The Very Weak Pigeonhole Principle and the hard side of \( \Sigma_1 \)-induction.

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1 Introduction

In this technical report we discuss a proof of the characteristic formula of the very weak pigeonhole principle and show that any prove would require \( O(n) \) inferences per inductive step. The Characteristic formula when instantiated is a \( \Sigma_1 \) statement. The proof schema it is derived from using a single parameter. This work highlights the unexpected complexity which can result from cut-elimination and proof transformation in a schematic setting.

2 Impossibility result for \( C^*(n) \)

Let us consider the clause set \( C^*(n) \) resulting from the following inductive definition.

\[
\begin{align*}
Top(n+1) & \implies \forall x((n + 1) = f(S(x)) \lor f(x) < (n + 1)) \land \\
\forall x((n + 1) = f(x) \lor f(x) < (n + 1)) \land \\
Next(n+1)

Top(0) & \implies Next(0) \land 0 = f(0) \lor 0 = f(S(0))
\end{align*}
\]

\[
Next(n+1) \implies \forall x((\neg(n + 1) = f(x)) \lor (\neg(n + 1) = f(S(x)))) \land \\
\forall x((\neg f(x) < (n + 1)) \lor n = f(x) \lor f(x) < n) \land \\
\forall x((\neg f(S(x)) < (n + 1)) \lor n = f(S(x)) \lor f(x) < n) \land \\
Next(n)
\]

\[
Next(0) \implies (\neg f(0) < 0) \land \forall x((\neg 0 = f(x)) \lor (\neg 0 = f(S(x))))
\]

Being that this clause set was extracted from a schematic proof it is always unsatisfiable. By \( S(\cdot) \) and \( 0 \) we denote the term signature of the individual sort. For the following results we need a few auxiliary definitions.
The following definition can be used to translate between the individual sort and the numeric sort.

\[ Sw(n + 1) \implies S(Sw(n)) \]
\[ Sw(0) \implies 0 \]

The following two definitions define clause sequences derivable from the above clause set.

\[ Seq_1(S(x), n) \implies (n = f(S(S(x))) \lor f(S(x)) < n) \land Seq_1(x, n) \]
\[ Seq_1(0, n) \implies (n = f(S(0)) \lor f(0) < n) \]
\[ Seq_2(S(x), n) \implies (n = f(S(x)) \lor f(S(x)) < n) \land Seq_2(x, n) \]
\[ Seq_2(0, n) \implies (n = f(0) \lor f(0) < n) \]

We will now consider the following clause set

\[ C(n) = Seq_1(Sw(n), n) \land Seq_2(Sw(n), n) \land Next(n) \]

**Lemma 1.** \( C(n) \) is derivable from \( C^*(n) \)

**Theorem 1.** For all \( n > 1 \), \( C(n) \vdash \) is provable.

Proof.

we will prove by induction on \( n \). Thus, by branch we are referring to the application \( \lor : \). If a branch contains both a negative and a positive instances of a literal (that is \( P \) and \( \neg P \) then it may be closed. let \( n = 1 \), then

\[ C(1) = Seq_1(S(0), 1) \land Seq_2(S(0), 1) \land Next(1) \]

and we must show that \( C(1) \vdash \) is provable. Unrolled \( C(1) \) results in the sequent \( \Gamma \vdash \) which contains the following formulas:

\[ (1 = f(S(0))) \lor f(S(0)) < 1) \]
(1)
\[ (1 = f(S(0))) \lor f(0) < 1) \]
(2)
\[ (1 = f(S(0))) \lor f(S(0)) < 1) \]
(3)
\[ (1 = f(0)) \lor f(0) < 1) \]
(4)
\[ \forall x((-1 = f(x)) \lor (-1 = f(S(x)))) \]
(5)
\[ \forall x((-f(x) < 1) \lor 0 = f(x) \lor f(x) < 0) \]
(6)
\[ \forall x((-f(S(x)) < 1) \lor 0 = f(S(x)) \lor f(x) < 0) \]
(7)
\[ (-f(0) < 0) \]
(8)
\[ \forall x((-0 = f(x)) \lor (-0 = f(S(x)))) \]
(9)
Now let us consider the 16 sequents constructed from the clauses (1)–(4) where \( \Delta \) contains clauses (5)–(9). These sequents are the leaves of the tree found in Figure 1.

\[
1 = f(S(S(0))), 1 = f(S(0)), 1 = f(S(0))), 1 = f(0), \Delta \vdash \tag{10}
\]
\[
1 = f(S(S(0))), f(0) < 1, 1 = f(S(0))), 1 = f(0), \Delta \vdash \tag{11}
\]
\[
f(S(0)) < 1, 1 = f(S(0))), 1 = f(S(0))), 1 = f(0), \Delta \vdash \tag{12}
\]
\[
1 = f(S(S(0))), 1 = f(S(0))), f(0) < 1, 1 = f(0), \Delta \vdash \tag{13}
\]
\[
1 = f(S(S(0))), f(0) < 1, f(S(0))) < 1, 1 = f(0), \Delta \vdash \tag{14}
\]
\[
1 = f(S(S(0))), f(0) < 1, f(S(0))) < 1, 1 = f(0), \Delta \vdash \tag{15}
\]
\[
f(S(0)) < 1, 1 = f(S(0))), f(0) < 1, 1 = f(0), \Delta \vdash \tag{16}
\]
\[
1 = f(S(S(0))), 1 = f(S(0))), f(0) < 1, 1 = f(0), \Delta \vdash \tag{17}
\]
\[
1 = f(S(S(0))), f(0) < 1, 1 = f(S(0))), 1 = f(0), \Delta \vdash \tag{18}
\]
\[
f(S(0)) < 1, 1 = f(S(0))), f(0) < 1, 1 = f(0), \Delta \vdash \tag{19}
\]
\[
f(S(0)) < 1, 1 = f(S(0))), f(0) < 1, 1 = f(0), \Delta \vdash \tag{20}
\]
\[
f(S(0)) < 1, 1 = f(S(0))), f(0) < 1, 1 = f(0), \Delta \vdash \tag{21}
\]
\[
f(S(0)) < 1, 1 = f(S(0))), 1 = f(S(0))), 1 = f(0), \Delta \vdash \tag{22}
\]
\[
f(S(0)) < 1, 1 = f(S(0))), f(0) < 1, 1 = f(0), \Delta \vdash \tag{23}
\]
\[
f(S(0)) < 1, 1 = f(S(0))), f(0) < 1, 1 = f(0), \Delta \vdash \tag{24}
\]
\[
f(S(0)) < 1, 1 = f(S(0))), f(0) < 1, 1 = f(0), \Delta \vdash \tag{25}
\]

If a branch contains both \( 1 = f(S(0)) \) and \( 1 = f(0) \) or \( 1 = f(S(0))) \) and \( 1 = f(S(S(0))) \) then it can be closed using (5) from \( \Delta \). For example consider branch (10)

\[
1 = f(S(S(0))) \vdash 1 = f(S(0)) \quad 1 = f(0) \vdash 1 = f(0)
\]
\[
1 = f(S(S(0))), -1 = f(S(0)), \Delta \vdash \quad 1 = f(0), -1 = f(0), \Delta \vdash
\]
\[
1 = f(S(0))), 1 = f(0), (\neg 1 = f(0)) \lor (\neg 1 = f(S(0))) \lor (\neg 1 = f(S(0))), \Delta \vdash
\]
\[
1 = f(S(S(0))), 1 = f(S(0))), 1 = f(S(0))), 1 = f(0), \Delta \vdash
\]

This leaves the following branches:

\[
1 = f(S(S(0))), f(0) < 1, f(S(0))) < 1, 1 = f(0), \Delta \vdash \tag{26}
\]
\[
f(S(0)) < 1, f(0) < 1, 1 = f(S(0))), 1 = f(0), \Delta \vdash \tag{27}
\]
\[
f(S(0)) < 1, 1 = f(S(0))), 1 = f(S(0))), f(0) < 1, \Delta \vdash \tag{28}
\]
\[
f(S(0)) < 1, 1 = f(S(0))), f(0) < 1, 1 = f(S(0))), \Delta \vdash \tag{29}
\]
\[
f(S(0)) < 1, 1 = f(S(0))), f(S(0))) < 1, f(0) < 1, \Delta \vdash \tag{30}
\]
\[
f(S(0)) < 1, f(0) < 1, f(S(0))) < 1, f(0) < 1, \Delta \vdash \tag{31}
\]
Fig. 1. Proof tree construction.
We can simplify these sequents by application of weakening and contraction rules so that they are all of the form

\[ f(0) < 1, f(S(0)) < 1, \Delta \vdash \]  \hspace{1cm} \hspace{1cm} (32)

Using these two literals and (6) & (7) we derive a sequent containing the following three clauses:

\[ 0 = f(0) \lor f(0) < 0 \]  \hspace{1cm} \hspace{1cm} (33)
\[ 0 = f(S(0)) \lor f(S(0)) < 0 \]  \hspace{1cm} \hspace{1cm} (34)
\[ 0 = f(S(0)) \lor f(0) < 0 \]  \hspace{1cm} \hspace{1cm} (35)

We will refer to the sequent containing (33), (34), and (35) as \( S \) and can construct the following proof tree starting at \( S \):

\[
\begin{array}{cccccc}
36 & 37 & \lor : l & 38 & 39 & \lor : l \\
(35) & (35) & (35) & (35) & (35) & (35) \\
40 & 41 & \lor : l & 42 & 43 & \lor : l \\
(34) & (34) & (34) & (34) & (34) & (34) \\
(33) & (33) & (33) & (33) & (33) & (33)
\end{array}
\]

Where the leaves are the following sequents

\[ 0 = f(0), 0 = f(S(0)), 0 = f(S(0)), \Delta \vdash \]  \hspace{1cm} \hspace{1cm} (36)
\[ 0 = f(0), 0 = f(S(0)), 0 = f(S(0)), \Delta \vdash \]  \hspace{1cm} \hspace{1cm} (37)
\[ 0 = f(0), 0 = f(S(0)), f(0) < 0, \Delta \vdash \]  \hspace{1cm} \hspace{1cm} (38)
\[ 0 = f(0), f(S(0)) < 0, f(0) < 0, \Delta \vdash \]  \hspace{1cm} \hspace{1cm} (39)
\[ f(0) < 0, 0 = f(S(0)), 0 = f(S(0)), \Delta \vdash \]  \hspace{1cm} \hspace{1cm} (40)
\[ f(0) < 0, f(S(0)) < 0, 0 = f(S(0)), \Delta \vdash \]  \hspace{1cm} \hspace{1cm} (41)
\[ f(0) < 0, 0 = f(S(0)) f(0) < 0, \Delta \vdash \]  \hspace{1cm} \hspace{1cm} (42)
\[ f(0) < 0, f(S(0)) < 0, f(0) < 0, \Delta \vdash \]  \hspace{1cm} \hspace{1cm} (43)

Sequent (38)–(43) contain \( f(0) < 0 \), and thus can be closed using (8). This leaves sequents (36) and (37) which can be closed using (9). Thus we have shown that \( C(1) \vdash \) is provable. For the step case we assume that \( C(n) \vdash \) is provable and show that \( C(n + 1) \vdash \) is provable.

Let us consider the sequent \( Seq_1(Sw(n), n), Seq_2(Sw(n), n), Next(n) \vdash \) which can be constructed from \( C^*(n + 1) \vdash \) using the clauses

\[ \forall x (\neg f(x) < (n + 1)) \lor n = f(x) \lor f(x) < n \]  \hspace{1cm} \hspace{1cm} (44)
\[ \forall x (\neg f(S(x)) < (n + 1)) \lor n = f(S(x)) \lor f(x) < n \]  \hspace{1cm} \hspace{1cm} (45)

found in \( Next(n + 1) \). We denote the following literal set by \( \Gamma \):
\[ f(0) < (n + 1) , f(S(0))) < (n + 1) , \cdots , f(S(w(n))) < (n + 1) \]  
(46)

These observations entail the soundness of the following proof tree:

\[
\vdots
\]

\[ \Gamma, \text{Next}(n + 1) \vdash \]

Notice the similarities between \( \Gamma, \text{Next}(n + 1) \vdash \) and (32). Essentially, if we consider the sequent \( \text{Seq}_1(Sw(n+1), n+1), \text{Seq}_2(Sw(n+1), n+1), \text{Next}(n+1) \vdash \), apply \( \lor : l \) to each clause of \( \text{Seq}_1(Sw(n+1), n+1) \) and \( \text{Seq}_2(Sw(n+1), n+1) \) one, close all branches where \( \forall x(((n+1) = f(x)) \lor -(n + 1) = f(S(x))) \) can be applied, and finally apply weakenings and contraction as in the basecase \( \Gamma, \text{Next}(n + 1) \vdash \) would be the resulting sequent. To see that this would be the case we would have to show that the sequents at the leaves of this construction can either be closed by \( \forall x(((n+1) = f(x)) \lor -(n + 1) = f(S(x))) \) or they contain \( \Gamma \). This follows directly from the definition of \( \text{Seq}_1 \) and \( \text{Seq}_2 \). Notice that \( \text{Seq}_1(k, n + 1) \) and \( \text{Seq}_2(k, n + 1) \) for some \( 0 \leq k \leq n + 1 \) add the following two formula to the sequent:

\[
(n + 1) = f(S(k)) \lor f(k) < (n + 1) \]  
(47)

\[
(n + 1) = f(k) \lor f(k) < (n + 1) \]  
(48)

From these two formula we can make the following four sequents:

\[
(n + 1) = f(S(k)), (n + 1) = f(k), \Delta \vdash \]  
(49)

\[
(n + 1) = f(S(k)), f(k) < (n + 1), \Delta \vdash \]  
(50)

\[
f(k) < (n + 1), f(k) < (n + 1), \Delta \vdash \]  
(51)

\[
f(k) < (n + 1), (n + 1) = f(k), \Delta \vdash \]  
(52)

Notice that only (49) can be closed by \( \forall x(((n+1) = f(x)) \lor -(n + 1) = f(S(x))) \), but every other branch contains at least one instance of \( (n + 1) = f(k) \). Thus, every branch which not closed will contain at least one instance \( (n + 1) = f(k) \) for each \( 0 \leq k \leq n + 1 \). This is precisely \( \Gamma \). Thus it follows that the proof tree

\[
\vdots
\]

\[ \vdots \]

\[ \Gamma, \text{Next}(n + 1) \vdash \]

\[
\vdots
\]

\[ \vdots \]

\[ \text{Seq}_1(Sw(n + 1), n + 1), \text{Seq}_2(Sw(n + 1), n + 1), \text{Next}(n + 1) \vdash \]
is a sound derivation. The root of this derivation is precisely $C(n + 1) \vdash$. By
the induction hypothesis that $C(n) \vdash$ is provable which is the leave sequent of
every open branch we see that $C(n + 1) \vdash$ is also provable.

Now that we have shown $C(n) \vdash$ to be provable for all $n$ we show that any
smaller clause set is not provable. While this can be done easily by removing
clauses such as (8) from the clause set it is not as informational as manipulating
the induction hypothesis that $C$ every open branch we see that $C(n + 1) \vdash$ is also provable.

\[ \square \]

These definition allow us to remove particular clauses from the two sequences
defined earlier. We will now consider the following two clause sets

\[ C_1(n, k) = Seq_1(Sw(n), n, k) \land Seq_2(Sw(n), n) \land Next(n) \]
\[ C_2(n, k) = Seq_1(Sw(n), n) \land Seq'_2(Sw(n), n, k) \land Next(n) \]

where $0 \leq k \leq n$. Using these definitions we prove the following theorem:

**Theorem 2.** For all $n > 1$ and $0 \leq k \leq n$, $C_1(n, k) \vdash$ and $C_2(n, k) \vdash$ are not provable.

**Proof.**

We will prove by induction on $n$ and by case distinction on $k$. Let $n = 1$, then
we are considering the following four sequents $C_1(1, 0) \vdash$, $C_2(1, 0) \vdash$, $C_1(1, 1) \vdash$,
$C_2(1, 1) \vdash$. When $\Delta$ denotes clauses (5)–(9), we can represent the sequents as follows:

\[ seq_1(Sw(1), 1, 0) \land Seq_2(Sw(1), 1), 0 \Delta \vdash \] (53)
\[ seq_1(Sw(1), 1, 1) \land Seq_2(Sw(1), 1), 1 \Delta \vdash \] (54)
\[ Seq_1(Sw(1), 1), Seq'_2(Sw(1), 1, 0), \Delta \vdash \] (55)
\[ Seq_1(Sw(1), 1), Seq'_2(Sw(1), 1, 1), \Delta \vdash \] (56)

We can replace $seq_1, seq'_1, seq_2,$ and $seq'_2$ by formula as follows:
As was done in the case of Theorem 1 we can apply $\lor : l$ to each of the formula derived above and consider which branches can be closed by formula of $\Delta$ and which cannot be. In this case $\Delta$ is $\text{next}(1)$. Below we highlight the branches derived from the above sequents which cannot be closed using (5).

\[(53)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(0) \lor f(0) < 1\]

\[(54)\]
\[1 = f(S(0)) \lor f(0) < 1\]
\[1 = f(S(0)) \lor f(0) < 1\]
\[1 = f(0) \lor f(0) < 1\]

\[(55)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(0) \lor f(S(0)) < 1\]

\[(56)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(0) < 1\]
\[1 = f(0) \lor f(S(0)) < 1\]

\[(57)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(0) \lor f(0) < 1\]

\[(58)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(0) < 1\]
\[1 = f(0) \lor f(0) < 1\]

\[(59)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(0) \lor f(0) < 1\]

\[(60)\]
\[1 = f(S(0)) \lor f(0) < 1\]
\[1 = f(S(0)) \lor f(0) < 1\]
\[1 = f(0) \lor f(0) < 1\]

\[(61)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(0) \lor f(S(0)) < 1\]

\[(62)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(0) < 1\]
\[1 = f(0) \lor f(S(0)) < 1\]

\[(63)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(0) \lor f(0) < 1\]

\[(64)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(0) \lor f(0) < 1\]

\[(65)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(0) \lor f(S(0)) < 1\]

\[(66)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(0) \lor f(S(0)) < 1\]

\[(67)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(0) \lor f(S(0)) < 1\]

\[(68)\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(S(0)) \lor f(S(0)) < 1\]
\[1 = f(0) \lor f(S(0)) < 1\]

\[(69)\]
\[f(S(0)) < 1 , f(S(0)) < 1 , 1 = f(0), \Delta \vdash\]

\[(70)\]
\[f(0) < 1 , 1 = f(S(0)), f(0) < 1, \Delta \vdash\]

\[(71)\]
\[f(S(0)) < 1 , 1 = f(S(0)), f(S(0)) < 1, \Delta \vdash\]
\[ 1 = f(S(S(0))), f(0) < 1, f(0) < 1, \Delta \vdash \quad (72) \]

Unlike the basecase in of Theorem 1 applying weakening and contraction results in more than one branch. In particular, this results in the following:

\[(54) \& (56)\]

\[f(0) < 1, \Delta \vdash \quad (73)\]

\[(53) \& (55)\]

\[f(S(0)) < 1, \Delta \vdash \quad (74)\]

Now let us consider the result of apply (6) and (7) to (73) and (74). The result is as follows

\[(54) \& (56)\]

\[0 = f(0) \lor f(0) < 0, \Delta \vdash \quad (75)\]

\[(53) \& (55)\]

\[(0 = f(S(0)) \lor f(0) < 0), (0 = f(S(0)) \lor f(S(0)) < 0), \Delta \vdash \quad (76)\]

The branch \(0 = f(0), \Delta \vdash\) entails from (75) and cannot be closed. Also the branch \(0 = f(S(0)), f(S(0)) < 0, \Delta \vdash\) and cannot be closed. Thus, none of the sequents \(C_1(1,0) \vdash, C_2(1,0) \vdash, C_1(1,1) \vdash, C_2(1,1) \vdash\) are provable.

Now let use assume that the sequents \(C_1(n,k) \vdash\) and \(C_2(n,k) \vdash\) are not provable for each \(0 \leq k \leq n\), we now show that the clause sets \(C_1(n+1,k) \vdash\) and \(C_2(n+1,k) \vdash\) are not provable for any \(0 \leq k \leq n+1\).

We prove the step case by a case distinction on \(k\): when \(k = n+1\), the missing clause from \(C_1(n+1,n+1)\) is

\[n + 1 = f(S(sw(n+1))) \lor f(sw(n+1)) < n + 1 \quad (77)\]

and the missing clause of \(C_2(n+1,n+1)\) is

\[n + 1 = f(sw(n+1)) \lor f(sw(n+1)) < n + 1 \quad (78)\]

In the case of \(C_1(n+1,n+1)\) the only clause containing \(f(sw(n+1)) < n+1\) is (78) and in the case of \(C_2(n+1,n+1)\) the only clause containing \(f(sw(n+1)) < n+1\) is (77). Now consider the branches which have the following form:

\[f(0) < n + 1, \ldots, f(sw(n)) < n + 1, n + 1 = f(sw(n+1)) \quad (79)\]

\[f(0) < n + 1, \ldots, f(sw(n)) < n + 1, n + 1 = f(S(sw(n+1))) \quad (80)\]
Being that neither branch contains the literal \( f(sw(n + 1)) < n + 1 \) we cannot derive the following formula:

\[
\begin{align*}
  n &= f(sw(n + 1)) \lor f(sw(n + 1)) < n \\
  n &= f(S(sw(n))) \lor f(sw(n)) < n
\end{align*}
\]

(81)

(82)

While (81) isn’t really of interest \( n = f(S(sw(n))) \lor f(sw(n)) < n \) is a clause of \( C(n) \) which cannot be derived from

\[
f(0) < n + 1, \, \cdots, \, f(sw(n)) < n + 1
\]

and thus on this branch we can only derive \( C_1(n, n) \vdash \) and by the induction hypothesis \( C_1(n, n) \vdash \) is not provable. Now what is left to show is that the same can be done of \( k < n + 1 \). The missing clause from \( C_1(n + 1, k) \) is

\[
\begin{align*}
  n + 1 &= f(S(sw(k))) \lor f(sw(k)) < n + 1
\end{align*}
\]

(83)

and the missing clause of \( C_2(n + 1, k) \) is

\[
\begin{align*}
  n + 1 &= f(sw(k)) \lor f(sw(k)) < n + 1
\end{align*}
\]

(84)

Like in the previous case, for \( C_1(n + 1, k) \) the only clause containing \( f(sw(k)) < n + 1 \) is (84) and in the case of \( C_2(n + 1, k) \) the only clause containing \( f(sw(k)) < n + 1 \) is (83). Now consider the branches which have the following form:

\[
\begin{align*}
  f(0) < n + 1, \, \cdots, \, n + 1 = f(sw(k)) \lor \cdots, \, f(sw(n + 1)) < n + 1 \\
  f(0) < n + 1, \, \cdots, \, n + 1 = f(S(sw(k))) \lor \cdots, \, f(sw(n + 1)) < n + 1
\end{align*}
\]

(85)

(86)

Being that neither branch contains the literal \( f(sw(k)) < n + 1 \) we cannot derive the following clauses:

\[
\begin{align*}
  n &= f(sw(k)) \lor f(sw(k)) < n \\
  n &= f(S(sw(k - 1))) \lor f(sw(k - 1)) < n
\end{align*}
\]

(87)

(88)

If \( k = 0 \) we only need to consider line (87) in which case we can only derive \( C_2(n, 0) \) on the branch which is satisfiable by the induction hypothesis. If \( 0 < k < n + 1 \) then we can only derive the sequent.

\[
Seq_1'(Sw(n), n, k) \land Seq_2'(Sw(n), n, k) \land Next(n) \vdash
\]

which is missing two clauses instead of just one and being that both \( C_1(n, k) \) and \( C_2(n, k) \) are not provable a subset of either one of them is also not provable. Thus we have proven the theorem by induction.
What Theorem 1 & 2 tell us is that $C^*(n) \vdash$ is only provable when we have $n$ instances of each of the clauses from $Top(n)$. These clauses only show up it $Top(n)$ and after deriving the $n$ instances we still need to perform an induction. Thus any program realizing a proof of $C^*(n) \vdash$ will be of the following form

$$
P(n + 1, k) \rightarrow G(n, k, P(n, k))
$$

$$
P(0, k) \rightarrow H(k)
$$

$$
H(n + 1) \rightarrow G'(n, H(n))
$$

$$
H(0) \rightarrow R
$$

This also implies that some of the sequents in the proof are not finite in size. Thus we can state the following corollary.

**Corollary 1.** A recursive proof of $C^*(n) \vdash$ cannot be formalized as a LOOP$_1$ program.

This results imply that the formalism of [1] cannot be used to formalize a refutation of the clause set constructed from $Top(n)$. Furthermore, $C^*(\bar{n}) \vdash$ where $\bar{n}$ is a numeral is a $\Sigma_1$ statement. Thus any proof of the sequent $C^*(\bar{n}) \vdash$ requires $\Sigma_1$-induction. The study carried out by Janek Vierlings in his master thesis points towards the possible incompleteness of the superposition calculus used by the formalism developed in [2]. The fact that we require $n$ instances of the clauses indexed by $n$ makes this clause set a prime example for such an incompleteness result. Thus, this example is most likely out of the scope of all current formalisms.

**References**