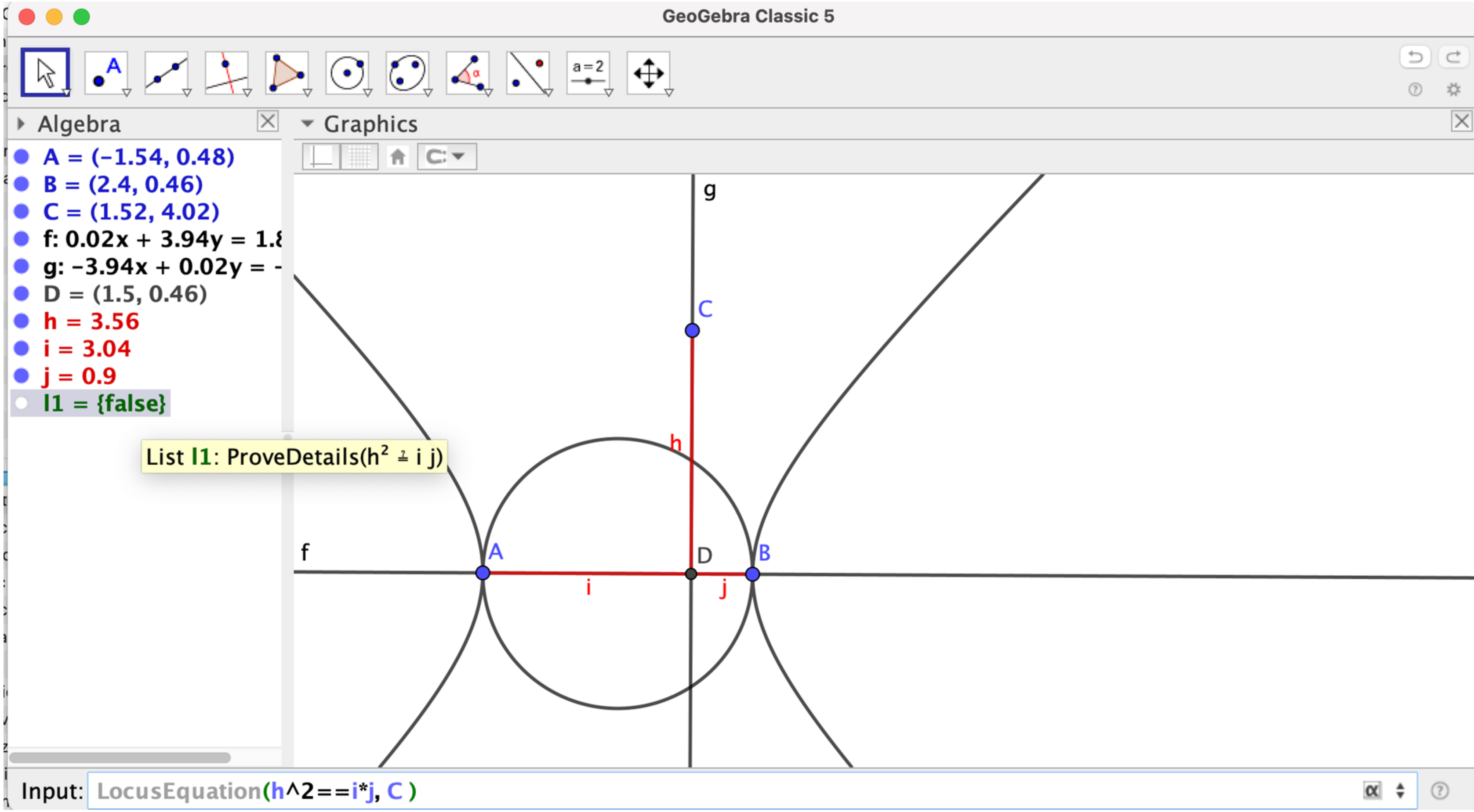
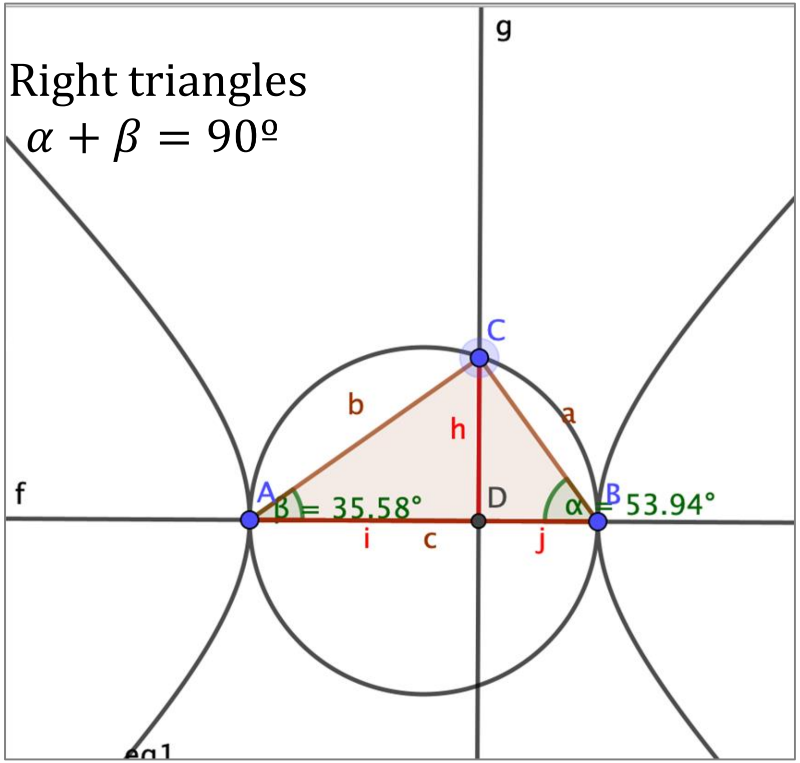
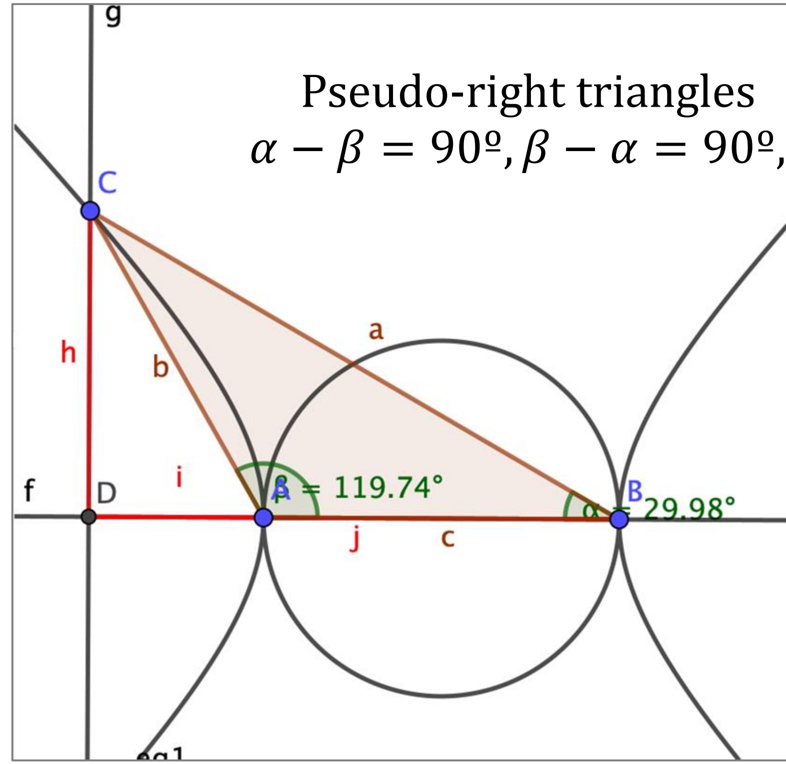


Discovering the Altitude's theorem



Re-discovering the Altitude's theorem





Etayo-Gordejuela, F., de Lucas-Sanz, N., Recio, T., Velez, M.P. (2021) Inventando teoremas con GeoGebra: un nuevo teorema de la altura. Boletín de la Soc. Puig Adam, 111, 8-27

Clough conjecture

This article provided an illustration of the explanatory and discovery functions of proof with an original geometric conjecture made by a Grade 11 student. After logically explaining (proving) the result geometrically and algebraically, the result is generalised to other polygons

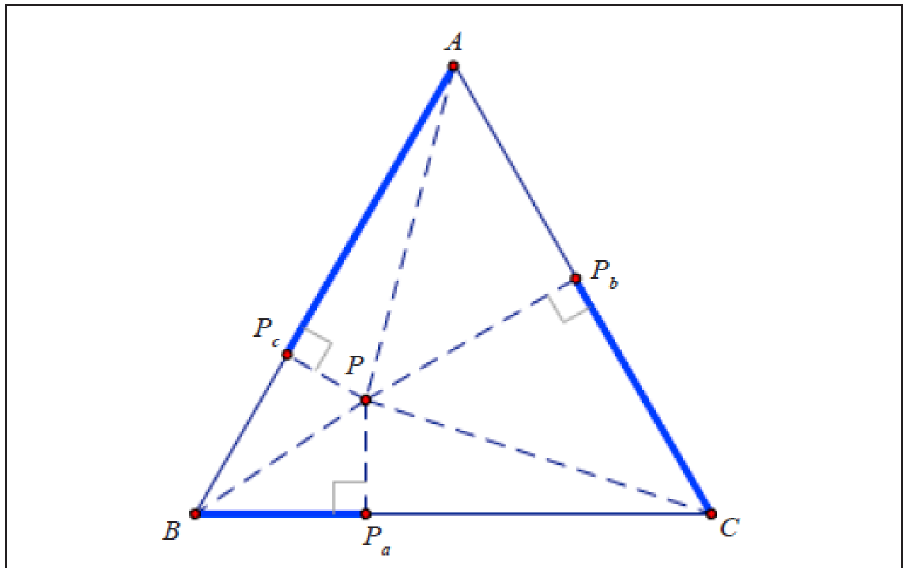
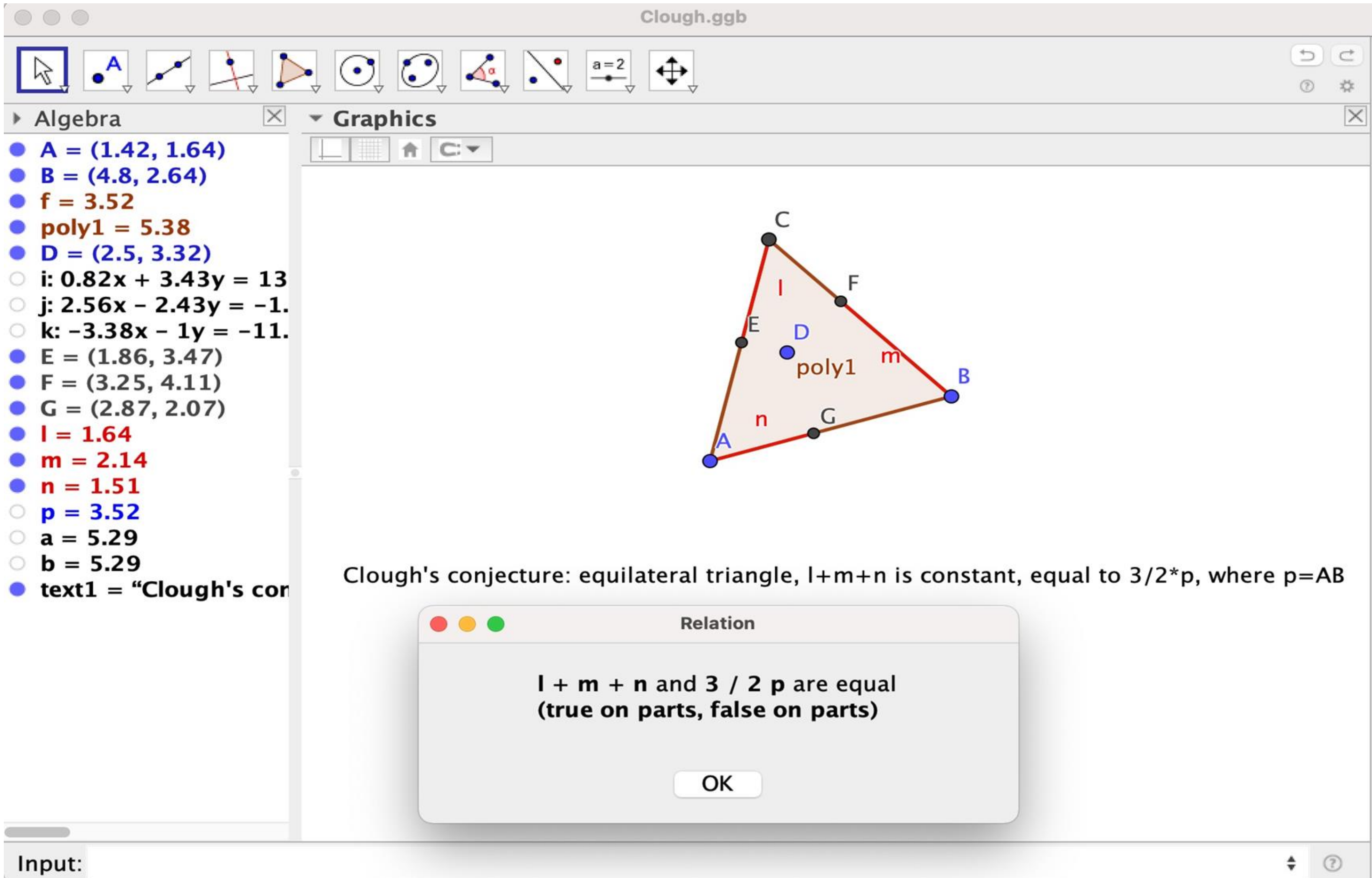


FIGURE 1: Clough's conjecture: $AP_c + BP_a + CP_b$ is constant.

De Villiers, M.((2012) An illustration of the explanatory and discovery functions of proof. Pythagoras, 33(3), 193.
https://doi.org/10.4102/pythagoras.v33i3.193

Clough conjecture with GeoGebra Discovery



Conclusions

- ❑ GGb Discovery is a powerful tool to deal with **open-ended problems**
- ❑ GGb Discovery leads us to **new geometric challenges**
- ❑ GGb Discovery gives a very rich context for developing **human reasoning skills**

But …

- ✓ What is the purpose of developing more and more performing ADG programs?
- ✓ In what context are we interested in having software that finds, e.g. the inequality between the sum of the sides of a triangle and the radius of the circumcircle?

Reflection

Gila Hanna and Xiaoheng (Kitty) Yan (2021) **Opening a discussion on teaching proof with automated theorem provers,** For the Learning of Mathematics, Nov. 2021.

GeoGebra's automated proving tools
GeoGebra …has gained in popularity over the last twenty years and is now widely used… GeoGebra has recently added an Automated Reasoning Tool (ART) to help students conjecture that a given property holds for a specific geometric object and then to find a proof that their conjecture is true. If that is not the case and the property does not hold, ART can also help students make the necessary changes to the original conjecture (Hohenwarter, Kovács, & Recio, 2019, p. 216).

Reflection

Gila Hanna and Xiaoheng (Kitty) Yan (2021) **Opening a discussion on teaching proof with automated theorem provers,** For the Learning of Mathematics, Nov. 2021.

Since the developers of GeoGebra added reasoning tools to their software, they have published a large number of papers in scholarly journals describing the potential of those tools for secondary- school learning…These additions appear to benefit students at both the undergraduate and the secondary level.

It is perhaps too early for empirical studies of classroom experience using the enhancements to GeoGebra… While it is reasonable to expect proof technology to foster students' proving abilities, and there is certainly supporting anecdotal evidence, its potential advantages have not yet been systematically assessed.

Reflection

Gila Hanna and Xiaoheng (Kitty) Yan (2021) **Opening …**

Proof assistants that meet the requirements of these stakeholders

(the curriculum decision makers (who specify the standard of mathematical validation at a given grade), the teachers (who orchestrate learning and decide what counts as a proof in relation to a standard), and the learners (who are simultaneously constructing an understanding of proof and of the related content) Balacheff & Boy de la Tour

will never be developed in the absence of initiative on the part of mathematics educators and a demonstrated demand fuelled by increased use. Secondly, success also requires new and effective teaching strategies. These two efforts stand in a reciprocal relationship, so that the full benefit of proof assistants will be seen only over time as new teaching strategies effect the demand for new tool features and vice versa. The responsibility for both efforts rests squarely on the shoulders of educators

"The key is to make a start, beginning with exploratory studies of the potential of these new tools at both the secondary and post-secondary levels."

https://en.wikipedia.org/wiki/Gila_Hanna

Thank you!

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Tomás Recio, Zoltán Kovács, Francisco Botana, Robert Vadja, Antonio Montes, Pavel Pech, Philippe Richard, Steven Van Vaerenberg , PV, …