A Cyclic Tree Construction for Recursive Resolution Refutations

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1 Introduction

I outline the construction of finite saturated tree, how to decorate them, and finally provide a resolution refutation of the NIA-schema’s characteristic clause set. The report ends with a short passage concerning how this formalism can be used to construct a completeness proof.

2 Point Spaces, Curves, and Recursive Trees

Definition 1. We define the set \( \mathcal{PR} \) of primitive recursive functions, \( f : \mathbb{N} \to \mathbb{N} \), as follows:

1. \( 0 \in \mathcal{PR} \)
2. if \( x \in \mathcal{PR} \), then \( s(x) \in \mathcal{PR} \)
3. If \( x_1, \ldots, x_n \in \mathcal{PR} \), then we define \( P^n_i(x_1, \ldots, x_n) = x_i \), for each \( 1 \leq i \leq n \).
4. If \( x_1, \ldots, x_n, g_1, \ldots, g_k, f \in \mathcal{PR} \) such that \( g_1, \ldots, g_k \) are \( n \)-ary and \( f \) is \( k \)-ary then the function \( f(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)) \in \mathcal{PR} \)
5. If \( x_1, \ldots, x_n, g, f \in \mathcal{PR} \) such that \( g \) is \( n \)-ary and \( f \) is \( n+2 \)-ary then the function

\[
\begin{align*}
h(0, x_1, \ldots, x_n) &= g(x_1, \ldots, x_n) \\
h(s(k), x_1, \ldots, x_n) &= f(k, h(k, x_1, \ldots, x_n), x_1, \ldots, x_n)
\end{align*}
\]

is in \( \mathcal{PR} \)

essentially the functions constructed in Definition 1 will be the term signature used for the construction of term spaces. We assume a countable set of parameter symbols \( \Sigma = \{ k_i \}_{i=0}^{\infty} \) used for term construction. Terms constructed from cases (1) and (2) of Definition 1 which are parameter free will be referred to ground terms \( \mathcal{G} \), that is all terms built from the signature \( \{ s(\cdot), 0 \} \). We will denote ground terms by overline lowercase latin characters, i.e. \( \overline{t} \). The set of terms constructible from the signature \( \mathcal{PR} \) together with \( \mathcal{P} \) will be referred to as \( \mathcal{T} \). In particular we will consider spaces of \( m \)-tuples of terms denoted by \( \mathcal{T}^m \) for \( m \in \mathbb{N} \) which we will refer to as term spaces. Term spaces are assumed to be ordered by a well-founded ordering \( \leq_m \) (by \( \prec_m \) we are referring to the irreflexive version of \( \leq_m \)). Note that by \( \leq_m \) we mean there is an ordering each \( m \) where \( m \) is the dimension of a term space. When representing a particular tuple \( v \in \mathcal{T}^m \) we will
write \( v = [n_1, \ldots, n_m] \). Let \( v \in T^m \) and \( t \in T \), by \( V(v) \) and \( V(t) \) we refer to the parameters of \( P \) used in the construction of \( v \) and \( t \).

We need to consider paths through a term space \( T^m \). This implies some sort of external ordering/mechanism for deciding the next point in a term space which is on a path. In order to construct such paths, we define path symbols \( \mathcal{R} = \{ \rho_i \}_{i=0}^{\infty} \) which are ordered by an independent well-founded ordering \( \leq_R \) (by \( \prec_R \) we are referring to the irreflexive version of \( \leq_R \), that is independent from the well founded ordering on the associated term space.

Each path symbol in \( \mathcal{R} \) is associated with a term space of a fixed size and a list of so called “scratch pad” terms of a fixed size. We define a total function, referred to as an arity function \( A : \mathcal{R} \to (\mathbb{N}, \mathbb{N}) \), mapping each path symbol to a pair natural numbers such that \( A(x, 0) \) and \( A(x, 1) \) refer to the first and second value in the pair \( A(x) \), respectively. The value of \( A(x, 0) \) is the size of the term space associated with \( x \) and \( A(x, 1) \) is the size of the scratch pad.

**Definition 2.** The point space \( \mathcal{P}^*(A) \) over an arity function \( A \) is the space of all 3-tuples of the form \((\rho, v, \{a_1, \ldots, a_{A(\rho, 1)}\})\), where \( \rho \in \mathcal{R} \), \( v \in T^A(x, 0) \) and \( a_1, \ldots, a_{A(\rho, 1)} \in T \). We refer to the 3-tuples of \( \mathcal{P}^*(A) \) as points.

Notice that the point space is not defined over a single term space but a countably infinite sequence of term spaces. This is necessary for the ordering and expansion rewrite rules. These rewrite rules allow us to define curves through a point space.

**Definition 3.** A rewrite rule \( \Rightarrow \) : \( \mathcal{P}^*(A) \to \mathcal{P}^*(A) \) is well formed if for every \( r, d \in \mathcal{P}^*(A) \), if \( r \Rightarrow d \) then \( V(d) \subseteq V(r) \).

There are two important well formed rewrite rules needed for the work outlined in this paper. They are as follows:

**Definition 4.** We define the well formed rewrite rules order traverse \( \overset{\rightarrow}{\Rightarrow} \): \( \mathcal{P}^*(A) \to \mathcal{P}^*(A) \) and expansion \( \overset{\leftarrow}{\Rightarrow} : \mathcal{P}^*(A) \to \mathcal{P}^*(A) \) over \( r, d \in \mathcal{P}^*(A) \), s.t. \( r = (\rho, v, \{a_1, \ldots, a_{A(\rho, 1)}\}) \), \( d = (\rho', v', \{b_1, \ldots, b_{A(\rho', 1)}\}) \), as follows:

- \( r \overset{\rightarrow}{\Rightarrow} d \) if \( A(\rho, 0) = A(\rho', 0) \), and
  - \( \rho' \prec_R \rho \) and \( v' \leq_{A(\rho, 0)} v \), or
  - \( \rho \leq_R \rho' \) and \( v' \leq_{A(\rho', 0)} v \)

- \( r \overset{\leftarrow}{\Rightarrow} d \) if \( \rho' \leq_R \rho \) and \( A(\rho, 0) + 1 = A(\rho', 0) \)

Note that \( \overset{\rightarrow}{\Rightarrow} \) is invariant the precise relationship between \( \rho \) and \( \rho' \) in that it only requires them to be ordered by \( \leq_R \) while \( \overset{\leftarrow}{\Rightarrow} \) requires that \( \rho' \) be smaller than \( \rho \) over \( \leq_R \). This implies that mutual recursion (i.e. cycling) is allowed but only between path symbols defined over term spaces of the same dimension. This will be essential for the construction of resolution derivations. Also, we would like to note that the condition \( A(\rho, 0) + 1 = A(\rho', 0) \) can be generalized to \( A(\rho, 0) + w = A(\rho', 0) \) for \( w > 0 \) w.l.o.g.

**Definition 5.** A substitution space \( S \), is defined as \( S = \{ \sigma : \mathcal{P} \to \mathcal{G} \} \).
Fig. 1. The relationships defined on points of the point space using our two meta-rewrite rules. Though the points are not directly in a term space we illustrate it as if this was the case to represent points defined over same term space as associated.

Substitution spaces allow us to define *grounded point spaces* and *curves*.

**Definition 6.** A **grounded point space** \( P^\bullet(A, \sigma) \) is a point space where all of the points are normalized with respect to the substitution \( \sigma \). A **pre-curve** \( C \) is defined over the substitution space with each substitution being identified with a grounded point space \( P^\bullet(A, \sigma) \). That is

\[
C = \{ A \mid A \subset P^\bullet(A, \sigma) , \sigma \in \mathcal{S} , |A| \in \mathbb{N} \}
\]

A pre-curve is applied to substitutions as one would apply a function to numbers, i.e. \( C(\sigma) = A \). A pre-curve \( C \) is said to be a curve if there exists a point \( p \in P^\bullet(A) \) such that for every substitution \( \sigma \), \( p \sigma \in C(\sigma) \) and for every \( q \), s.t. \( q \sigma \in C(\sigma) \) and \( p \neq q \), either \( p \sigma \rightarrow q \sigma \) or \( p \sigma \rightarrow q \sigma \). For a curve \( C \) fixed to a point \( p \), we write \( [C] = p \).

In the next definition we introduce a very important type of point set which will be essential to constructing resolution derivations.

**Definition 7.** Let \( C \) be a curve. We will refer to \( \mathcal{B}(C) = \{ \sigma \mid |C(\sigma)| = 1 , \sigma \in \mathcal{S} \} \) as the basecases of the curve \( C \).

We can further restrict the types of curves we are considering by restricting the type of point sets of a curve. To define such restrictions we need to define the **set of points of a curve at \( \sigma \)**, that is, \( P(C, \sigma) = \{ p \mid p \sigma \in C(\sigma) \} \).
Definition 8. Let $C$ be a curve over the point space $\mathcal{P}^*(A)$. We refer to $C$ as regular if $\mathcal{P}(C, S) = \{ p \mid p \in \mathcal{P}(C, \sigma), \sigma \in S \}$ is finite. The set of all regular curves over a point space $\mathcal{P}^*(A)$ will be denoted by $\mathcal{C}^*(A)$.

Notice that for all regular curves there are finitely many ways the rewrite rules $\equiv$ and $\rightarrow$ can be applied. So far only one point in $\mathcal{P}(C, S)$ is associated with a curve, that is the point $[C]$.

We now define branches as regular curves $C'$ of a regular curve $C$ such that there exists $\sigma, \theta$, $[C'] \theta \in C(\sigma)$ and $[C'] \theta + [C] \sigma$. Here $C$ is the trunk that branches attach to at points. A tree is a pairing of a trunk and a set of branches where subtree is defined in the standard way. Note that given a trunk $C$ then a curve $C'$ cannot branch from $[C]$. Such a branch would result in an infinite loop. So far this provides an intuitive notion of branches, trunks, and trees. The following definition provides the formal concept.

Definition 9. A tree $\langle C, B \rangle$ is a pairing of a regular curve $C$ and a set of branches $B$, that is a pair of a point and a tree. The set of branches is formally defined as a subset of $\mathcal{B}^*(C)$, defined as follows:

$$\left\{ (p, \langle C', B' \rangle) \mid \begin{array}{l} \exists \sigma, \theta \text{ s.t. } p \in \mathcal{P}(C, \sigma) \text{ and } [C'] \theta = p \sigma \\text{ and } [C'] \theta + [C] \sigma \end{array} \right\}.$$

Essentially trees can be define recursively as follows:

Definition 10. The set of all trees $T^*(A)$ over $\mathcal{P}^*(A)$ is defined recursively as follows:

- If $C \in \mathcal{C}^*(A)$ then $\langle C, \emptyset \rangle$ is a tree.
- If $\langle C, B \rangle$ is a tree, $C' \in \mathcal{C}^*(A)$ and there exists $\sigma, \theta$ such that $q \in \mathcal{P}(C, \sigma)$, $[C'] \theta = q \sigma$, $q + [C]$, and $B' \subset \mathcal{B}^*(C')$, then $\langle C, B \cup (q, \langle C', B' \rangle) \rangle$ is a tree.

Now we can restrict the types of trees we will consider.

Definition 11. A tree $T$ will be called finite if it contains a finite number of curves. A tree $T$ will be referred to as saturated if for every curve $C$ in the tree and every $p \in \mathcal{P}(C, S)$ and $p + [C]$, there is a curve $C'$ in the tree such that in the subtree of $T$ where $C'$ is the trunk the branch set contains $(p, \langle C, B \rangle)$ where $C'$ is the trunk of $T'$. The set of all finite saturated trees over an arity function $A$ will be denoted by $\mathcal{F}(T^*)$.

Definition 12 (Tree Normalization). Let $T = \langle C, B \rangle \in \mathcal{F}(T^*)$, $\sigma \in S$. We recursively define tree normalization $tn(T, \sigma)$ as follows:

- Let $C(\sigma) = A$. For each $\sigma \in A$ such that $(p, \langle C', B' \rangle) \in B$ we can compute the substitution $\theta$ such that $[C'] \theta = p$. We now compute $tn(\langle C', B' \rangle, \theta^\prime)$
- Let $C(\sigma) = A$. If $B = \emptyset$ then the recursion terminates.

Now we can formalize the main theorem of this section.
**Theorem 1.** Let $T \in FS(T^*)$ and $\sigma \in S$. Then $tn(T, \sigma)$ terminates at the basecases of the curves in $T$.

**Proof.** If normalization does not end at a basecase than there are two possible outcomes, either it terminates at point which is not a basecase or it does not terminate. Terminating at a point which is not a basecase violates Definition 6 concerning the construction of curves. Thus, such a tree is not well-formed. Non-termination also violates Definition 6 because This would implies an infinite descending chain of rewrite applications.

**Example 1.** The Nia-schema’s refutation requires four path symbols $\{p_4, p_3, p_2, p_1\}$ and an arity function $A$ containing the following mappings, $A(p_4) = (1, 0), A(p_3) = (3, 1), A(p_2) = (3, 1)$, and $A(p_1) = (4, 1)$. The points of the point space $P^*(A, \{n\})$ we need are as follows:

- $p_1 = (p_4, [n], \{\})$
- $p_2 = (p_3, [n, n, n], \{0\})$
- $p_3 = (p_3, [n, m, k], \{t\})$
- $p_4 = (p_3, [n, m, r], \{0\})$
- $p_5 = (p_2, [n, m, r], \{w\})$
- $p_6 = (p_3, [n, s, n], \{0\})$
- $p_7 = (p_1, [n, m, w, (w \cdot r)], \{r\})$
- $p_8 = (p_2, [n, m, t], \{w\})$
- $p_9 = (p_1, [n, s, w, r], \{t\})$
- $p_{10} = (p_1, [n, s, w, m], \{k\})$

The trunk of a finite saturated tree is the following regular curve

$$C_1 = \{p_1, p_2\} \mid \sigma \in S.$$  

where $[C_1] = p_1$. We now have the base tree $\langle C_1, \varnothing \rangle$. We can now attach a curve $C_2$ as a branch to the tree $\langle C_1, \varnothing \rangle$ where $[C_2] = p_3$. The definition of the curve $C_2$ is much more complex than the definition of $C_1$ and is as following

$$C_2 = \left\{ \begin{array}{c} \{p_1, p_4, p_5\} \\ \{p_3, p_5\} \end{array} \right\} \mid \sigma \in \Sigma_1$$

$$\sigma \in \Sigma_2$$

$$\sigma \in \Sigma_3$$

$$\Sigma_1 = \left\{ \begin{array}{c} \sigma \in \hat{S}, \\ n\sigma = \pi, \\ m\sigma = \pi, \\ k\sigma = s(m), \\ n\sigma = \pi, \\ m\sigma = \pi, \\ \hat{r} \sigma = s(k), \end{array} \right\}$$

$$\Sigma_2 = \left\{ \begin{array}{c} \sigma \in \hat{S}, \\ m\sigma = s(m), \end{array} \right\}$$

$$\Sigma_3 = \left\{ \begin{array}{c} \sigma \in \hat{S}, \\ k\sigma = 0, \end{array} \right\}$$

Notice that there are three points in $C_2$ upon which we can attach curves. In two cases we attach the curve $C_2$ to itself, $p_4$ and $p_5$, and in the other case we need a new curve $C_3$. So far the tree we have constructed is

$$\langle C_1, \left\{ \begin{array}{c} \langle p_2, \langle C_2, \left\{ \langle p_4, \langle C_2, B \rangle \rangle, \right. \rangle, \rangle, \left\{ \langle p_5, \langle C_3, \varnothing \rangle \rangle \right. \rangle, \rangle, \rangle, \rangle \rangle. \rangle \rangle.$$
Notice that for two subtree we have placed $B$ instead of the correct subtrees. The branches attached at this point repeat infinitely often but in a regular fashion. Essentially, in a regular fashion. Now we construct $C_3$

$$C_3 = \left\{ \begin{array}{ll}
\{p_5, p_7, p_6\} & \sigma \in \Sigma_4 \\
\{p_5, p_6, p_9\} & \sigma \in \Sigma_5 \\
\{p_5, p_9\} & \sigma \in \Sigma_6
\end{array} \right\}$$

$$\Sigma_4 = \left\{ \begin{array}{ll}
\sigma \in \mathcal{S}, & \\
\eta \sigma = \pi, & m \sigma = s(\overline{\pi}), \\
\tau \sigma = \tau, & r \sigma = s(\overline{\tau}), \\
w \sigma = \overline{w},
\end{array} \right\}$$

$$\Sigma_5 = \left\{ \begin{array}{ll}
\sigma \in \mathcal{S}, & \\
\eta \sigma = \pi, & m \sigma = s(\overline{\pi}), \\
r \sigma = 0, & t \sigma = 0, \\
w \sigma = \overline{w}, & s \sigma = \overline{s}
\end{array} \right\}$$

$$\Sigma_6 = \left\{ \begin{array}{ll}
\sigma \in \mathcal{S}, & \\
\eta \sigma = \pi, & m \sigma = 0, \\
r \sigma = 0, & t \sigma = 0, \\
w \sigma = \overline{w}, & s \sigma = 0
\end{array} \right\}$$

We attach the curve $C_3$ to the point $p_8$, $C_2$, to point $p_6$ and a new curve $C_4$ to points $p_7$ and $p_9$. The resulting tree is as follows

$$\left\langle C_1, \left\langle p_2, \left\langle C_2, \left\langle p_5, \left\langle C_3, \left\langle p_4, \langle C_2, B \rangle \rangle, \left\langle p_6, \langle C_2, B \rangle \rangle, \left\langle p_7, \langle C_4, \mathcal{S} \rangle \rangle, \left\langle p_8, \langle C_3, B' \rangle \rangle, \left\langle p_9, \langle C_4, \mathcal{S} \rangle \rangle \right\rangle \right\rangle \right\rangle \right\rangle \right\rangle \right\rangle.$$
Now in the next section, we define resolution derivations and decorations of finite saturated trees

3 Decorations of Finite Saturated Trees

In order to properly use the finite saturated trees introduced in the previous section we need to not only decorate the point sets but the points themselves as well. The necessity of such point decoration comes from the fact that a resolution derivation might not only need a “scratch pad” for the numeric terms, but also one for the individual terms. So, far we have not consider this issue and will address it in this section.

Definition 13. A model is a set of non-contradictory literals $\mathbf{M}$. We allow so called model variables within our resolution derivations.

Definition 14 (Point Decoration). Let $p \in \mathcal{P}^*(A)$ and $t_1, \ldots, t_k$ be terms of the individual set and $\mathbf{M}$ be a model. A point Decoration of $p$ is a pair $(p, t_1, \ldots, t_k, \mathbf{M})$.

Definition 15 (Resolution Derivation). A resolution derivation can be built recursively as follows:

- A point decoration is a resolution derivation.
- A clause $C$ composed with a model or a model variable is a resolution derivation.
- Given resolution derivations $R_1$ and $R_2$ and a substitution $\sigma$

$$
\frac{R_1 R_2}{\text{es}(R_1) \backslash \{P\} \circ \text{es}(R_2) \backslash \{\neg P\}} \text{res}(\sigma)
$$

is a resolution derivation if $Q\sigma = P$ and $E\sigma = \neg P$ for literal $Q$ in $R_1$ and $E$ in $R_2$. 
Definition 16 (Decoration of a Point Set of a Curve). Let $C \in C^*(A)$, $\sigma$ a substitution, and $R$ a resolution derivation such that for each point in $q\sigma \in C(\sigma)$ such that $q \notin [C]$ there is a point decoration in $R$ matching $q$ and $V(R) \subseteq V(P(C, \sigma))$. Then a decoration of $C(\sigma)$ by $R$ is a pair $(\sigma, R)$. If more then one substitution is decorated by $R$, then we can write $(S, R)$ where $S$ is a set of substitutions.

Definition 17 (Decoration of a Tree). Let $T \in FS(T^*)$ and $RD$ a set of resolution derivations. A decoration of $T$ by $RD$, $\text{dec}^{RD, T}$, is a set of distinct triples $(C, \sigma, R)$, where $C \subseteq T$, $\sigma$ is a substitution, and $R \in RD$ such that $(\sigma, R)$ is a decoration of the point set $C(\sigma)$. If for every $C \subseteq T$, $S\sigma_1 \cap \ldots \cap S\sigma_k = C$ then we refer to the decoration as festive.

From now on we will only consider festive decorations of finite saturated trees. The following is an example of a festive decoration of the tree constructed in the previous section.

Example 2. We use the following definition in the decoration

$$m(n+1, 0) = \max(s(m(n, 0)), m(n, 0))$$
$$m(0, 0) = \max(s(0, 0))$$

Also, the decoration concerns the refutation of the clausal form of the following recursive clause formula:

$$\hat{\omega}(n+1) \implies \alpha_1 \land \hat{\chi}(s(n), \alpha) \land \hat{\phi}(n+1)$$
$$\hat{\omega}(0) \implies \alpha_1 \land f(\alpha_2) = 0 \land (\neg s(\alpha_3) \leq \alpha_4 \lor \neg f(\alpha_3) = 0 \lor \neg f(\alpha_4) = 0$$
$$\hat{\chi}(n+1, \alpha) \implies f(\alpha) = (n+1) \lor \hat{\chi}(n, \alpha)$$
$$\hat{\chi}(0, \alpha) \implies f(\alpha) = 0$$
$$\hat{\phi}(n+1) \implies$$
$$\neg \max(\alpha_1, \alpha_2) \leq (\alpha_3 \lor \alpha_2) \leq \alpha_3) \land$$
$$\neg \max(\alpha_4, \alpha_5) \leq (\alpha_6 \lor \alpha_4) \leq \alpha_6) \land$$
$$\hat{\phi}(n) \land$$
$$\neg s(\alpha_7) \leq \alpha_8 \lor \neg f(\alpha_8) = (n+1) \lor \neg f(\alpha_7) = (n+1)$$
$$\hat{\phi}(0) \implies s(b) \leq a \lor \neg f(a) = 0 \lor \neg f(b) = 0$$

It’s clausal form is as follows:
\[ \vdash a \leq a \]
\[ \max(a, b) \leq c \vdash a \leq c \]
\[ \max(a, b) \leq c \vdash b \leq c \]
\[ s(b) \leq a, f(a) = 0, f(b) = 0 \vdash \]
\[ s(b) \leq a, f(a) = n + 1, f(b) = n + 1 \vdash \]
\[ \vdash f(a) = 0, \cdots f(a) = n + 1 \]

The decoration of the point sets of the curve \( C_1 \) is as follows for all substitutions in \( S \).
\[ (C_1, S, ([\rho_3, \{n, n, n\}], \{0\}), X, \vdash) \]

The decoration of the point sets of the curve \( C_2 \) is as follows:
\[ (C_2, \Sigma_1, R_1) \]
where \( R_1 \) is
\[ \frac{((\rho_2, [n, m, r], \{w\}), \{f(x) = m \vdash \circ \mathcal{M}\}) \quad ((\rho_3, [n, m, r], \{0\}), \{\vdash f(x) = m \circ \mathcal{M}\})}{\mathcal{M} \text{ res}(\{x \leftarrow m(r, 0)\})} \]

\[ (C_2, \Sigma_2, R_2) \]
where \( R_2 \) is
\[ \frac{((\rho_2, [n, m, r], \{w\}), \{f(x) = m \vdash \circ \mathcal{C}\}) \quad ((\rho_3, [n, m, r], \{0\}), \{\vdash f(x) = m \circ \mathcal{C}\})}{\mathcal{M} \text{ res}(\{x \leftarrow m(r, 0)\})} \]

\[ (C_2, \Sigma_3, R_3) \]
where \( R_3 \) is
\[ \frac{((\rho_2, [n, m, r], \{w\}), \{f(x) = m \vdash \circ \mathcal{M}\}) \quad \vdash f(X) = m \circ \mathcal{M}}{\mathcal{M} \text{ res}(\{x \leftarrow m(r, 0)\})} \]

The decoration of the point sets of the curve \( C_3 \) is as follows:
\[ (C_3, \Sigma_4, R_4) \]
where \( R_4 \) is
\[ \frac{((\rho_1, [n, m, w, \{w \vdash r\}], \{r\}), \{f(x) = m \vdash \circ \mathcal{M}\}) \quad s(x) \leq \vdash y \circ \mathcal{M}}{\mathcal{M} \text{ res}(\{x \leftarrow m(r, 0)\}, \{y \leftarrow m(w, 0)\})} \]

\[ \vdots \]
\[ \frac{((\rho_2, [n, m, t], \{w\}), \{\vdash f(x) = m \circ \mathcal{M}\})}{\mathcal{M} \text{ res}(\{x \leftarrow m(r, 0)\})} \]
\((C_3, \Sigma_5, R_5)\) where \(R_5\) is

\[
\begin{align*}
&\left(\rho_1, [n, s, w, r], \{t\}\right), \\
&\{f(x) = m \vdash s(x) \leq y \circ \mathcal{M}\} \\
&\vdots \\
&\mathcal{M} \quad \rho_2([n, s, n], \{0\}, \{\neg f(x) = m \circ \mathcal{M}\}) \\
&\mathcal{M} \quad \text{res}\left(\{x \leftarrow m(0, 0)\}\right)
\end{align*}
\]

\((C_3, \Sigma_6, R_6)\) where \(R_6\) is

\[
\begin{align*}
&\rho_1([n, s, m, w], \{k\}, \\
&\{f(x) = m \vdash s(x) \leq y \circ \mathcal{M}\} \\
&\vdots \\
&\vdash f(x) = m \circ \mathcal{M} \\
&\mathcal{M} \quad \text{res}\left(\{x \leftarrow m(0, 0)\}\right)
\end{align*}
\]

And finally, the decoration of the point sets of the curve \(C_4\) is as follows:

\((C_4, \Sigma_7, R_7)\) where \(R_7\) is

\[
\begin{align*}
&\left(\rho_1, [n, n, n], \{\lambda s, t\}\right), \\
&\text{max}(s(x), x) \leq y \circ \mathcal{M} \\
&\mathcal{M} \quad \text{res}\left(\{x \leftarrow m(t, 0)\}\right) \\
&\mathcal{M} \quad \text{res}\left(\{y \leftarrow m(w, 0)\}\right)
\end{align*}
\]

\((C_4, \Sigma_8, R_8)\) where \(R_8\) is

\[
\begin{align*}
&\vdash \text{max}(s(x), x) \leq y \circ \mathcal{M} \\
&\text{max}(s(x), x) \leq y \vdash \circ \mathcal{M} \\
&\mathcal{M} \quad \text{res}\left(\{x \leftarrow m(t, 0)\}\right) \\
&\mathcal{M} \quad \text{res}\left(\{y \leftarrow m(w, 0)\}\right)
\end{align*}
\]

Let \(\mathbf{RD} = \left\{\left(\rho_1, [n, n, n], \{\lambda 0\}, \vdash\right)\right\} \cup \{R_i\}_{i=1}^8\) and \(T\) the finite saturated tree constructed in the previous section, then

\[
\text{dec}(T, \mathbf{RD}) =
\begin{cases}
(C_1, \Sigma, \left(\rho_3, [n, n, n], \{0\}, \vdash\right)) \\
(C_2, \Sigma, R_1) \\
(C_2, \Sigma, R_2) \\
(C_3, \Sigma, R_3) \\
(C_3, \Sigma, R_4) \\
(C_3, \Sigma, R_5) \\
(C_3, \Sigma, R_6) \\
(C_4, \Sigma, R_8) \\
(C_4, \Sigma, R_7) \\
\end{cases}
\]

is a festive decoration of \(T\). Any point set not referenced in \(\text{dec}(T, \mathbf{RD})\) is decorated by the empty clause because it is not reachable by tree normalization.