Construction of all Polynomial Relations among Dedekind Eta Functions of Level N*

Ralf Hemmecke† Silviu Radu‡

21-Dec-2017

Abstract

We describe an algorithm that, given a positive integer N, computes a Gröbner basis of the ideal of polynomial relations among Dedekind η-functions of level N, i. e., among the elements of $\{\eta(\delta_1\tau), \ldots, \eta(\delta_n\tau)\}$ where $1 = \delta_1 < \delta_2 \cdots < \delta_n = N$ are the positive divisors of N.

More precisely, we find a finite generating set (which is also a Gröbner basis) of the ideal $\ker\phi$ where

$$\phi : \mathbb{Q}[E_1, \ldots, E_n] \to \mathbb{Q}[\eta(\delta_1\tau), \ldots, \eta(\delta_n\tau)], \quad E_k \mapsto \eta(\delta_k\tau), \quad k = 1, \ldots, n.$$

1 Introduction

In many publications one finds directly or indirectly lists of relations among Dedekind η-functions, see, for example, [Köh11]. Somos on his website `http://eta.math.georgetown.edu/etal` gives quite a huge number of such relations together with references to the literature. In this article, we not only provide means to compute or check new relations, but rather describe a method to compute a basis for the ideal of *all* possible polynomial relations among Dedekind η-functions of a certain level. Our method adapts the ideas of [KZ08] to η-functions.

Since our basis will be a Gröbner basis, it is easy to express a given relation as a combination of the Gröbner basis elements by merely reducing the relation to zero and keeping track of the reduction steps.

After listing the notations used in this article and the exact problem specification, we continue in Chapter 4 with four reduction steps of the problem, that roughly say, that any relation among η-functions can be expressed by a relation among η-quotients that are modular functions. In Chapter 5, we reduce further and then can say that any η-relation can be expressed by a relation among η-quotients that are modular functions and have at most a pole at infinity. In

*The second author was supported by grant SFB F50-06 of the Austrian Science Fund (FWF).

†hemmecke@risc.jku.at

‡sradu@risc.jku.at

Chapter 6, we show that the quotients of η-functions of a certain level that are modular functions and only have at most a pole at infinity, can be generated by only finitely many elements of this kind and that finding such generators is constructive. Eventually, we show in Chapter 7 how these finitely many elements can be turned in finitely many steps into a Gröbner basis for the ideal of relations among η-functions. We demonstrate our method by an example in Chapter 8, and show how our findings relate to the table given by Somos.

Our article does not primarily focus on efficiency of the computation, but rather on its effectiveness, i. e., that there *exists an algorithm* to compute a Gröbner basis for the relations among Dedekind η-functions.

2 Notation

For a set $E = \{E_1, \ldots, E_n\}$ of indeterminates let us abbreviate the polynomial ring $\mathbb{Q}[E_1, \ldots, E_n]$ by $\mathbb{Q}[E]$. Furthermore, we use multi-index notation, i. e., if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, then we simply write E^α instead of $E_1^{\alpha_1} \cdots E_n^{\alpha_n}$.

Let $\mathbb{H} = \{c \in \mathbb{C} \mid \Im(c) > 0\}$ denote the upper complex half-plane.

Let

$$\eta : \mathbb{H} \to \mathbb{C}, \quad \tau \mapsto \exp\left(\frac{\pi i \tau}{12}\right) \prod_{n=1}^{\infty} (1 - q^n) \text{ with } q = q(\tau) = \exp(2\pi i \tau)$$

denote the Dedekind eta function.

In the following N denotes a positive integer and $1 = \delta_1 < \delta_2 \cdots < \delta_n = N$ the positive divisors of N. Let $\Delta := \{\delta_1, \ldots, \delta_n\}$. For convenience, we allow to index n-dimensional vectors by the divisors of N, instead of the usual index set $\{1, \ldots, n\}$. For $\delta \in \Delta$ we consider the functions

$$\eta_\delta : \mathbb{H} \to \mathbb{C}, \quad \tau \mapsto \eta(\delta\tau)$$

None of these functions is identically zero. We denote for any integer k by η_δ^k the function

$$\eta_\delta^k : \mathbb{H} \to \mathbb{C}, \quad \tau \mapsto \eta(\delta\tau)^k.$$

We define $R(N)$ to be the set of integer tuples $r = (r_{\delta_1}, \ldots, r_{\delta_n}) \in \mathbb{Z}^n$. By $R^*(N)$ we denote the subset of all tuples $r = (r_\delta)_{\delta \in \Delta}$ of $R(N)$ that fulfil the following conditions.

$$\sum_{\delta \in \Delta} r_\delta = 0 \tag{1}$$

$$\sum_{\delta \in \Delta} \delta r_\delta \equiv 0 \pmod{24} \tag{2}$$

$$\sum_{\delta \in \Delta} (N/\delta) r_\delta \equiv 0 \pmod{24} \tag{3}$$

$$\sqrt{\prod_{\delta \in \Delta} \delta^{r_\delta}} \in \mathbb{Q} \tag{4}$$

Note that $R^*(N)$ is an additive monoid. It even is an additive group.

If L is a ring and S is a subset of an L-module, we denote by $\langle S \rangle_L$ the set of L-linear combinations of elements of S. If L is a field, then $\langle S \rangle_L$ is a vector space. If $S \subset L$, then $\langle S \rangle_L$ is an ideal of L.

We, furthermore, define the following groups.

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{Z} \wedge ad - bc = 1 \right\}$$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \; c|N \right\}$$

3 The Problem

We are interested in computing a generating set of the kernel of the homomorphism

$$\phi : \mathbb{Q}[E_{\delta_1}, \ldots, E_{\delta_n}] \to \mathbb{Q}[\eta_{\delta_1}, \ldots, \eta_{\delta_n}], \quad \forall \delta \in \Delta : E_\delta \mapsto \eta_\delta.$$

$\ker \phi$ is an ideal of $\mathbb{Q}[E_{\delta_1}, \ldots, E_{\delta_n}]$.

In this article, we call an element of $\ker \phi$ a *polynomial relation* or just *relation*, i. e., a *relation* is a polynomial that when the variables are replaced by the respective (η-)functions gives zero.

By Hilbert's basis theorem, the ideal $\ker \phi$ is finitely generated. In the following, we show how to compute these generators.

In order to do this, we extend ϕ to Laurent polynomials.

$$\Phi : \mathbb{Q}[E_{\delta_1}, \ldots, E_{\delta_n}, E_{\delta_1}^{-1}, \ldots, E_{\delta_n}^{-1}] \to \mathbb{Q}[\eta_{\delta_1}, \ldots, \eta_{\delta_n}, \eta_{\delta_1}^{-1}, \ldots, \eta_{\delta_n}^{-1}]$$
$$\forall \delta \in \Delta : E_\delta \mapsto \eta_\delta.$$

In order to ease notation, we let $E = \{E_{\delta_1}, \ldots, E_{\delta_n}\}$, and let $L = \mathbb{Q}[E, E^{-1}]$ denote the Laurent polynomial ring in the variables E.

First, we focus on the kernel of Φ.

Let L^* be the set of $\mathbb{Q}$-linear combinations of monomials $E^r \in L$ with $r \in R^*(N)$. In the following we show that $\ker \Phi = \langle L^* \cap \ker \Phi \rangle_L$.

Note that $\Phi|_{\mathbb{Q}[E]} = \phi$ and $\ker \phi = \ker \Phi \cap \mathbb{Q}[E]$.

4 Reduction of the problem

Let $L_k^{(1)} \subset L$ denote the $\mathbb{Q}$-vector space generated by the monomials E^r with

$$\sum_{\delta \in \Delta} r_\delta = k.$$

Claim 1.

$$\ker \Phi = \left\langle L_0^{(1)} \cap \ker \Phi \right\rangle_L.$$

Proof. An element $p \in \ker \Phi$ can be written as a finite sum of elements, i. e.,

$$p = \sum_{k=k_1}^{k_2} p_k$$

where $p_k \in L_k^{(1)}$. By [Rad16, Section 2] it follows that $\Phi(p) = 0$ if and only if $\Phi(p_k) = 0$ for every $k_1 \leq k \leq k_2$. Thus, it is sufficient to prove for all $k_1 \leq k \leq k_2$ that if $p_k \in L_k^{(1)} \cap \ker \Phi$, then $p_k \in \left\langle L_0^{(1)} \cap \ker \Phi \right\rangle_L$. Let k be such that $k_1 \leq k \leq k_2$, $r \in R(N)$, $E^r \in L_k^{(1)}$, and $p_k \in L_k^{(1)} \cap \ker \Phi$. Then $E^{-r} \in L_{-k}^{(1)}$ and $E^{-r} p_k \in L_0^{(1)} \cap \ker \Phi$. Therefore, $p_k = E^r(E^{-r} p_k) \in \left\langle L_0^{(1)} \cap \ker \Phi \right\rangle_L$. □

Note that elements in $L_0^{(1)}$ are $\mathbb{Q}$-linear combinations of monomials of the form E^r such that $r \in R(N)$ fulfils (1).

For $k \in \{0, \ldots, 23\}$ we define $L_k^{(2)}$ to be the $\mathbb{Q}$-vector subspace of $L_0^{(1)}$ spanned by those monomials E^r which satisfy

$$\sum_{\delta \in \Delta} \delta \, r_\delta \equiv k \pmod{24}.$$

Claim 2.

$$\ker \Phi = \left\langle L_0^{(2)} \cap \ker \Phi \right\rangle_L .$$

Proof. By Claim 1 it is sufficient to show that if $p \in L_0^{(1)} \cap \ker \Phi$, then $p \in \left\langle L_0^{(2)} \cap \ker \Phi \right\rangle_L$. Let $p = \sum_{k=0}^{23} p_k \in L_0^{(1)} \cap \ker \Phi$ where $p_k \in L_k^{(2)}$. Since $p \in \ker \Phi$, it follows by [Rad16, Section 3] that $p_k \in \ker \Phi$ for every $k \in \{0, \ldots, 23\}$.

Let $k \in \{0, \ldots, 23\}$, $r \in R(N)$, $E^r \in L_k^{(2)}$, and $p_k \in L_k^{(1)} \cap \ker \Phi$. Then $E^{-r} \in L_{23-k}^{(2)}$ and $E^{-r} p_k \in L_0^{(2)} \cap \ker \Phi$. Thus, $p_k = E^r(E^{-r} p_k) \in \left\langle L_0^{(2)} \cap \ker \Phi \right\rangle_L$. □

Note that elements in $L_0^{(2)}$ are $\mathbb{Q}$-linear combinations of monomials of the form E^r such that $r \in R(N)$ fulfils (1) and (2).

For $k \in \{0, \ldots, 23\}$ we define $L_k^{(3)}$ to be the $\mathbb{Q}$-vector subspace of $L_0^{(2)}$ spanned by those monomials E^r which satisfy

$$\sum_{\delta \in \Delta} \frac{N}{\delta} \, r_\delta \equiv k \pmod{24}.$$

Then the following Claim and its proof are nearly identical to what has been shown above, only that we replace $L_k^{(2)}$ by $L_k^{(3)}$ and $L_0^{(1)}$ by $L_0^{(2)}$.

Claim 3.

$$\ker \Phi = \left\langle L_0^{(3)} \cap \ker \Phi \right\rangle_L .$$

Proof. By Claim 2 it is sufficient to show that if $p \in L_0^{(2)} \cap \ker\Phi$, then $p \in \left\langle L_0^{(3)} \cap \ker\Phi \right\rangle_L$. Let $p = \sum_{k=0}^{23} p_k \in L_0^{(2)} \cap \ker\Phi$ where $p_k \in L_k^{(3)}$. Since $p \in \ker\Phi$, it follows by [Rad16, Section 3] that $p_k \in \ker\Phi$.

Let $k \in \{0, \dots, 23\}$, $r \in R(N)$, $E^r \in L_k^{(3)}$, and $p_k \in L_k^{(2)} \cap \ker\Phi$. Then $E^{-r} \in L_{23-k}^{(3)}$ and $E^{-r} p_k \in L_0^{(3)} \cap \ker\Phi$. Thus, $p_k = E^r(E^{-r}p_k) \in \left\langle L_0^{(3)} \cap \ker\Phi \right\rangle_L$. ☐

Note that elements in $L_0^{(3)}$ are $\mathbb{Q}$-linear combinations of monomials of the form E^r such that $r \in R(N)$ fulfils (1), (2), and (3).

Let $\pi_1, \dots, \pi_s$ be the primes dividing N and let $u_{\delta j} \in \mathbb{N}$ for $\delta \in \Delta$ and $j \in \{1, \dots, s\}$ be defined by the prime factorization of δ, i. e., $\delta = \prod_{j=1}^{l} \pi_j^{u_{\delta j}}$. We define functions $\varepsilon_j : R(N) \to F_2$ from $R(N)$ into the Galois field F_2 of order 2 by

$$r \mapsto \sum_{\delta \in \Delta} r_\delta \, u_{\delta j}.$$

The function $\varepsilon : R(N) \to F_2^s$ is defined by $r \mapsto (\varepsilon_1(r), \dots, \varepsilon_s(r))$.

For $e \in F_2^s$ we denote by $L_e^{(4)}$ the $\mathbb{Q}$-vector subspace of $L_0^{(3)}$ generated by those terms $E^r \in L_0^{(3)}$, with the property $r \in R(N)$ and $e = \varepsilon(r)$. Clearly, for $r, r' \in R(N)$ it holds $\varepsilon(r + r') = \varepsilon(r) + \varepsilon(r')$.

Note that $L_e^{(4)}$ corresponds to the set $S_3(e)$ as defined in [Rad16, Section 4].

Claim 4.

$$\ker\Phi = \left\langle L_0^{(4)} \cap \ker\Phi \right\rangle_L .$$

Proof. By Claim 3 it is sufficient to show that if $p \in L_0^{(3)} \cap \ker\Phi$, then $p \in \left\langle L_0^{(4)} \cap \ker\Phi \right\rangle_L$. Let $p = \sum_{e \in F_2^s} p_e \in L_0^{(3)} \cap \ker\Phi$ with $p_e \in L_e^{(4)}$. By [Rad16, Section 4] it follows that $p_e \in \ker\Phi$ for all $e \in F_2^s$. Now fix $e \in F_2^s$ and let $E^r, E^{r'} \in L_e^{(4)}$ be two monomials. By additivity of ε, we conclude $E^{r+r'} \in L_0^{(4)}$. Let $r \in R(N)$, $E^r \in L_e^{(4)}$. Then $E^r p_e \in L_0^{(4)} \cap \ker\Phi$. Therefore, $p_e = E^{-r}(E^r p_e) \in \left\langle L_0^{(4)} \cap \ker\Phi \right\rangle_L$. ☐

Note that $L_0^{(4)}$ is a $\mathbb{Q}$-linear combination of monomials of the form E^r such that $r \in R(N)$ fulfils (1), (2), (3), and (4), i. e., $r \in R^*(N)$, therefore, we define $L^* := L_0^{(4)} = \langle E^r \mid r \in R^*(N) \rangle_Q \subset L$.

5 From $R^*(N)$ to $R^\infty(N)$

Because $R^*(N)$ is an additive monoid, L^* is a ring and we can write $L^* = Q[E^r \mid r \in R^*(N)]$. In this section we are going to define a (finitely generated) submonoid $R^\infty(N) \subset R^*(N)$ such that

$$\ker\Phi = \langle L^\infty \cap \ker\Phi \rangle_L$$

where $L^\infty := \mathbb{Q}[E^r \mid r \in R^\infty(N)]$. Motivation for passing from $R^*(N)$ to $R^\infty(N)$ is that it eventually allows us to feed Laurent series that are related to $R^\infty(N)$ into a computer algebra system and actually compute a basis of all the relations among the η-functions of level N.

Informally speaking, $R^\infty(N)$ correspond to the set of η-quotients that have poles (if any) only at infinity.

Definition 5.1. *For any $c, \delta \in \Delta$, $r \in R(N)$ let us define*

$$
\begin{aligned}
a_N(c,\delta) &:= \frac{N/c}{\gcd(N/c,c)} \frac{\gcd(c,\delta)^2}{\delta}, \\
\operatorname{ord}_c^N(r) &:= \frac{1}{24} \sum_{\delta\in\Delta} a_N(c,\delta)\, r_\delta,
\end{aligned}
\tag{5}
$$

and

$$
g_r(\tau) := \prod_{\delta\in\Delta} \eta(\delta\tau)^{r_\delta}. \tag{6}
$$

With $\operatorname{ord}_\gamma^N$ as defined in [Rad15], Theorem 23 of [Rad15] turns into

Theorem 5.2. *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ with $c \in \Delta$. If $r \in R^*(N)$, then $\operatorname{ord}_\gamma^N(g_r) = \operatorname{ord}_c^N(r)$.*

For a proof we refer to [Lig75, Proposition 3.2.8].

Similar to the "valence matrix" in [New57], we define an (integer) matrix that is indexed by the positive divisors of N. The rows (indexed by c) correspond to cusps $\frac{a}{c}$ for some $a \in \mathbb{Z}$ with $\gcd(a,c) = 1$.

$$
A_N := \Big(a_N(c,\delta)\Big)_{c,\delta\in\Delta}.
$$

Note that Newman only deals with N that are squarefree. The non-squarefree case is compensated by the additional quotient $\gcd(N/c,c)$, compare with Notation 3.2.6 in [Lig75].

According to [Rad16, Lemma 5.3], there are $\varphi(\gcd(N/c,c))$ different cusps $\frac{a}{c}$ of $\Gamma_0(N)$ that correspond to a divisor c of N (i.e., to the row with index c in A_N) where φ is Euler's totient function. As a preparation for Lemma 5.5, we introduce a row vector

$$
V_N = (\varphi(\gcd(N/c,c)))_{c\in\Delta}
$$

and functions

$$
v_\delta(N,c) := \varphi(\gcd(N/c,c))\, a_N(c,\delta)
$$

such that $V_N A_N = (\sum_{c\in\Delta} v_\delta(N,c))_{\delta\in\Delta}$.

For the proof of Lemma 5.4, we need an auxiliary result.

Lemma 5.3. *For any $0 \neq \alpha \in \mathbb{N}$, $0 \leq m \leq \alpha$, we have $\sum_{d|p^\alpha} v_{p^m}(p^\alpha, d) = p^\alpha + p^{\alpha-1}$.*

Proof.

$$\begin{aligned}\sum_{k=0}^{\alpha} v_{p^m}(p^\alpha, p^k) &= v_{p^m}(p^\alpha, 1) + v_{p^m}(p^\alpha, p^\alpha) + \sum_{k=1}^{\alpha-1} V_{p^m}(p^\alpha, p^k)\\ &= p^{\alpha-m} + p^m + \sum_{k=1}^{\alpha-1} \varphi(\gcd(p^{\alpha-k}, p^k)) \frac{p^{\alpha-k}}{\gcd(p^{\alpha-k}, p^k)} \frac{\gcd(p^k, p^m)^2}{p^m}\\ &= p^{\alpha-m} + p^m + \sum_{k=1}^{\alpha-1} \varphi(p^{\min(\alpha-k,k)}) p^{\alpha-k-\min(\alpha-k,k)+2\min(k,m)-m}\\ &= p^{\alpha-m} + p^m + \sum_{k=1}^{\alpha-1} (p-1) p^{\alpha-1-k+2\min(k,m)-m}\end{aligned}$$

For $m = 0$ we get $\sum_{k=0}^{\alpha} v_{p^m}(p^\alpha, p^k) = p^\alpha + 1 + \sum_{k=1}^{\alpha-1}(p-1)p^{\alpha-1-k} = p^\alpha + p^{\alpha-1}$. If $0 < m \leq \alpha$, then

$$\begin{aligned}\sum_{k=0}^{\alpha} v_{p^m}(p^\alpha, p^k) &= p^{\alpha-m} + p^m + \sum_{k=1}^{m-1}(p-1)p^{\alpha-1+k-m} + \sum_{k=m}^{\alpha-1}(p-1)p^{\alpha-1-k+m}\\ &= p^{\alpha-m} + p^m + (p-1)p^{\alpha-m}\sum_{k=0}^{m-2} p^k + (p-1)p^m \sum_{k=0}^{\alpha-m-1} p^k\\ &= p^{\alpha-m} + p^m + p^{\alpha-m}(p^{m-1} - 1) + p^m(p^{\alpha-m} - 1)\\ &= p^{\alpha-m} + p^m + p^{\alpha-1} - p^{\alpha-m} + p^\alpha - p^m = p^\alpha + p^{\alpha-1}\end{aligned}$$

□

Lemma 5.4. $V_N A_N = N \prod_{p|N}\left(1 + \frac{1}{p}\right) \cdot (1, \ldots, 1)$.

Proof. We have to show that the value of $\sum_{c\in\Delta} v_\delta(N, c)$ is independent of δ. Clearly, if p is a prime that divides N, i.e., $N = N'p^\alpha$ for some $\alpha > 0$, and $\delta = \delta' p^m$, $c = c'p^k$ with $\gcd(p, N') = \gcd(p, \delta') = \gcd(p, c') = 1$, then $v_\delta(N, c) = v_{\delta'}(N', c')\, v_{p^m}(p^\alpha, p^k)$.

Since the divisors of N can be written as a disjoint union according to the respective power of p they contain, we can write

$$\begin{aligned}\sum_{c|N} v_\delta(N, c) &= \sum_{c'|N'} \sum_{d|p^\alpha} v_\delta(N, c'd) = \sum_{c'|N'} \sum_{d|p^\alpha} v_{\delta'}(N', c') v_{p^m}(p^\alpha, d)\\ &= \sum_{c'|N'} v_{\delta'}(N', c') \cdot \sum_{d|p^\alpha} v_{p^m}(p^\alpha, d).\end{aligned}$$

Together with $\sum_{d|p^\alpha} v_{p^m}(p^\alpha, d) = p^\alpha(1 + \frac{1}{p})$, $0 \leq m \leq \alpha$, the result follows by induction over the number of prime divisors of N. □

Lemma 5.5. *The matrix A_N is invertible.*

Proof. The proof is along the lines of the proof of Lemma 3 in [New57]. Suppose $\det(A_N) = 0$. Then there is a vector $r \in R(N)$ such that $A_N r = 0$ and $r_\delta \neq 0$ for at least one $\delta \in \Delta$. Since $V_N A_N r = 0$, we conclude from Lemma 5.4 that (1) holds for such an r. We may assume that each component of r is a multiple of 24, thus, we can take $r \in R^*(N)$. Then the conditions of Theorem 1 of [New59] are fulfilled. Thus, the function g_r is a modular function for $\Gamma_0(N)$. g_r does neither have zeroes nor poles on the upper complex half-plane, since η does not have zeroes or poles. The cusps of $\Gamma_0(N)$ can be assumed to be of the form $\frac{a}{c}$ with $a, c \in \mathbb{Z}$, $\gcd(a, c) = 1$, $c \in \Delta$, cf. [Rad16, Lemma 5.3]. Note that $c = N$ corresponds to the cusp at infinity. From $A_N r = 0$, (5) and Theorem 5.2 follows that the function g_r has zero order at all the cusps. Thus, it must be constant, i. e., $g_r = 1$. By Theorem 4 of [New57] follows that r is the zero vector and, thus, the Lemma is proved. □

Let

$$R^\infty(N) := \left\{ r \in R^*(N) \;\middle|\; \forall c \in \mathbb{N} : (0 < c < N \wedge c|N \implies \operatorname{ord}_c^N(r) \geq 0) \right\}.$$

Note that the set $\{g_r \mid r \in R^\infty(N)\}$ is the same as $E^\infty(N)$ in [Rad15].

Let $K \in \mathbb{N}$ be the (positive) least common multiple of all denominators of the entries of A_N^{-1}. Let $\varrho = 24KA_N^{-1}(1, \ldots, 1, 0)^T$. Obviously, $\varrho \in R^*(N)$ and by construction $\operatorname{ord}_c^N(\varrho) = K$ for every $c \in \Delta$ with $c \neq N$ and $\operatorname{ord}_N^N(\varrho) = 0$. Thus, $\varrho \in R^\infty(N)$ and for any $r \in R^*(N)$ there exists $d \in \mathbb{N}$ such that

$$r + d\varrho \in R^\infty(N). \tag{7}$$

Lemma 5.6.

$$\ker \Phi = \langle L^\infty \cap \ker \Phi \rangle_L$$

Proof. It is sufficient to show that if $p \in L^* \cap \ker \Phi$, then $p \in \langle L^\infty \cap \ker \Phi \rangle_L$. Let E^r be a monomial of p. Then by (7) there is $d_r \in \mathbb{N}$ such that $(E^\varrho)^{d_r} E^r \in L^\infty$. If we choose d as the maximum of all such d_r for all monomials of p, then clearly, $(E^\varrho)^d p \in L^\infty \cap \ker \Phi$. Therefore, $p = E^{-d\varrho}(E^{d\varrho} p) \in \langle L^\infty \cap \ker \Phi \rangle_L$ □

6 L^∞ is finitely generated

We have shown that $\ker \Phi$ is generated by elements of $L^\infty = \mathbb{Q}[E^r \mid r \in R^\infty(N)]$. Now we show that L^∞ can be generated (as a polynomial ring over $\mathbb{Q}$) by finitely many elements.

Lemma 6.1. *$R^\infty(N)$ is a finitely generated (additive) monoid.*

Proof. In order to apply Lemma 2.6.8 from [DLHK13], we construct a matrix A by stacking matrices B_N, $-B_N$, and $-A_N^\infty$ on top of each other.

The matrix B_N encodes the conditions for $r \in R^*(N)$ and A_N^∞ encodes the conditions about the orders for cusps not at infinity, i. e., that $r \in R^\infty(N)$.

For the conditions (2), (3), and (4) we introduce additional variables b_∞, b_0, and $b_1, \dots, b_s$ in order to turn the "mod 24" and the square root condition into an integer problem. These additional variables enter our problem transformation for [DLHK13, Lemma 2.6.8], but are otherwise irrelevant for us. Let $z = (r_{\delta_1}, \dots, r_{\delta_n}, b_\infty, b_0, b_1, \dots b_s)^T$ be the column vector that correspond to the r-variables and the additional variables. We transform the question about a finite generating set for $R^\infty(N)$ into a problem about the (integer) solutions of the system $Az \le 0$.

We define

$$B_N := \begin{bmatrix} 1 & \cdots & 1 & 0 & 0 & 0 & \dots & 0 \\ \delta_1 & \cdots & \delta_n & 24 & 0 & 0 & \dots & 0 \\ N/\delta_1 & \cdots & N/\delta_n & 0 & 24 & 0 & \dots & 0 \\ u_{\delta_1 1} & \cdots & u_{\delta_n 1} & 0 & 0 & 2 & \dots & 0 \\ \vdots & & \vdots & 0 & 0 & 0 & \ddots & 0 \\ u_{\delta_1 s} & \cdots & u_{\delta_n s} & 0 & 0 & 0 & \dots & 2 \end{bmatrix} \tag{8}$$

where s and the $u_{\delta j}$ are defined as in the text before Claim 4 by the prime factorization of the divisors of N. Then $B_N z = 0$ corresponds to the condition $r \in R^*(N)$ from Section 2. Furthermore, with

$$A_N^\infty := \begin{bmatrix} a_N(\delta_1, \delta_1) & \dots & a_N(\delta_1, \delta_N) & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_N(\delta_{n-1}, \delta_1) & \dots & a_N(\delta_{n-1}, \delta_N) & 0 & \dots & 0 \end{bmatrix} \tag{9}$$

the inequality $A_N^\infty z \ge 0$ for $r \in R^*(N)$ encodes $\operatorname{ord}_c^N(r) \ge 0$ for every $c \in \Delta$ with $c \ne N$.

By [DLHK13, Lemma 2.6.8] we can conclude that there are finitely many $\varrho_1, \dots, \varrho_k \in R^\infty(N)$ such that $\langle \varrho_1, \dots, \varrho_k \rangle_{\mathbb{N}} = R^\infty(N)$. $\square$

Let $\varrho_1, \dots, \varrho_k$ be such that $\langle \varrho_1, \dots, \varrho_k \rangle_{\mathbb{N}} = R^\infty(N)$. For $\kappa \in \{1, \dots, k\}$ let $m_\kappa := E^{\varrho_\kappa}$. Then $L^\infty = \mathbb{Q}[E^r \mid r \in R^\infty(N)] = \mathbb{Q}[m_1, \dots, m_k]$.

7 From ring to ideal

By Lemma 5.6 we now have $\ker \Phi = \langle \mathbb{Q}[m_1, \dots, m_k] \cap \ker \Phi \rangle_L$, i. e., any relation among η-functions of a certain level, can be expressed as an L-linear combination of polynomials of a finite number of η-quotients corresponding to $m_1, \dots, m_k$ whose coefficients are in $\mathbb{Q}$. In other words, we would like to find polynomials $p \in \mathbb{Q}[m_1, \dots, m_k]$ such that $\Phi(p) = 0$. In order to do this, we first transform the problem in such a way that we can employ the algorithm AB from [Rad15]. It leads to temporarily working with an ideal in the polynomial ring $\mathbb{Q}[Z, M]$ only to later eliminate the Z-variables to obtain an ideal $J^{(M)}$. Substitution of the indeterminates $M_1, \ldots, M_k$ by the respective $m_1, \ldots, m_k$ eventually gives a better representation for $\ker \Phi$.

The output of AB is used to find all polynomial relations among the m_κ that are in $\ker \Phi$. In fact, instead of AB, we use the algorithm samba from [Hem18]. It gives essentially the same output as AB, but can be exploited in its extended form to avoid the computation of an elimination ideal.

For $\kappa \in \{1, \ldots, k\}$ let $\mathfrak{m}_\kappa := \Phi(m_\kappa)$. Since $\varrho_\kappa \in R^\infty(N)$, $\mathfrak{m}_\kappa$ can be represented as a Laurent series $\mathbb{Q}((q))$ in $q = \exp(2\pi i \tau)$. We denote by $\operatorname{ord}_q(\mathfrak{f})$ the smallest power of q that appears in $\mathfrak{f} \in \mathbb{Q}((q))$ with a non-zero coefficient. Note that $\operatorname{ord}_q(\mathfrak{m}_\kappa) = \operatorname{ord}_N^N(\varrho_\kappa)$. $\mathbb{Q}[\mathfrak{m}_1, \ldots, \mathfrak{m}_k]$ is a subring of $\mathbb{Q}((q))$ and, by construction via $R^\infty(N)$, has the the property that if $\mathfrak{f} \in \mathbb{Q}[\mathfrak{m}_1, \ldots, \mathfrak{m}_k]$ and $\operatorname{ord}_q(\mathfrak{f}) > 0$, then $\mathfrak{f} = 0$. Thus, we can apply algorithm samba from [Hem18] to $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ and obtain elements $\mathfrak{z}_1, \ldots, \mathfrak{z}_l \in \mathbb{Q}[\mathfrak{m}_1, \ldots, \mathfrak{m}_k]$ with $\mathfrak{z}_1 = 1$ such that

$$\mathbb{Q}[\mathfrak{m}_1, \ldots, \mathfrak{m}_k] = \langle \mathfrak{z}_1, \ldots, \mathfrak{z}_l \rangle_{\mathbb{Q}[\mathfrak{m}_1]} . \tag{10}$$

Since samba is applied over $\mathbb{Q}$, i. e.., with a field as coefficients, we can assume that $0 = \operatorname{ord}_q(\mathfrak{z}_1) > \operatorname{ord}_q(\mathfrak{z}_2) > \ldots > \operatorname{ord}_q(\mathfrak{z}_l)$. Because of line 10 of samba, the sequence $\mathfrak{z}_2, \ldots, \mathfrak{z}_l$ is $\mathfrak{m}_1$-reduced in the sense of Definition 8 of [Rad15]. In other words, there are no non-trivial $\mathbb{Q}[\mathfrak{m}_1]$-linear relations among the $\mathfrak{z}_1, \ldots, \mathfrak{z}_l$, i. e., if $v_1, \ldots, v_l \in \mathbb{Q}[M_1]$ with $v_1(\mathfrak{m}_1)\mathfrak{z}_1 + \cdots v_l(\mathfrak{m}_1)\mathfrak{z}_l = 0$, then $v_1 = \cdots = v_l = 0$.

Let $\mathbb{Q}[Z, M]$ denote the polynomial ring $\mathbb{Q}[Z_1, \ldots, Z_l, M_1, \ldots, M_k]$. As a consequence of (10), there are polynomials ($\kappa \in \{1, \ldots, k\}$, $j, j', \lambda \in \{1, \ldots, l\}$)

$$v_{\kappa\lambda}, v_{jj'\lambda} \in \mathbb{Q}[M_1], \tag{11}$$

$$p_\kappa := M_\kappa - \sum_{\lambda=1}^{l} v_{\kappa\lambda}(M_1) Z_\lambda \in \mathbb{Q}[Z, M], \tag{12}$$

$$p_{jj'} := Z_j Z_{j'} - \sum_{\lambda=1}^{l} v_{jj'\lambda}(M_1) Z_\lambda \in \mathbb{Q}[Z, M] \tag{13}$$

such that (by plugging in the corresponding Laurent series)

$$p_\kappa(\mathfrak{z}, \mathfrak{m}) = 0, \qquad p_{jj'}(\mathfrak{z}, \mathfrak{m}) = 0.$$

The polynomials $v_{\kappa\lambda}, v_{jj'\lambda}$ can easily be obtained by reducing $\mathfrak{m}_\kappa$ and $\mathfrak{z}_j \mathfrak{z}_{j'}$ to zero by the module basis elements $\mathfrak{z}_1, \ldots, \mathfrak{z}_l$ and keeping track of the cofactors in this reduction. Note that even though this reduction to zero deals with Laurent series, these are Laurent series coming from $R^\infty(N)$ and, thus, it is a finite process. The reduction can stop, if an element of positive order is obtained.

We can form the ideal $J^{(Z,M)}$ in $\mathbb{Q}[Z, M]$ generated by

$$\{p_1, \ldots, p_k\} \cup \{p_{jj'} \mid j, j' \in \{1, \ldots, l\}\} . \tag{14}$$

This ideal contains every relation among the $\mathfrak{m}_\kappa$ and $\mathfrak{z}_j$. For a proof, suppose $f \in \mathbb{Q}[Z, M]$ with $f(\mathfrak{z}, \mathfrak{m}) = 0$. Then, using (12) and (13), we can reduce f to a polynomial f' of the form $f' = v_1(M_1)Z_1 + \cdots v_l(M_1)Z_l$ with $v_1, \ldots, v_l \in \mathbb{Q}[M_1]$ and $f'(\mathfrak{z}, \mathfrak{m}) = 0$. By a remark above, $v_1 = \cdots = v_l = 0$ and thus $f \in J^{(Z,M)}$.

The intersection of $J^{(Z,M)}$ with $\mathbb{Q}[M]$ gives an ideal $J^{(M)}$ that represents all relations among the $\mathfrak{m}_\kappa$. In principle, generators for the ideal $J^{(M)}$ can be obtained by computing a Gröbner basis (see [Buc65] or [BW93]) of (14) with respect to an elimination term ordering. However, employing an extended form of the algorithm samba allows us to avoid such a Gröbner basis computation. The extended form of samba keeps track of all the transformations during its run and thus yields not only $\mathfrak{z}_1, \ldots, \mathfrak{z}_l$, but also polynomials $f_\lambda \in \mathbb{Q}[M]$ such that

$$\mathfrak{z}_\lambda = f_\lambda(\mathfrak{m}) = f_\lambda(\mathfrak{m}_1, \ldots, \mathfrak{m}_k) \tag{15}$$

for every $\lambda \in \{1, \ldots, l\}$. By replacing each indeterminate Z_λ by f_λ ($\lambda \in \{1, \ldots, l\}$), we can transform (12) and (13) into

$$h_\kappa := M_\kappa - \sum_{\lambda=1}^{l} v_{\kappa\lambda} f_\lambda \in \mathbb{Q}[M], \tag{16}$$

$$h_{jj'} := f_j f_{j'} - \sum_{\lambda=1}^{l} v_{jj'\lambda} f_\lambda \in \mathbb{Q}[M]. \tag{17}$$

Then,

$$H^{(M)} := \{h_1, \ldots, h_k\} \cup \{h_{jj'} \mid j, j' \in \{1, \ldots, l\}\} \subseteq \mathbb{Q}[M]. \tag{18}$$

is a set of generators for the ideal of relations among the $\mathfrak{m}_\kappa$. In other words, $h(\mathfrak{m}) = 0$ for every $h \in \langle H^{(M)} \rangle_{\mathbb{Q}[M]}$. Clearly, $\langle H^{(M)} \rangle_{\mathbb{Q}[M]} \subseteq J^{(M)}$. In order to show $\langle H^{(M)} \rangle_{\mathbb{Q}[M]} \supseteq J^{(M)}$, take $h \in J^{(M)}$. Because $J^{(M)} \subset J^{(Z,M)}$, there exist polynomials $w_k, w_{jj'} \in \mathbb{Q}[Z, M]$ ($\kappa \in \{1, \ldots, k\}$, $j, j' \in \{1, \ldots, l\}$) such that

$$h = \sum_{\kappa=1}^{k} w_\kappa p_\kappa + \sum_{j=1}^{l} \sum_{j'=1}^{l} w_{jj'} p_{jj'}. \tag{19}$$

Because of (15), each indeterminate Z_λ in (19) can be replaced by f_λ ($\lambda \in \{1, \ldots, l\}$). Since with this replacements p_κ becomes h_κ and $p_{jj'}$ becomes $h_{jj'}$, (19) turns into an equation that shows $h \in \langle H^{(M)} \rangle_{\mathbb{Q}[M]}$. Therefore, $J^{(M)} = \langle H^{(M)} \rangle_{\mathbb{Q}[M]}$.

If we plug in the $m_\kappa (= E^{\varrho_\kappa})$ for the M_κ in the polynomials of $H^{(M)}$, we obtain the set

$$H^L := \{h_1(m), \ldots, h_k(m)\} \cup \{h_{jj'}(m) \mid j, j' \in \{1, \ldots, l\}\} \subseteq L. \tag{20}$$

Clearly, $H^L \subset \ker \Phi$. By construction of H^L and from Lemma 5.6, we get

$$\ker \Phi = \langle \mathbb{Q}[m_1, \ldots, m_k] \cap \ker \Phi \rangle_L = \langle H^L \rangle_L .$$

We are left with the problem of computing a generating set for the intersection $\ker\Phi \cap \mathbb{Q}[E] = \ker\phi$. A solution to this problem is well-known in the computer algebra community.

Let us denote by $P = \mathbb{Q}[E, Y]$ the polynomial ring in the variables $E = \{E_\delta \mid \delta \in \Delta\}$ and $Y = \{Y_\delta \mid \delta \in \Delta\}$. Let $U = \{1 - E_\delta Y_\delta \mid \delta \in \Delta\}$ and $I = \langle U \rangle_P$ be the ideal generated by the elements of U. By [Sim94, Proposition 7.1], $\ker\chi = I$ for the surjective homomorphism $\chi : P \to L$ with $\chi(E_\delta) = E_\delta$ and $\chi(Y_\delta) = E_\delta^{-1}$ for every $\delta \in \Delta$, i. e., $P/I \cong L$.

Let $\chi' : L \to P$ be such that $\chi'(E_\delta) = E_\delta$, $\chi'(E_\delta^{-1}) = Y_\delta$, i. e., $\chi(\chi'(f)) = f$ for every $f \in L$. Then $\ker\phi = \ker\Phi \cap \mathbb{Q}[E] = \left\langle \chi'(H^L) \cup U \right\rangle_P \cap \mathbb{Q}[E]$.

A generating set for the latter intersection can be computed by Buchberger's algorithm applied to $\chi'(H^L) \cup U$ with respect to a term ordering such that monomials with variables exclusively from the set E are smaller than any monomial involving variables from Y. Then the polynomials $g_1, \ldots, g_t$ in this Gröbner basis that only involve variables from the set E form a Gröbner basis G of all the relations among the η-quotients of level N.

8 Implementation and Computation

We have implemented all the above steps in the computer algebra system FriCAS[1]. The computation of a basis of $R^\infty(N)$ can be done by 4ti2[2]. For (bigger) Gröbner basis computations, we have used the slimgb implementation of Singular[3] via its interface through SageMath[4].

Somos presents on the website `http://eta.math.georgetown.edu/etal` a list of identities for η-functions for various levels. For example, there are 120 identities for level 8. In our approach, we compute 5 polynomials in $\mathbb{Q}[E_1, E_2, E_4, E_8]$, namely

$$
\begin{aligned}
g_1 &= E_1^8 E_2^6 E_4^{10} - E_1^{12} E_2^8 E_8^4 - 4E_1^4 E_2^8 E_4^8 E_8^4 + 32E_2^{10} E_4^6 E_8^8 - 16E_1^{12} E_8^{12} - 256E_1^4 E_4^8 E_8^{12},\\
g_2 &= E_2^{10} E_4^8 - E_1^{12} E_4^2 E_8^4 - 8E_1^4 E_4^{10} E_8^4 - 4E_1^8 E_2^2 E_8^8,\\
g_3 &= E_2^8 E_4^{10} - E_1^4 E_2^{10} E_8^4 - 4E_1^8 E_4^2 E_8^8 - 32E_1^4 E_2^2 E_8^{12},\\
g_4 &= E_2^{12} - E_1^8 E_4^4 - 8E_1^4 E_2^2 E_4^2 E_8^4,\\
g_5 &= E_4^{12} - E_1^4 E_2^2 E_4^2 E_8^4 - 4E_2^4 E_8^8,
\end{aligned}
$$

such that by substituting η_δ for the respective E_δ, the function $g_k(\eta_1, \eta_2, \eta_4, \eta_8)$ is the zero function on $\mathbb{H}$ for every $k \in \{1, 2, 3, 4, 5\}$.

Let us demonstrate the steps to arrive at these polynomials. In order to

[1] FriCAS 1.3.2 [Fri]
[2] 4ti2 1.6.7 [tt]
[3] Singular 4.1.0 [DGPS16]
[4] SageMath 8.0 [Dev17]

compute a basis for $R^\infty(8)$ we set up the matrices B_8 and A_N^∞.

$$B_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 24 & 0 & 0 \\ 8 & 4 & 2 & 1 & 0 & 24 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 2 \end{bmatrix} \qquad A_8^\infty = \begin{bmatrix} 8 & 4 & 2 & 1 & 0 & 0 & 0 \\ 2 & 4 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 2 & 0 & 0 & 0 \end{bmatrix}$$

Then give the system $B_8 z = 0$, $A_8^\infty z \geq 0$ to 4ti2-zsolve and obtain the (truncated) vectors

$$\varrho_1 = (4, -2, 2, -4)^T \qquad \varrho_2 = (-4, 10, -2, -4)^T \qquad \varrho_3 = (0, -4, 12, -8)^T$$

which represent the η-quotients according to (6) with the following Laurent series expansions at $\tau = i\infty$.

$$\begin{aligned} \mathfrak{m}_1 &= q^{-1} - 4 + 4\,q + 2\,q^3 - 8\,q^5 - q^7 + 20\,q^9 + O(q^{10}) \\ \mathfrak{m}_2 &= q^{-1} + 4\,q + 2\,q^3 - 8\,q^5 - q^7 + 20\,q^9 + O(q^{10}) \\ \mathfrak{m}_3 &= q^{-1} + 4 + 4\,q + 2\,q^3 - 8\,q^5 - q^7 + 20\,q^9 + O(q^{10}) \end{aligned}$$

The application of samba yields $\mathfrak{z}_1 = 1$ as the only generator according to (10). Because of $f_1 = 1$, we then get from (12), (13), (16), and (17): $h_1 = p_1 = 0$, $h_2 = p_2 = M_2 - M_1 - 8$, $h_3 = p_3 = M_3 - M_1 - 4$ and $h_{1,1} = p_{1,1} = 0$. Then, we replace every M_κ by $m_\kappa = E^{\varrho_\kappa}$ and, by using χ', write Y_δ^{-e} instead of E_δ^e if $e := \varrho_{\kappa,\delta} < 0$. After removing zeros, (20) becomes $\chi'(H^L) = \{h_2(m), h_3(m)\}$ where

$$h_2(m) = \underbrace{Y_1^4 E_2^{10} Y_4^2 Y_8^4}_{m_2} - \underbrace{E_1^4 Y_2^2 E_4^2 Y_8^4}_{m_1} - 8, \quad h_3(m) = \underbrace{Y_2^4 E_4^{12} Y_8^8}_{m_3} - \underbrace{E_1^4 Y_2^2 E_4^2 Y_8^4}_{m_1} - 4.$$

With $U = \{Y_1E_1 - 1, Y_2E_2 - 1, Y_4E_4 - 1, Y_8E_8 - 1\}$, we are left to compute a Gröbner basis for the ideal $\langle \chi'(H^L) \cup U \rangle_P$. The Gröbner basis with respect to the elimination block-ordering (degrevlex in both Y and E variables) consists of 703 elements and (when printed) would be about 650 lines. However, there are only 5 polynomials among those elements, namely $G = \{g_1, \dots, g_5\}$ listed above that do not contain a Y indeterminate. Since G is a generating set for $\ker \phi$, every other (polynomial) relation among η-functions of level 8 can be expressed as a $\mathbb{Q}[E]$-linear combination of the elements of G. In fact, G is a Gröbner basis with respect to a degree reverse lexicographical term ordering, and thus, for any given polynomial $f \in \mathbb{Q}[E_1, E_2, E_4, E_8]$, we can *algorithmically* decide whether it is in $\ker \phi$ by simply reducing f with the Gröbner basis G. The polynomial f is in $\ker \phi$ if and only if the reduction modulo G gives 0. By keeping track of the cofactors in that reduction, we can express f as a $\mathbb{Q}[E]$-linear combination of $g_1, \dots, g_5$.

The identities in the table of Somos can easily be translated from their representation in terms of q and u_δ, where u_δ corresponds to the Euler function $\prod_{n=1}^\infty (1 - q^{\delta n})$, to polynomials in $\mathbb{Q}[E_1, E_2, E_4, E_8]$, and then expressed in terms of G.

For example, Somos' identity

$$q_{8,12,24} := -u_2^{12} + u_1^8 u_4^4 + 8qu_4^{12} - 32q^2 u_2^4 u_8^8$$

translates to

$$q^E_{8,12,24} := -E_2^{12} + E_1^8 E_4^4 + 8E_4^{12} - 32E_2^4 E_8^8$$

and can be expressed as

$$q^E_{8,12,24} = -g_4 + 8g_5. \tag{21}$$

There are other identities in the table of Somos, namely

$$\begin{aligned} t_{8,12,24} &:= -u_2^{12} + u_1^8\, u_4^4 + 8\, q\, u_1^4\, u_2^2\, u_4^2\, u_8^4, \\ t_{8,12,48} &:= -u_4^{12} + u_1^4\, u_2^2\, u_4^2\, u_8^4 + 4\, q\, u_2^4 u_8^8. \end{aligned}$$

They correspond to $-g_4$ and $-g_5$, respectively. The above relation (21) is

$$q_{8,12,24} = t_{8,12,24} - 8\, q\, t_{8,12,48}$$

in the notation of Somos.

The additional factor q in the above relation comes from the fact that the identities in Somos' table do not exactly correspond to relations in Dedekind η-functions, but rather might have a common factor of a (fractional) power of q cancelled.

We can do such a reduction for all the 120 identities from the table of Somos, i. e., express them in terms of G. In fact, at `http://www.risc.jku.at/people/hemmecke` we give a list of the respective Gröbner basis elements for various levels and how relations from Somos' table can be expressed by them.

We can use the 120 identities from the table of Somos and compute a (degrevlex) Gröbner basis in $\mathbb{Q}[E]$ of them. That also leads to 5 polynomials, namely, $E_1^4 E_8^4 g_1, g_2, g_3, g_4, g_5$. In other words, these 120 identities do not generate (in $\mathbb{Q}[E]$) the ideal of *all* relations. For this case, dividing the first polynomial by $E_1^4 E_8^4$ would lead to a Gröbner basis of all relations. However, in general, such a postprocessing would not be a proof that the full ideal of relations is obtained.

Clearly, we can apply Buchberger's algorithm over $\mathbb{Q}[E]$ or (with the respective elements of U added) over $\mathbb{Q}[Y, E]$ also to the relations of Somos' tables of other levels. However, although it might give relations that are not in the table, they are not essentially new, since a Gröbner basis computation does not change the ideal that is already given by the input polynomials. Furthermore, in contrast to our derivation, it would not prove that the ideal of *all* relations has been found.

For example, for level 34, our method produces a Gröbner basis G_{34} of 59 elements, whereas in Somos' table is only one element, namely $x_{34,14,129}$. $x_{34,14,129}$ corresponds to the element of smallest degree of G_{34}. In other words, G_{34} contains essentially new relations.

9 Conclusion

Our method, theoretically, solves the problem of finding polynomial relations among Dedekind η-functions completely. Furthermore, all steps can be programmed on a computer, i. e., the method is *constructive*.

Unfortunately, the more divisors are involved in the computation, the bigger is the effort to compute the respective Gröbner basis. The relations for levels 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 25, 26, 27, 28, 32, 34, 35, 36, 40, 44, 45, 49, 50, 54, 63, 64, 121, 169 are relatively easy to compute, i. e., in less than 5 hours and often much faster. For 24, 30, 56 the Gröbner basis computation is quite lengthy. It took 12.2, 59.9, 16.6 hours, respectively. The computation, in particular the elimination of the Y variables may be quite memory consuming. For level 56 more than 100 GB where used during the computation, although the final Gröbner basis can be stored in about 1 MB.

References

[Buc65] Bruno Buchberger. *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal.* PhD thesis, Univ. Innsbruck, Dept. of Math., Innsbruck, Austria, 1965.

[BW93] Thomas Becker and Volker Weispfenning. *Gröbner Bases. A Computational Approach to Commutative Algebra*, volume 141 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993.

[Dev17] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.0)*, 2017. `http://www.sagemath.org`.

[DGPS16] Wolfram Decker, Gert-Martin Greuel, Gerhard Pfister, and Hans Schönemann. SINGULAR 4-1-0 — A computer algebra system for polynomial computations. `http://www.singular.uni-kl.de`, 2016.

[DLHK13] Jesús A. De Loera, Raymond Hemmecke, and Matthias Köppe. *Algebraic and Geometric Ideas in the Theory of Discrete Optimization.* MOS-SIAM Series on Optimization. Society for Industrial and Applied Mathematics, 2013.

[Fri] FriCAS—an advanced computer algebra system. Available at `http://fricas.sf.net`.

[Hem18] Ralf Hemmecke. Dancing samba with Ramanujan partition congruences. *Journal of Symbolic Computation*, 84:14–24, 2018.

[Köh11] Günter Köhler. *Eta Products and Theta Series Identities.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, Heidelberg, 2011.

[KZ08] Manuel Kauers and Burkhard Zimmermann. Computing the algebraic relations of C-finite sequences and multisequences. *Journal of Symbolic Computation*, 43(11):787–803, 2008.

[Lig75] G. Ligozat. *Courbes Modulaires de Genre 1.* U.E.R. Mathématique, Université Paris XI, Orsay, 1975. Publication Mathématique d'Orsay, No. 75 7411.

[New57] Morris Newman. Construction and application of a class of modular functions. *Proceedings of the London Mathematical Society*, s3-7(1):334–350, 1957.

[New59] Morris Newman. Construction and application of a class of modular functions (II). *Proceedings of the London Mathematical Society*, s3-9(3):373–387, 1959.

[Rad15] Cristian-Silviu Radu. An algorithmic approach to Ramanujan-Kolberg identities. *Journal of Symbolic Computation*, 68, Part 1:225–253, 2015.

[Rad16] Cristian-Silviu Radu. An algorithm to prove algebraic relations involving eta quotients. *Annals of Combinatorics*, pages 1–16, 2016. to appear.

[Sim94] Charles C. Sims. *Computation with finitely presented groups*, volume 48 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, 1994.

[tt] 4ti2 team. 4ti2—a software package for algebraic, geometric and combinatorial problems on linear spaces. Available at `http://www.4ti2.de`.