THE NUMBER OF REALIZATIONS OF A LAMAN GRAPH

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Abstract. Laman graphs model planar frameworks that are rigid for a general choice of distances between the vertices. There are finitely many ways, up to isometries, to realize a Laman graph in the plane. Such realizations can be seen as solutions of systems of quadratic equations prescribing the distances between pairs of points. Using ideas from algebraic and tropical geometry, we provide a recursion formula for the number of complex solutions of such systems.

Introduction

Suppose that we are given a graph G with edges E. We consider the set of all possible realizations of the graph in the plane such that the lengths of the edges coincide with some edge labeling $\lambda\colon E \to \mathbb{R}_{\geq 0}$. Edges and vertices are allowed to overlap in such a realization. For example, suppose that G has four vertices and is a complete graph minus one edge. Figure 1 shows all possible realizations of G up to rotations and translations, for a given edge labeling.

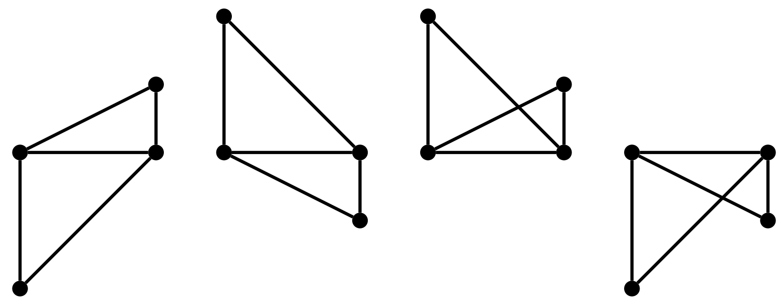

Figure 1. Realizations of a graph up to rotations and translations.

We say that a property holds for a general edge labeling if it holds for all edge labelings belonging to a dense open subset of the vector space of all edge labelings. In this paper we address the following problem:

> *For a given graph determine the number of realizations up to rotations and translations for a general edge labeling.*

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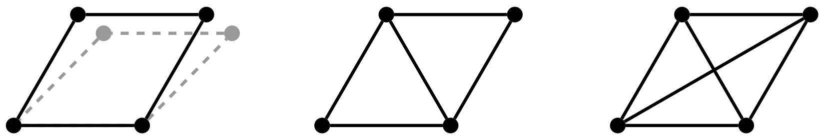
The realizations of a graph can be considered as structures in the plane, which are constituted by rods connected by rotational joints. If a graph with an edge labeling admits infinitely (finitely) many realizations up to rotations and translations, then the corresponding planar structure is flexible (rigid), see Figure 2.



(a) flexible (b) rigid (c) rigid (overdetermined)

FIGURE 2. Graphs and their state of rigidity

Historical notes. The study of rigid structures, also called frameworks, was originally motivated by mechanics and architecture, and it goes back at least to the 19th century, to the works of James Clerk Maxwell, August Ritter, Karl Culmann, Luigi Cremona, August Föppl, and Lebrecht Henneberg. Nowadays, there is still a considerable interest in rigidity theory [GSS93] due to various applications in natural science and engineering; for an exemplary overview, see the conference proceedings "Rigidity Theory and Applications" [TD02]. Let us just highlight three application areas that are covered there: In materials science the rigidity of crystals, noncrystalline solids, glasses, silicates, etc. is studied; among the numerous publications in this area we can mention here only a few [BS13, JH97]. In biotechnology one is interested in possible conformations of proteins and cyclic molecules [JRKT01], and in particular in the enumeration of such conformations [LML+14, EM99]. In robotics one aims at computing the configurations of mechanisms, such as 6R chains or Stewart-Gough platforms. For the former, the 16 solutions of the inverse kinematic problem have been found by using very elegant arguments from algebraic geometry [Sel05, Section 11.5.1]. For the latter, the 40 complex assembly modes have been determined by algebraic geometry [RV95] or by computer algebra [FL95]; Dietmaier [Die98] showed that there is also an assignment of the parameters such that all 40 solutions are real. Recently, connections between rigidity theory and incidence problems have been established [Raz16].

Laman's characterization. A graph is called generically rigid (or isostatic) if a general edge labeling yields a rigid realization. No edge in a generically rigid graph can be removed without losing rigidity, that is why such graphs are also called minimally rigid in the literature. Note that the graph in Figure 2c is not generically rigid, while the one in Figure 2b is. Gerard Laman [Lam70] characterized this property in terms of the number of edges and vertices of the graph and its subgraphs, hence such objects are also known as Laman graphs.

State of the art. All realizations of a Laman graph with an edge labeling can be recovered as the solution set of a system of algebraic equations, where the edge labels can be seen as parameters. Here, we are interested in the number of complex solutions of such a system, up to an equivalence relation coming from direct planar isometries; this number is the same for any general choice of parameters, so we call it the *Laman number* of the graph. For some graphs up to 8 vertices, this number had been computed using random values for the parameters [JO12] — this means that it is very likely, but not absolutely certain, that these computations give the true numbers. Upper and lower bounds on the maximal Laman number for graphs with up to 10 vertices were found by analyzing the Newton polytopes of the equations and their mixed volumes [ETV09] using techniques from [ST10]. It has been proven [BS04] that the Laman number of a Laman graph with n vertices is at most $\binom{2n-4}{n-2}$ when n goes to infinity.

Our contribution. Our main result is a combinatorial algorithm that computes the number of complex realizations of any given Laman graph. This is much more efficient than just solving the corresponding nonlinear system of equations.

We found it convenient to see systems of equations related to Laman graphs as special cases of a slightly more general type of systems, determined by *bigraphs*. Roughly stated, a bigraph is a pair of graphs whose edges are in bijection. Every graph can be turned into a bigraph by duplication and it is possible to extend the notion of Laman number also to bigraphs. The majority of these newly introduced systems does not have geometric significance: they are merely introduced to have a suitable structure to set up a recursion scheme. Our main result (Theorem 4.15) is a recursion formula expressing the Laman number of a bigraph in terms of Laman numbers of smaller bigraphs. Using this formula we succeeded in computing the exact Laman numbers of graphs with up to 18 vertices — a task that was absolutely out of reach with the previously known methods.

The idea for proving the recursion formula is inspired by tropical geometry (see [MS15] or [Stu02, Chapter 9]): we consider the equation system over the field of Puiseux series, and the inspection of the valuations of the possible solutions allows us to endow every bigraph with some combinatorial data that prescribes how the recursion should proceed.

Structure of the paper. Section 1 contains the statement of the problem and a proof of the equivalence of generic rigidity and Laman's condition in our setting. This section is meant for a general mathematical audience and requires almost no prerequisite. Section 2 analyzes the system of equations defined by a bigraph, and Section 3 provides a scheme for a recursion formula for the number of solutions of the system. Here, we employ some standard techniques in algebraic geometry, so the reader should be acquainted with the basic concepts in this area. In Section 4, we specialize the general scheme provided at the end of Section 3 and we give two recursion formulas for the Laman number. One of them leads to an algorithm that is employed in Section 5 to derive some new results on the number of realizations of Laman graphs. The last two sections are again meant for a general audience, and they require only the knowledge of the objects and the results in Sections 2 and 3, but not of the proof techniques used there.

1. Laman graphs

In this section, by a *graph* we mean a finite, connected, undirected graph without self-loops or multiple edges. By writing $G = (V, E)$ we denote that the graph G has vertices V and edges E. An (unoriented) edge e between two vertices u and v is denoted by $\{u, v\}$.

Definition 1.1. A *labeling* of a graph $G = (V, E)$ is a function $\lambda \colon E \longrightarrow \mathbb{R}$. The pair (G, λ) is called a *labeled graph.*

Definition 1.2. Let $G = (V, E)$ be a graph. A *realization* of G is a function $\rho \colon V \longrightarrow \mathbb{R}^2$. Let λ be a labeling of G: we say that a realization ρ is *compatible with* λ if for each $e \in E$ the Euclidean distance between its endpoints agrees with its label:

$$\lambda(e) = \left\|\rho(u) - \rho(v)\right\|^2, \quad \text{where } e = \{u, v\}. \tag{1}$$

Definition 1.3. A labeled graph (G, λ) is *realizable* if and only if there exists a realization ρ that is compatible with λ.

Definition 1.4. We say that two realizations ρ_1 and ρ_2 of a graph G are *equivalent* if and only if there exists a direct Euclidean isometry σ of $\mathbb{R}^2$ such that $\rho_1 = \sigma \circ \rho_2$; recall that a direct Euclidean isometry is an affine-linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that preserves the distance and orientation in $\mathbb{R}^2$.

Definition 1.5. A labeled graph (G, λ) is called *rigid* if it satisfies the following properties:

- ▷ (G, λ) is realizable;
- ▷ there are only finitely many realizations compatible with λ, up to equivalence.

Our main interest is to count the number of realizations of generically rigid graphs, namely graphs for which almost all realizable labelings induce rigidity. Unfortunately, in the real setting, this number is not well-defined, since it may depend on the actual labeling and not only on the graph.

In order to define a number that depends only on the graph, we switch to a complex setting. By this we mean that we allow complex labelings $\lambda \colon E \longrightarrow \mathbb{C}$ and complex realizations $\rho \colon V \longrightarrow \mathbb{C}^2$. In this case, the compatibility condition (1) becomes

$$\lambda(e) = \big\langle \rho(u) - \rho(v), \rho(u) - \rho(v) \big\rangle, \quad e = \{u, v\},$$

where $\langle x, y \rangle = x_1 y_1 + x_2 y_2$. Moreover, we consider "direct complex isometries", namely maps of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto A \begin{pmatrix} x \\ y \end{pmatrix} + b, \quad A \in \mathbb{C}^{2 \times 2} \text{ and } b \in \mathbb{C}^2,$$

where A is an orthogonal matrix with determinant 1. Notice that if we are given a labeling $\lambda \colon E \longrightarrow \mathbb{R}$ for a graph G and two realizations of G into $\mathbb{R}^2$ that are not equivalent under real direct isometries, then they are not equivalent under complex isometries either. This means that counting the number of non-equivalent realizations in $\mathbb{C}^2$ delivers an upper bound for the number of non-equivalent realizations in $\mathbb{R}^2$.

Terminology. Given a graph $G = (V, E)$, the set of possible labelings $\lambda\colon E \longrightarrow \mathbb{C}$ forms a vector space, that we denote $\mathbb{C}^E$. In this way we are able to address the components of a vector λ in $\mathbb{C}^E$ by edges $e \in E$, namely to write $\lambda = (\lambda_e)_{e \in E}$. Since $\mathbb{C}^E$ is a vector space, it is meaningful to speak about properties holding for a *general* labeling: a property $\mathscr{P}$ holds for a general labeling if the set

$$\bigl\{\lambda \in \mathbb{C}^E \,:\, \mathscr{P}(\lambda) \text{ does not hold}\bigr\}$$

is contained in an algebraic proper subset of $\mathbb{C}^E$, i.e. a subset strictly contained in $\mathbb{C}^E$ and defined by polynomial equations. Whenever, in an argument, we say "fix a general labeling λ", we mean that we allow λ to satisfy any finite number of properties that hold in general and that may be needed.

Definition 1.6. A graph G is called *generically realizable* if for a general labeling λ the labeled graph (G, λ) is realizable. A graph G is called *generically rigid* if for a general labeling λ the labeled graph (G, λ) is rigid.

Remark 1.7. If a graph G is generically realizable, then every subgraph G' of G is generically realizable. In fact, every general labeling λ' for G' can be extended to a general labeling λ for G, and since by hypothesis the latter has a compatible realization, also the former admits one.

We now introduce the main objects of interest in this section.

Definition 1.8. A *Laman graph* is a graph $G = (V, E)$ such that $|E| = 2|V| - 3$, and for every subgraph $G' = (V', E')$ it holds $|E'| \leq 2|V'| - 3$.

We are going to see (Theorem 1.10) that Laman graphs are exactly the generically rigid ones. Many different characterizations of this property have appeared in the literature, for example by construction steps [Hen03] (see Theorem 1.10), or in terms of spanning trees after doubling one edge [LY82] or after adding an edge [Rec84], or in terms of three trees such that each vertex of the graph is covered by two trees [Cra06]. These characterizations can be used for decision algorithms on the minimal rigidity of a given graph [Ber05, JH97, DK09].

For any graph $G = (V, E)$, there is a natural map r_G from the set $\mathbb{C}^{2|V|}$ of its realizations to the set $\mathbb{C}^E$ of its labelings:

$$r_G\colon \mathbb{C}^{2|V|} \longrightarrow \mathbb{C}^E, \quad (x_v, y_v)_{v \in V} \longmapsto \bigl((x_u - x_v)^2 + (y_u - y_v)^2\bigr)_{\{u,v\} \in E}.$$

Each fiber of r_G, i.e. a preimage $r_G^{-1}(p)$ of a single point $p \in \mathbb{C}^E$, is invariant under the group of direct complex isometries. We define a subspace $\mathbb{C}^{2|V|-3} \subseteq \mathbb{C}^{2|V|}$ as follows: choose two distinguished vertices $\bar{u}$ and $\bar{v}$ with $\{\bar{u}, \bar{v}\} \in E$, and consider the linear subspace defined by the equations $x_{\bar{u}} = y_{\bar{u}} = 0$ and $x_{\bar{v}} = 0$. In this way the subspace $\mathbb{C}^{2|V|-3}$ intersects every orbit of the action of isometries on a fiber of r_G in exactly two points: in fact, the equations do not allow any further translation or rotation; however, for any labeling $\lambda\colon E \longrightarrow \mathbb{C}$ and for every realization in $\mathbb{C}^{2|V|-3}$

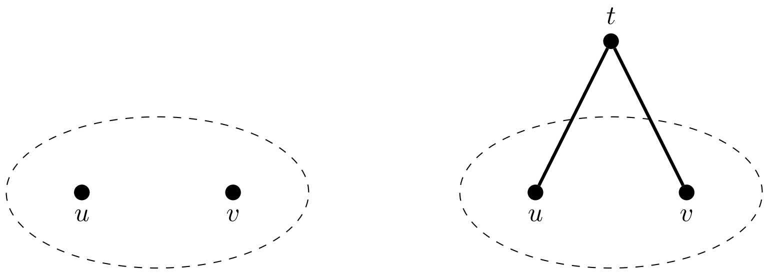


FIGURE 3. The first Henneberg rule: given any two vertices u and v (which may be connected by an edge or not), we add a vertex t and the two edges $\{u, t\}$ and $\{v, t\}$.

compatible with λ there exists another realization, obtained by multiplying the first one by -1, which is equivalent, but gives a different point in $\mathbb{C}^{2|V|-3}$. The restriction of r_G to $\mathbb{C}^{2|V|-3}$ gives the map

$$h_G \colon \mathbb{C}^{2|V|-3} \longrightarrow \mathbb{C}^E.$$

The following statement follows from the construction of h_G; notice that the choice of $\bar{u}$ and $\bar{v}$ has no influence on the result. Recall that a map $f \colon X \longrightarrow Y$ between algebraic sets is called *dominant* if $Y \setminus f(X)$ is contained in an algebraic proper subset of Y.

Lemma 1.9. *A graph G is generically rigid if and only if h_G is dominant and a general fiber of h_G is finite. This is equivalent to saying that h_G is dominant and $2|V| = |E| + 3$.*

Proof. It is enough to notice that if h_G is dominant, then the dimension of the general fiber is $2|V| - 3 - |E|$. □

We now report a proof of Laman's theorem characterizing generically rigid graphs. We do not claim any originality for this proof. In fact, we follow very closely Laman's argument, in particular in the first two implications. We include this proof for the sake of self-containedness, but we also recommend the reading of the original paper [Lam70]; moreover, for our purposes we need a result that implies the existence of only a finite number of complex realizations, while the original statement deals with the real setting, and proves that a given realization does not admit infinitesimal deformations.

Theorem 1.10. *Let $G = (V, E)$ be a graph. Then the following three conditions are equivalent:*

(a) *G is a Laman graph;*
(b) *G is generically rigid;*
(c) *G can be constructed by iterating the two* Henneberg rules *(see Figures 3 and 4), starting from the graph that consists of two vertices connected by an edge.*

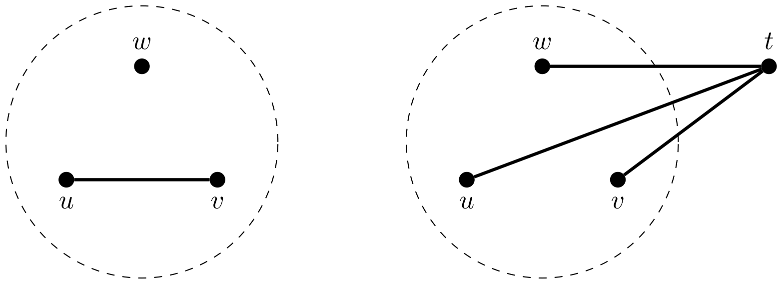


FIGURE 4. The second Henneberg rule: given any three vertices u, v, and w such that u and v are connected by an edge, we remove the edge $\{u, v\}$, we add a vertex t and the three edges $\{u, t\}$, $\{v, t\}$, and $\{w, t\}$.

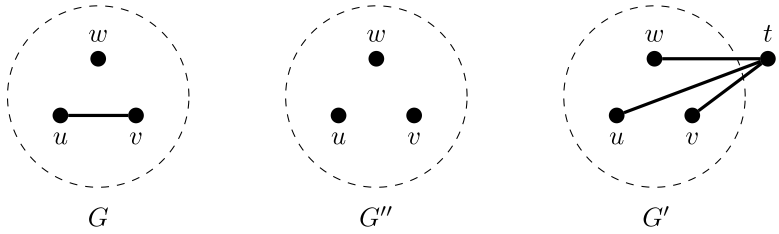
Proof. (b) $\Longrightarrow$ (a): Assume that G is generically rigid. Then every subgraph $G' = (V', E')$ is generically realizable (see Remark 1.7), and so the map $h_{G'}$ is dominant. Therefore, the dimension of the codomain is bounded by the dimension of the domain, which says $2|V'| - 3 \geq |E'|$. The equality in the previous formula for the whole graph G follows from Lemma 1.9.

(a) $\Longrightarrow$ (c): We prove the statement by induction on the number of vertices. The induction base with two vertices is clear. Assume that G is a Laman graph with at least 3 vertices. By [Lam70, Proposition 6.1], the graph G has a vertex of degree 2 or 3. If G has a vertex of degree 2, then the subgraph G' obtained by removing this vertex and its two adjacent edges is a Laman graph by [Lam70, Theorem 6.3]. By induction hypothesis, G' can be constructed by Henneberg rules, and then G can be constructed from G' by the first Henneberg rule. Assume now that G has a vertex v of degree 3. By [Lam70, Theorem 6.4], there are two vertices u and w connected with v such that the graph G' obtained by removing v and its three adjacent edges and then adding the edge $\{u, w\}$ is Laman. By induction hypothesis, G' can be constructed by Henneberg rules, and then G can be constructed from G' by the second Henneberg rule.

(c) $\Longrightarrow$ (b): We prove the statement by induction on the number of Henneberg rules. The induction base is the case of the one-edge graph, which is generically rigid. By induction hypothesis we assume that $G = (V, E)$ is generically rigid. Perform a Henneberg rule on G and let G' be the result. We intend to show that G' is generically rigid, too.

As far as the first Henneberg rule is concerned, we observe that for any realization of G, compatible with a general labeling λ, and for any labeling λ' extending λ we can always construct exactly two realizations of G' that are compatible with λ'.

Let us now assume that G' is constructed via the second Henneberg rule. Call t the new vertex of G', and denote the three vertices to which it is connected by u, v and w. Let G'' be the graph obtained by removing from G the same edge e that is removed in G'. Without loss of generality we assume $e = \{u, v\}$.



We first show that G' is generically realizable (see Definition 1.6). Let $N\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$ be the quadratic form corresponding to the bilinear form $\langle \cdot, \cdot \rangle$.

Fix a general labeling for G''. We define the algebraic set $C \subseteq \mathbb{C}^3$ as the set of all points (a, b, c) such that there is a compatible realization ρ of G'' satisfying

$$N\big(\rho(u) - \rho(v)\big) = a, \quad N\big(\rho(u) - \rho(w)\big) = c, \quad N\big(\rho(v) - \rho(w)\big) = b.$$

For a general $a_0 \in \mathbb{C}$, there exist finitely many, up to equivalence, points (a_0, b, c) in C, namely the "lengths" of the triangle (u, v, w) that come from the finitely many realizations of G. It follows that $\dim(C) \geq 1$.

A complex version of a classical result in distance geometry (see [ETV13, Theorem 2.4]) states that four points p_0, p_1, p_2, $p_3 \in \mathbb{C}^2$ fulfill

$$\begin{aligned} N(p_0 - p_1) &= x, & N(p_0 - p_2) &= y, & N(p_0 - p_3) &= z, \\ N(p_1 - p_2) &= a, & N(p_2 - p_3) &= b, & N(p_1 - p_3) &= c, \end{aligned}$$

if and only if the following Cayley-Menger determinant

$$F(a, b, c, x, y, z) := \det \begin{bmatrix} 0 & a & c & x & 1 \\ a & 0 & b & y & 1 \\ c & b & 0 & z & 1 \\ x & y & z & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

vanishes. We define

$$U := \bigcup_{p=(a,b,c)\in C} S_p\,, \quad \text{where} \quad S_p := \big\{(x, y, z) \in \mathbb{C}^3 : \; F(x, y, z, a, b, c) = 0\big\}$$

and

$$\begin{aligned} e_p := (\; & abc : a(a-b-c) : b(b-a-c) : c(c-a-b) : a-b-c : \\ & b-a-c : c-a-b : a : b : c \;) \in \mathbb{P}^9_{\mathbb{C}}. \end{aligned}$$

The point $e_p \in \mathbb{P}^9_{\mathbb{C}}$ with $p = (a, b, c)$ has coordinates given by the coefficients of $F(a, b, c, x, y, z)$, considered as a polynomial in x, y and z. Because of this, the point e_p determines S_p uniquely as a surface. The function $\mathbb{C}^3 \backslash \{0\} \longrightarrow \mathbb{P}^9_{\mathbb{C}}$ sending $p \mapsto e_p$ is injective, and hence the family $(S_p)_{p\in C}$ of surfaces is not constant. It follows that the algebraic set U has dimension 3, and thus a general point $(x, y, z) \in \mathbb{C}^3$ lies in U. If we extend the general labeling of G'' by assigning a general triple $(x, y, z) \in U$ as labels to the three new edges, then we get at least one realization of G'. It follows that G' is generically realizable.

Since $|V'| = |V| + 1$ and $|E'| = |E| + 2$, it follows that $2|V| = |E'| + 3$. Since, as we have just shown, the map $h_{G'}$ is dominant, the graph G' is generically rigid by Lemma 1.9. $\square$

For a given Laman graph, we are interested in the number of realizations in $\mathbb{C}^2$ that are compatible with a general labeling, up to equivalence. Theorem 1.10 and Lemma 1.9 imply that in this case the map h_G is dominant and its degree $\deg(h_G)$ is finite. Recall that in this context the *degree* of h_G is the cardinality of a fiber $h_G^{-1}(p)$ where p is a general point in the image of h_G. Moreover $\deg(h_G)$ is twice the number of realizations of G compatible with a general labeling, up to equivalence.

We now construct a similar map whose degree is exactly the number of equivalence classes.

For this purpose we employ a different way than in h_G to get rid of "translations" and "rotations": first, for the translations, we take a quotient of vector spaces, which can be interpreted as setting $x_{\bar{u}} = y_{\bar{u}} = 0$ as for h_G, or alternatively as moving the barycenter to the origin; second, we use projective coordinates to address the rotations. More precisely, in order to study the system of equations

$$\bigl((x_u - x_v)^2 + (y_u - y_v)^2 \;=\; \lambda_{uv}\bigr)_{\{u,v\} \in E}$$

which defines a realization of a Laman graph, we can regard the vectors $(x_u)_{u \in V}$ and $(y_u)_{u \in V}$ as elements of the space $\mathbb{C}^V / \langle x_u = x_v \text{ for all } u, v \in V \rangle$. In this way, we are allowed to add arbitrary constants to all components x_u or to all components y_u without changing the representative in the quotient; hence these vectors are invariant under translations. Moreover, if one performs the change of variables

$$(x_v)_{v \in V}, (y_v)_{v \in V} \quad \longmapsto \quad (x_v + i\, y_v)_{v \in V}, (x_v - i\, y_v)_{v \in V}, \tag{2}$$

then the previous system of equations becomes

$$\bigl((x_u - x_v)(y_u - y_v) \;=\; \lambda_{uv}\bigr)_{\{u,v\} \in E}$$

and the action of a complex rotation turns into the multiplication of the x_u-coordinates by a scalar in $\mathbb{C}$, and of the y_u-coordinates by its inverse. Thus, by considering $(x_u)_{u \in V}$ and $(y_u)_{u \in V}$ as coordinates in the projective space, the points we obtain are invariant under complex rotations.

In order to employ these two strategies, we define

$$\mathbb{P}_{\mathbb{C}}^{|V|-2} := \mathbb{P}\Bigl(\mathbb{C}^V / \bigl\langle (1, \ldots, 1) \bigr\rangle\Bigr) = \mathbb{P}\Bigl(\mathbb{C}^V / \bigl\langle (x_v)_{v \in V} \,:\, x_u = x_w \text{ for all } u, w \in V \bigr\rangle\Bigr)$$

and the map

$$\begin{array}{rccc} f_G\colon & \mathbb{P}_{\mathbb{C}}^{|V|-2} \times \mathbb{P}_{\mathbb{C}}^{|V|-2} & \dashrightarrow & \mathbb{P}_{\mathbb{C}}^{|E|-1} \\ & [(x_v)_{v \in V}], [(y_v)_{v \in V}] & \longmapsto & \Bigl((x_u - x_v)(y_u - y_v)\Bigr)_{\{u,v\} \in E} \end{array}, \tag{3}$$

where $[\,\cdot\,]$ denotes the point in $\mathbb{P}_{\mathbb{C}}^{|V|-2}$ determined by a vector in $\mathbb{C}^V$. Notice that the map f_G is well-defined, because the quantities $x_u - x_v$ depend, up to scalars, only on the points $[(x_v)_{v \in V}]$, and not on the particular choice of representatives (and similarly for $y_u - y_v$). Note that f_G may not be defined everywhere, which is conveyed by the notation $\dashrightarrow$.

Lemma 1.11. *For any Laman graph G the equality* $\deg(h_G) = 2\deg(f_G)$ *holds.*

Proof. Recall that the degree is computed by counting the number of preimages of a general point in the codomain. Let therefore $\lambda \in \mathbb{C}^E$ be a general labeling and let $\{\bar{u}, \bar{v}\}$ be the edge used to define h_G, so in particular we can suppose $\lambda\bigl(\{\bar{u}, \bar{v}\}\bigr) \neq 0$. We show that there is a 2:1 map η from $h_G^{-1}(\lambda)$ to $f_G^{-1}(\lambda)$, where $\lambda \in \mathbb{P}_{\mathbb{C}}^{|E|-1}$ is the point defined by the values of $\lambda \in \mathbb{C}^E$ as projective coordinates. The map η is defined according to the change of variables (2):

$$\eta\colon h_G^{-1}(\lambda) \longrightarrow f_G^{-1}(\lambda), \quad (x_v)_{v \in V}, (y_v)_{v \in V} \;\longmapsto\; \bigl[(x_v + i\, y_v)_{v \in V}\bigr], \bigl[(x_v - i\, y_v)_{v \in V}\bigr].$$

In other words, we just take the coordinates of the embedded vertices as projective coordinates and make a complex coordinate transformation, namely one that diagonalizes the linear part of the isometries. The map η is well-defined, since the quantities $(x_v + i\, y_v)_{v \in V}$ and $(x_v - i\, y_v)_{v \in V}$ are never all zero because of the definition of the map h_G. For $q \in \mathbb{P}_{\mathbb{C}}^{|V|-2} \times \mathbb{P}_{\mathbb{C}}^{|V|-2}$ of the form $q = \bigl([x_v]_{v \in V}, [y_v]_{v \in V}\bigr)$ and such that $x_{\bar{u}} \neq x_{\bar{v}}$ and $y_{\bar{u}} \neq y_{\bar{v}}$, we choose coordinates $(x_v)_{v \in V}, (y_v)_{v \in V}$ such that $x_{\bar{u}} = y_{\bar{u}} = 0$, $x_{\bar{v}} = 1$, and $y_{\bar{v}} = -1$; this is possible because we can add a constant vector without changing the point in $\mathbb{P}_{\mathbb{C}}^{|V|-2} \times \mathbb{P}_{\mathbb{C}}^{|V|-2}$. Then when $q \in f_G^{-1}(\lambda)$, every point in $\eta^{-1}(q)$ determines a realization of the form

$$\rho\colon V \longrightarrow \mathbb{C}^2, \qquad v \;\longmapsto\; \left(c\, \frac{x_v + y_v}{2}, c\, \frac{x_v - y_v}{2i} \right),$$

that must be compatible with λ. This forces $c \in \mathbb{C}$ to satisfy $c^2 = \lambda\bigl(\{\bar{u}, \bar{v}\}\bigr)$. There are exactly two such numbers c, and this proves the statement. ☐

Corollary 1.12. *The number of realizations of a Laman graph compatible with a general labeling, counted up to equivalence, is equal to the degree of the map f_G.*

2. Bigraphs and their equations

In this section we introduce the main concept of the paper, the one of *bigraph*. Bigraphs are pairs of graphs whose edges are in bijection; any graph determines a bigraph by simply duplicating it and considering the natural bijection between the edges. It is possible to associate to any bigraph a rational map as we did with the map f_G in Equation (3). The reason for this duplication is that in order to set up a recursive formula for the degree of f_G, we want to be able to handle independently the two factors $(x_u - x_v)$ and $(y_u - y_v)$ that appear in its specification. To do this we have to allow disconnected graphs with multiple edges.

Notice that if we allow graphs with multiedges, then we have to give away the possibility to encode an edge via an unordered pair of vertices. Instead, we consider the sets V and E of vertices and edges, respectively, to be arbitrary sets, related by a function $\tau\colon E \longrightarrow \mathcal{P}(V)$, where $\mathcal{P}$ denotes the power set, assigning to each edge its corresponding vertices. The image of an element $e \in E$ via τ can be either a set of cardinality two, when e connects two distinct vertices, or a singleton, when e is a self-loop.

The objects we are going to introduce are pairs of graphs for which there is a bijection between the two sets of edges. Because of the way we encode graphs, we can use exactly the same set for the edges of both graphs.

Definition 2.1. A *bigraph* is a pair of finite undirected graphs (G, H) — allowing several components, multiple edges and self-loops — where $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$. We denote by $\tau_G : \mathcal{E} \longrightarrow \mathcal{P}(V)$ and $\tau_H : \mathcal{E} \longrightarrow \mathcal{P}(W)$ the two maps assigning to each edge its vertices. The set $\mathcal{E}$ is called the set of *biedges*. For technical reasons, we need to order the vertices of edges in G or H; therefore, we assume that there is a total order $\prec$ given on the sets of vertices V and W. An example of a bigraph is provided in Figure 6.

Notice that a single graph $G = (V, E)$ can be turned into a bigraph by considering the pair (G, G), and by taking the set of biedges to be E; the total order $\prec$ is obtained by fixing any total order on V and duplicating it. Next, we extend a weakened version of the Laman condition to bigraphs.

Definition 2.2. For a graph $G = (V, E)$ we define the *dimension* of G as

$$\dim(G) := |V| - |\{\text{connected components of } G\}|.$$

Remark 2.3. Since a Laman graph is connected by assumption, the condition $2|V| = |E| + 3$ can be rewritten as $2\dim(G) = |E| + 1$.

Definition 2.4. Let $B = (G, H)$ be a bigraph with biedges $\mathcal{E}$, then we say that B is *pseudo-Laman* if

$$\dim(G) + \dim(H) = |\mathcal{E}| + 1.$$

It follows immediately from Remark 2.3 that if G is a Laman graph, then the bigraph (G, G) is pseudo-Laman.

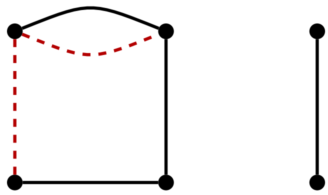
We introduce two operations that can be performed on a graph, starting from a subset of its edges: the subtraction of edges and the quotient by edges. We are going to use these constructions several times in our paper: subtraction is first used at the end of this section, while the quotient operation is mainly utilized starting from Section 3.

Definition 2.5. Let $G = (V, E)$ be a graph, and let $E' \subseteq E$. We define two new graphs, denoted $G \,/\, E'$ and $G \setminus E'$, as follows.

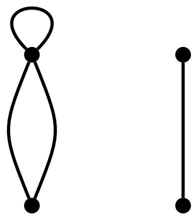
- ▷ Let G' be the subgraph of G determined by E'. Then we define $G \,/\, E'$ to be $G \,/\, G'$. Here by $G \,/\, G'$ we mean the graph obtained as follows: its vertices are the equivalence classes of the vertices of G modulo the relation dictating that two vertices u and v are equivalent if there exists a path in G' connecting them; its edges are determined by edges in $E \setminus E'$, more precisely an edge e in $E \setminus E'$ such that $\tau_G(e) = \{u, v\}$ defines an edge in the quotient connecting the equivalence classes of u and v if and only e is not an edge of G'.
- ▷ Let $\widehat{V}$ be the set of vertices of G that are endpoints of some edge not in E'. Set $\widehat{E} = E \setminus E'$. Define $G \setminus E' = (\widehat{V}, \widehat{E})$.

An example for the two operations is provided in Figure 5.

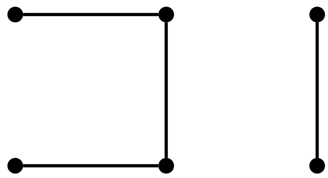
Via Definitions 2.6 and 2.7 we associate to each bigraph B a rational function f_B, as we did in Section 1 for graphs.



(a) A graph $G = (V, E)$ and a subset E' of edges, in dashed red.



(b) The graph $G \,/\, E'$.



(c) The graph $G \setminus E'$.

FIGURE 5. Example of the two constructions in Definition 2.5.

Definition 2.6. Let $B = (G, H)$ be a bigraph, where $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$. We set

$$\begin{aligned} \mathbb{P}_{\mathbb{C}}^{\dim(G)-1} &:= \mathbb{P}\big(\mathbb{C}^V \,/\, L_G\big), \\ \mathbb{P}_{\mathbb{C}}^{\dim(H)-1} &:= \mathbb{P}\big(\mathbb{C}^W \,/\, L_H\big), \end{aligned}$$

where

$$L_G := \left\langle (x_v)_{v \in V} \;:\; \begin{array}{c} x_u = x_t \text{ if and only if } u \text{ and } t \\ \text{are in the same connected component of } G \end{array} \right\rangle,$$

$$L_H := \left\langle (y_w)_{w \in W} \;:\; \begin{array}{c} y_u = y_t \text{ if and only if } u \text{ and } t \\ \text{are in the same connected component of } H \end{array} \right\rangle,$$

and $(x_v)_{v \in V}$ are the standard coordinates of $\mathbb{C}^V$ and similarly for $(y_w)_{w \in W}$.

Definition 2.7. Let $B = (G, H)$ be a bigraph, where $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$. Define

$$\begin{array}{rccc} f_B \colon & \mathbb{P}_{\mathbb{C}}^{\dim(G)-1} \times \mathbb{P}_{\mathbb{C}}^{\dim(H)-1} & \dashrightarrow & \mathbb{P}_{\mathbb{C}}^{|\mathcal{E}|-1} \\ & \big[(x_v)_{v \in V}\big], \big[(y_w)_{w \in W}\big] & \longmapsto & \big((x_u - x_v)(y_t - y_w)\big)_{e \in \mathcal{E}} \end{array},$$

where $\{u, v\} = \tau_G(e)$, $u \prec v$, and $\{t, w\} = \tau_H(e)$, $t \prec w$, with τ_G and τ_H as in Definition 2.1. Here and in the rest of the paper, if e is a self-loop say in G, then the corresponding polynomial in the definition of f_B is considered to be $x_u - x_u = 0$. As in Section 1, the square brackets $[\,\cdot\,]$ denote points in $\mathbb{P}_{\mathbb{C}}^{\dim(G)-1}$ or $\mathbb{P}_{\mathbb{C}}^{\dim(H)-1}$ determined by vectors in $\mathbb{C}^V$ or $\mathbb{C}^W$. As for the map f_G, the map f_B is well-defined because the quantities $(x_u - x_v)$ and $(y_t - y_w)$ depend only on points, and not on the chosen representatives. We call the map f_B the *rational map associated to* B.

In Definition 2.7 we impose $u \prec v$ and $t \prec w$ in the equations defining the map f_B. The reason for this is that we want that f_B, when B is of the form (G, G), coincides with f_G defined at the end of Section 1. If we do not specify the order in which the vertices appear in the expressions $(x_u - x_v)$ and $(y_t - y_w)$, we could end up with a map f_B for which one component is of the form $(x_u - x_v)(y_v - y_u)$, and not $(x_u - x_v)(y_u - y_v)$ as we would expect.

As in Section 1, we are mainly interested in the degree of the rational map associated to a bigraph.

Definition 2.8. Let B be a bigraph. If f_B is dominant, we define the *Laman number* $\mathrm{Lam}(B)$ of B as $\deg(f_B)$, which can hence be either a positive number, or ∞. Otherwise we set $\mathrm{Lam}(B)$ to zero.

Remark 2.9. Notice that if B is pseudo-Laman and $\mathrm{Lam}(B) > 0$, then $\mathrm{Lam}(B) \in \mathbb{N} \setminus \{0\}$.

If a bigraph has a self-loop or it is particularly simple, then its Laman number is zero or one, as shown by the following proposition.

Proposition 2.10. *Let $B = (G, H)$ be a bigraph.*

- ▷ *If G or H has a self-loop, then* $\mathrm{Lam}(B) = 0$.
- ▷ *If both G and H consist of a single edge that joins two vertices, then* $\mathrm{Lam}(B) = 1$.

Proof. If G or H has a self-loop, a direct inspection of the map f_B shows that one of its defining polynomials (the one corresponding to the self-loop) is zero, hence f_B cannot be dominant.

If both G and H consist of a single edge that joins two vertices, then the map f_B reduces to the map $\mathbb{P}^0_{\mathbb{C}} \times \mathbb{P}^0_{\mathbb{C}} \longrightarrow \mathbb{P}^0_{\mathbb{C}}$, which has degree 1. □

By simply unraveling the definitions we see that the number of realizations of a Laman graph can be expressed as Laman numbers; this is stated in the following proposition.

Proposition 2.11. *Let G be a Laman graph, then the Laman number of the bigraph (G, G) — where biedges are the edges of G — is equal to the number of different realizations compatible with a general labeling of G, up to direct complex isometries.*

Due to Proposition 2.11, the problem we want to address in this work is a special instance of the problem of computing the Laman number of a bigraph. Notice, however, that the Laman number of an arbitrary bigraph does not have an immediate geometric interpretation.

Remark 2.12. Let B be a bigraph with biedges $\mathcal{E}$ such that $\mathrm{Lam}(B) > 0$ and fix a biedge $\bar{e} \in \mathcal{E}$. Since f_B is a rational dominant map between varieties over $\mathbb{C}$, there is a Zariski open subset $\mathcal{U} \subseteq \mathbb{P}^{|\mathcal{E}|-1}_{\mathbb{C}}$ such that the preimage of any point $p \in \mathcal{U}$ under f_B consists of $\mathrm{Lam}(B)$ distinct points. In particular, we can suppose that p is of the form $(\lambda_e)_{e \in \mathcal{E}}$ with $\lambda_{\bar{e}} = 1$ and $(\lambda_e)_{e \in \mathcal{E} \setminus \{\bar{e}\}}$ a general point of $\mathbb{C}^{\mathcal{E} \setminus \{\bar{e}\}}$.

The computation of the degree of the maps f_B can therefore be accomplished by counting the cardinality of a general fiber. In the following we find it useful to work in an affine setting: this is why in Definition 2.14 we introduce the sets $Z_{\mathbb{C}}^{B}$. We are going to use the language of affine schemes, mainly to be able to manipulate the equations freely without being concerned about the reducedness of the ideal they generate; the reader not acquainted with scheme theory can harmlessly think about classical affine varieties, and indeed we are going to prove that the ideals we are concerned with are reduced. We first need to set some notation.

Definition 2.13. Let $B = (G, H)$ be a bigraph, where $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$. Define the two sets

$$P := \big\{(u, v) \in V^2 \,:\, \{u, v\} \in \tau_G(\mathcal{E}),\ u \neq v\big\},$$
$$Q := \big\{(t, w) \in W^2 \,:\, \{t, w\} \in \tau_H(\mathcal{E}),\ t \neq w\big\}.$$

Notice that the elements of P and Q are ordered, and not unordered pairs (and this is conveyed also by the different notation used). In particular, from the definition we see that if $(u, v) \in P$, then also $(v, u) \in P$, and similarly for Q. Moreover, we require the two elements in each pair to be different, and this is crucial in view of Definition 3.7.

Definition 2.14. Let $B = (G, H)$ be a bigraph with biedges $\mathcal{E}$ without self-loops. Fix a biedge $\bar{e} \in \mathcal{E}$. For a general point $(\lambda_e)_{e \in \mathcal{E} \setminus \{\bar{e}\}}$ in $\mathbb{C}^{\mathcal{E} \setminus \{\bar{e}\}}$, we define $Z_{\mathbb{C}}^{B}$ as the subscheme of $\mathbb{C}^P \times \mathbb{C}^Q$ defined by

$$\begin{cases} x_{\bar{u}\bar{v}} = y_{\bar{t}\bar{w}} = 1 & \bar{u} \prec \bar{v},\ \bar{t} \prec \bar{w}, \\ x_{uv}\, y_{tw} = \lambda_e & \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\},\ u \prec v,\ t \prec w, \\ \sum_{\mathscr{C}} x_{uv} = 0 & \text{for all cycles } \mathscr{C} \text{ in } G, \\ \sum_{\mathscr{D}} y_{tw} = 0 & \text{for all cycles } \mathscr{D} \text{ in } H, \end{cases}$$

where we take $(x_{uv})_{(u,v) \in P}$ and $(y_{tw})_{(t,w) \in Q}$ as coordinates and where

$$\{\bar{u}, \bar{v}\} = \tau_G(\bar{e}), \qquad \{u, v\} = \tau_G(e),$$
$$\{\bar{t}, \bar{w}\} = \tau_H(\bar{e}), \qquad \{t, w\} = \tau_H(e).$$

Here and in the following, when we write $\sum_{\mathscr{C}} x_{uv}$ for a cycle $\mathscr{C} = (u_0, u_1, \ldots, u_n = u_0)$ in G we mean the expression $x_{u_0 u_1} + \cdots + x_{u_{n-1} u_0}$ (and similarly for cycles in H). In particular, if $(u, v) \in P$, one can always consider the cycle (u, v, u), which implies the relation $x_{uv} = -x_{vu}$.

We drop the dependence of $Z_{\mathbb{C}}^{B}$ on $\bar{e}$ and $(\lambda_e)_{e \in \mathcal{E} \setminus \{\bar{e}\}}$ in the notation, since in the following it is clear from the context.

Example 2.15. Consider the bigraph (G, G) with set of biedges $\mathcal{E}$ as in Figure 6, that consists of two copies of the only Laman graph with 4 vertices. If we fix the biedge $\bar{e}$ to be the one associated to the two edges connecting 2 and 3, and we fix a

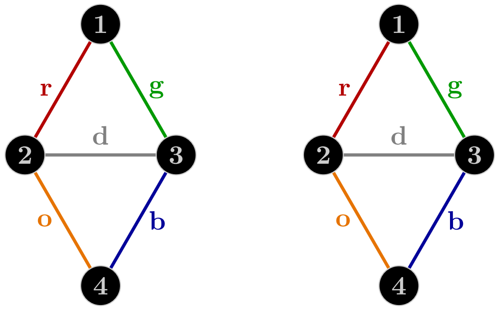


FIGURE 6. A bigraph that consists of two copies of the only Laman graph with 4 vertices. Edges on the left and on the right bearing the same label are associated to the same biedge.

general point $(\lambda_e)_{e\in\mathcal{E}\setminus\{\bar{e}\}}$, then the scheme $Z^B_{\mathbb{C}}$ is defined by the following equations:

$$x_{23} = y_{23} = 1,$$

$$\begin{aligned} x_{12}\,y_{12} &= \lambda_{\mathbf{r}}, & x_{12}+x_{21} = x_{13}+x_{31} = x_{23}+x_{32} = x_{24}+x_{42} = x_{34}+x_{43} &= 0,\\ x_{13}\,y_{13} &= \lambda_{\mathbf{g}}, & y_{12}+y_{21} = y_{13}+y_{1} = y_{23}+y_{32} = y_{24}+y_{42} = y_{34}+y_{43} &= 0,\\ x_{24}\,y_{24} &= \lambda_{\mathbf{o}}, & x_{12}+x_{23}+x_{31} = y_{12}+y_{23}+y_{31} &= 0,\\ x_{34}\,y_{34} &= \lambda_{\mathbf{b}}, & x_{24}+x_{43}+x_{32} = y_{24}+y_{43}+y_{32} &= 0. \end{aligned}$$

Note that we did not include redundant equations coming from cycles such as $(1,2,4,3,1)$.

In the following lemma we show that the sets $Z^B_{\mathbb{C}}$ can be used to compute the degree of the map f_B.

Lemma 2.16. *Let $B = (G, H)$ be a bigraph with biedges $\mathcal{E}$ without self-loops. Fix a biedge $\bar{e} \in \mathcal{E}$. Let $p \in \mathbb{P}^{|\mathcal{E}|-1}_{\mathbb{C}}$ be given by $p_{\bar{e}} = 1$ and $p_e = \lambda_e$ for all $e \in \mathcal{E}\setminus\{\bar{e}\}$. Then the schemes $f_B^{-1}(p)$ and $Z^B_{\mathbb{C}}$ are isomorphic. In particular, $Z^B_{\mathbb{C}}$ consists of* $\mathrm{Lam}(B)$ *distinct points.*

Proof. We define a morphism from $f_B^{-1}(p)$ to $Z^B_{\mathbb{C}}$ by sending a point

$$\big([(x_v)_{v\in V}], [(y_w)_{w\in W}]\big) \in f_B^{-1}(p)$$

to the point whose uv-coordinate is $(x_u - x_v)/(x_{\bar{u}} - x_{\bar{v}})$, where $\bar{u} \prec \bar{v}$, for all $(u, v) \in P$, and whose tw-coordinate is $(y_t - y_w)/(y_{\bar{t}} - y_{\bar{w}})$, where $\bar{t} \prec \bar{w}$, for all $(t, w) \in Q$.

We define a morphism from $Z^B_{\mathbb{C}}$ to $f_B^{-1}(p)$ as follows. For every component C of G, fix a rooted spanning tree T_C and denote its root by $r(C)$; similarly for H. We send a point $\big((x_{uv})_{(u,v)\in P}, (y_{tw})_{(t,w)\in Q}\big) \in Z^B_{\mathbb{C}}$ to the point $\big([(x_u)_{u\in V}], [(y_t)_{t\in W}]\big) \in f_B^{-1}(p)$ such that if a vertex $u \in V$ belongs to the connected component C, then $x_u = \sum_{i=0}^{n-1} x_{u_i u_{i+1}}$, where $(r(C) = u_0, \ldots, u_n = u)$ is the unique path in T_C from $r(C)$ to u, and similarly for the vertices $t \in W$.

A direct computation shows that both maps are well-defined, and are each other's inverse. From this the statement follows. $\square$

We conclude the section by proving a few results about the Laman number of a special kind of bigraphs, that is used in Section 4 to obtain the final algorithm.

Definition 2.17. Let G be a graph and let e be an edge of G. We say that e is a *bridge* if removing e increases the number of connected components of G.

Lemma 2.18. *Let $B = (G, H)$ be a pseudo-Laman bigraph with biedges $\mathcal{E}$ without self-loops and fix $\bar{e} \in \mathcal{E}$. If $\bar{e}$ is a bridge in both G and H, then* $\mathrm{Lam}(B) = 0$.

Proof. Suppose for a contradiction $\mathrm{Lam}(B) > 0$. Consider the equations defining $Z_{\mathbb{C}}^{B}$: since $\bar{e}$ is a bridge in both G and H, the variables $x_{\bar{u}\bar{v}}$ and $y_{\bar{t}\bar{w}}$, where $\{\bar{u}, \bar{v}\} = \tau_G(\bar{e})$ and $\{\bar{t}, \bar{w}\} = \tau_H(\bar{e})$, do not appear in any of the equations defined by cycles in G or in H except for the equations $x_{\bar{u}\bar{v}} = -x_{\bar{v}\bar{u}}$ and $y_{\bar{t}\bar{w}} = -y_{\bar{w}\bar{t}}$. Hence, the system of equations

$$\begin{cases} x_{uv}\, y_{tw} = \lambda_e, & \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\},\ u \prec v,\ t \prec w, \\ \sum_{\mathscr{C}} x_{uv} = 0, & \text{for all cycles } \mathscr{C} \text{ in } G \setminus \{\bar{e}\}, \\ \sum_{\mathscr{D}} y_{tw} = 0, & \text{for all cycles } \mathscr{D} \text{ in } H \setminus \{\bar{e}\} \end{cases}$$

defines an affine scheme $\widetilde{Z}$ isomorphic to $Z_{\mathbb{C}}^{B}$. One notices, however, that if (x_{uv}, y_{tw}) is a point in $\widetilde{Z}$, then for every $\eta \in \mathbb{C} \setminus \{0\}$ also the point $(\eta\, x_{uv}, \frac{1}{\eta} y_{tw})$ is in $\widetilde{Z}$, this implying that $Z_{\mathbb{C}}^{B}$ has infinite cardinality. This contradicts the pseudo-Laman assumption on B. $\square$

Lemma 2.19. *Let $B = (G, H)$ be a pseudo-Laman bigraph with biedges $\mathcal{E}$ without self-loops and fix $\bar{e} \in \mathcal{E}$. If $\bar{e}$ is a bridge in G, but not in H, then*

$$\mathrm{Lam}(B) \;=\; \mathrm{Lam}\big((G \setminus \{\bar{e}\}, H \setminus \{\bar{e}\})\big).$$

Proof. Consider another biedge $\tilde{e}$ and use it to define the scheme $Z_{\mathbb{C}}^{B}$. Its equations are hence:

$$Z_{\mathbb{C}}^{B}: \begin{cases} x_{\tilde{u}\tilde{v}} = y_{\tilde{t}\tilde{w}} = 1, & \tilde{u} \prec \tilde{v},\ \tilde{t} \prec \tilde{w}, \\ x_{uv}\, y_{tw} = \lambda_e, & \text{for all } e \in \mathcal{E} \setminus \{\tilde{e}\},\ u \prec v,\ t \prec w, \\ \sum_{\mathscr{C}} x_{uv} = 0, & \text{for all cycles } \mathscr{C} \text{ in } G, \\ \sum_{\mathscr{D}} y_{tw} = 0, & \text{for all cycles } \mathscr{D} \text{ in } H. \end{cases}$$

Now consider the bigraph $\widetilde{B} = \big(G \setminus \{\bar{e}\}, H \setminus \{\bar{e}\}\big)$: notice that we can still use $\tilde{e}$ to define the scheme $Z_{\mathbb{C}}^{\widetilde{B}}$. Its equations are:

$$Z_{\mathbb{C}}^{\widetilde{B}}: \begin{cases} x_{\tilde{u}\tilde{v}} = y_{\tilde{t}\tilde{w}} = 1, & \tilde{u} \prec \tilde{v},\ \tilde{t} \prec \tilde{w}, \\ x_{uv}\, y_{tw} = \lambda_e, & \text{for all } e \in \mathcal{E} \setminus \{\bar{e}, \tilde{e}\},\ u \prec v,\ t \prec w, \\ \sum_{\mathscr{C}} x_{uv} = 0, & \text{for all cycles } \mathscr{C} \text{ in } G \setminus \{\bar{e}\}, \\ \sum_{\mathscr{D}} y_{tw} = 0, & \text{for all cycles } \mathscr{D} \text{ in } H \setminus \{\bar{e}\}. \end{cases}$$

We are going to prove that $Z_{\mathbb{C}}^{B}$ and $Z_{\mathbb{C}}^{\widetilde{B}}$ are isomorphic, concluding the proof. Since $\bar{e}$ is a bridge in G, the coordinate $x_{\bar{u}\bar{v}}$ appears in the equations of $Z_{\mathbb{C}}^{B}$ only in $x_{\bar{u}\bar{v}}\, y_{\bar{t}\bar{w}} = \lambda_{\bar{e}}$, and in $x_{\bar{u}\bar{v}} = -x_{\bar{v}\bar{u}}$. This means that the image of $Z^B_{\mathbb{C}}$ under the projection from the coordinates $x_{\bar{u}\bar{v}}$, $x_{\bar{v}\bar{u}}$, $y_{\bar{t},\bar{w}}$ and $y_{\bar{w},\bar{t}}$ coincides with $Z^{\widetilde{B}}_{\mathbb{C}}$. Moreover, the projection is an isomorphism on $Z^B_{\mathbb{C}}$: in fact, the $y_{\bar{t},\bar{w}}$-coordinate can be recovered by a cycle condition (recall that $\bar{e}$ is not a bridge in H, so it appears in a cycle different from the trivial cycle $(\bar{t}, \bar{w}, \bar{t})$), and then the $x_{\bar{u},\bar{v}}$-coordinate can be recovered from the equation $x_{\bar{u}\bar{v}}\, y_{\bar{t}\bar{w}} = \lambda_{\bar{e}}$. □

Definition 2.20. Let $B = (G, H)$ be a bigraph with biedges $\mathcal{E}$ without self-loops and let $\bar{e} \in \mathcal{E}$ be fixed. Suppose that the graph G splits into disconnected subgraphs G'_1, G'_2 and that H splits into disconnected subgraphs H'_1, H'_2. Suppose further that $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{\bar{e}\}$ decomposes into three disjoint subsets such that

$$G'_1 = (V'_1, \mathcal{E}_1 \cup \{\bar{e}\}), \quad G'_2 = (V'_2, \mathcal{E}_2) \quad \text{and} \quad H'_1 = (W'_1, \mathcal{E}_1), \quad H'_2 = (W'_2, \mathcal{E}_2 \cup \{\bar{e}\}).$$

Under these assumptions we say that the bigraph B *untangles* via $\bar{e} \in \mathcal{E}$ into bigraphs B_1 and B_2, where

$$\begin{aligned} B_1 &:= \bigl(G'_1 \setminus \{\bar{e}\},\ H'_1\bigr), \\ B_2 &:= \bigl(G'_2,\ H'_2 \setminus \{\bar{e}\}\bigr). \end{aligned}$$

See Figure 8 for an example of a bigraph that untangles with respect to an edge (the gray vertical one).

Proposition 2.21. *Suppose that a bigraph $B = (G, H)$ with biedges $\mathcal{E}$ without self-loops untangles via $\bar{e} \in \mathcal{E}$ into bigraphs B_1 and B_2 such that $\bar{e}$ is neither a bridge in G nor in H, then*

$$\mathrm{Lam}(B) \;=\; \mathrm{Lam}(B_1) \cdot \mathrm{Lam}(B_2).$$

Proof. We use the notation from Definition 2.20. The hypothesis implies that

$$\dim(G) \;=\; \dim(G'_1) + \dim(G'_2) \quad \text{and} \quad \dim(H) \;=\; \dim(H'_1) + \dim(H'_2).$$

Set $\{\bar{u}, \bar{v}\} = \tau_G(\bar{e})$ and $\{\bar{t}, \bar{w}\} = \tau_H(\bar{e})$. Fix a biedge $e_1 \in \mathcal{E}_1$ and let $\{u_1, v_1\} = \tau_G(e_1)$ and $\{t_1, w_1\} = \tau_H(e_1)$. Similarly, fix a biedge $e_2 \in \mathcal{E}_2$ and let $\{u_2, v_2\} = \tau_G(e_2)$ and $\{t_2, w_2\} = \tau_H(e_2)$. We consider the following three rational maps:

$$\begin{array}{rcl}
\mathbb{P}^{\dim(G)-1}_{\mathbb{C}} & \dashrightarrow & \mathbb{P}^{\dim(G'_1)-1}_{\mathbb{C}} \times \mathbb{P}^{\dim(G'_2)-1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \\
\bigl([(x_v)_{v\in V}]\bigr) & \longmapsto & \bigl([(x_v)_{v\in V'_1}], [(x_v)_{v\in V'_2}], (x_{\bar{u}} - x_{\bar{v}} : x_{u_2} - x_{v_2})\bigr) \\
\mathbb{P}^{\dim(H)-1}_{\mathbb{C}} & \dashrightarrow & \mathbb{P}^{\dim(H'_1)-1}_{\mathbb{C}} \times \mathbb{P}^{\dim(H'_2)-1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \\
\bigl([(y_w)_{w\in W}]\bigr) & \longmapsto & \bigl([(y_w)_{w\in W'_1}], [(y_w)_{w\in W'_2}], (y_{\bar{t}} - y_{\bar{w}} : x_{t_1} - x_{w_1})\bigr) \\
\mathbb{P}^{|\mathcal{E}|-1}_{\mathbb{C}} & \dashrightarrow & \mathbb{P}^{|\mathcal{E}_1|-1}_{\mathbb{C}} \times \mathbb{P}^{|\mathcal{E}_2|-1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \\
(z_e)_{e\in\mathcal{E}} & \longmapsto & \bigl((z_e)_{e\in\mathcal{E}_1}, (z_e)_{e\in\mathcal{E}_2}, (z_{\bar{e}} : z_{e_1}), (z_{\bar{e}} : z_{e_2})\bigr)
\end{array}$$

One can check that these maps are birational. We define the rational map $\widehat{f}$ so that the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{P}_{\mathbb{C}}^{\dim(G)-1} \times \mathbb{P}_{\mathbb{C}}^{\dim(H)-1} \arrow[r, dashed, leftrightarrow] \arrow[d, dashed, "f_B"] & \begin{array}{c} \left(\mathbb{P}_{\mathbb{C}}^{\dim(G_1')-1} \times \mathbb{P}_{\mathbb{C}}^{\dim(G_2')-1} \times \mathbb{P}_{\mathbb{C}}^{1}\right) \\ \times \\ \left(\mathbb{P}_{\mathbb{C}}^{\dim(H_1')-1} \times \mathbb{P}_{\mathbb{C}}^{\dim(H_2')-1} \times \mathbb{P}_{\mathbb{C}}^{1}\right) \end{array} \arrow[d, dashed, "\widehat{f}"] \\
\mathbb{P}_{\mathbb{C}}^{|\mathcal{E}|-1} \arrow[r, dashed, leftrightarrow] & \mathbb{P}_{\mathbb{C}}^{|\mathcal{E}_1|-1} \times \mathbb{P}_{\mathbb{C}}^{|\mathcal{E}_2|-1} \times \mathbb{P}_{\mathbb{C}}^{1} \times \mathbb{P}_{\mathbb{C}}^{1}
\end{tikzcd}$$

It follows that $\deg(f_B) = \deg\big(\widehat{f}\big)$. Denote $[(x_v)_{v \in V_i'}]$ by $[X_i]$ for $i \in \{1, 2\}$, and denote $[(y_w)_{w \in W_i'}]$ by $[Y_i]$ for $i \in \{1, 2\}$. An explicit computation shows that $\widehat{f}$ sends a point

$$\Big([X_1], [X_2], (\mu_G : \nu_G)\Big), \Big([Y_1], [Y_2], (\mu_H : \nu_H)\Big)$$

to the point

$$\Big(\overbrace{f_{B_1}\big([X_1], [Y_1]\big)}^{\in \mathbb{P}_{\mathbb{C}}^{|\mathcal{E}_1|-1}}, \overbrace{f_{B_2}\big([X_2], [Y_2]\big)}^{\in \mathbb{P}_{\mathbb{C}}^{|\mathcal{E}_2|-1}},$$
$$\underbrace{\big(\mu_G\, \delta_G([X_1]) : \nu_G\, \delta_G([X_1])\big)}_{\in \mathbb{P}_{\mathbb{C}}^{1}}, \underbrace{\big(\mu_H\, \delta_H([Y_2]) : \nu_H\, \delta_H([Y_2])\big)}_{\in \mathbb{P}_{\mathbb{C}}^{1}}\Big),$$

where $\delta_G \colon \mathbb{P}_{\mathbb{C}}^{\dim(G_1')-1} \dashrightarrow \mathbb{C}$ and $\delta_H \colon \mathbb{P}_{\mathbb{C}}^{\dim(H_2')-1} \dashrightarrow \mathbb{C}$ are some rational functions. From the explicit form of $\widehat{f}$ we see that $\deg\big(\widehat{f}\big) = \deg\big(\widehat{f_1}\big) \cdot \deg\big(\widehat{f_2}\big)$, where the map $\widehat{f_1}$ is given by

$$\begin{array}{ccc}
\mathbb{P}_{\mathbb{C}}^{\dim(G_1')-1} \times \mathbb{P}_{\mathbb{C}}^{\dim(H_1')-1} \times \mathbb{P}_{\mathbb{C}}^{1} & \dashrightarrow & \mathbb{P}_{\mathbb{C}}^{|\mathcal{E}_1|-1} \times \mathbb{P}_{\mathbb{C}}^{1} \\
\big([X_1], [Y_1], (\mu_G : \nu_G)\big) & \longmapsto & f_{B_1}\big([X_1], [Y_1]\big), \big(\mu_G\, \delta_G([X_1]) : \nu_G\, \delta_G([X_1])\big)
\end{array}$$

and similarly for $\widehat{f_2}$. Note that for both $i \in \{1, 2\}$, the map $\widehat{f_i}$ is nothing but the restriction to a suitable open set of the map $f_{B_i} \times \mathrm{id}_{\mathbb{P}_{\mathbb{C}}^1}$, since the rational maps δ_G and δ_H do not have any other influence than restricting the domain of the map. This means that $\deg\big(\widehat{f_i}\big) = \deg(f_{B_i})$ for both $i \in \{1, 2\}$, which concludes the proof. $\square$

Lemma 2.22. *If a pseudo-Laman bigraph $B = (G, H)$ without self-loops untangles via $\bar{e} \in \mathcal{E}$ into bigraphs B_1 and B_2 such that $\bar{e}$ is a bridge in G but not in H, then* $\mathrm{Lam}(B) = 0$.

Proof. By Lemma 2.19, we know that $\mathrm{Lam}(B) = \mathrm{Lam}\big(\widetilde{B}\big)$, where $\widetilde{B}$ is the bigraph $\big(G \setminus \{\bar{e}\}, H \setminus \{\bar{e}\}\big)$. It follows that $\widetilde{B}$ is the disjoint union of B_1 and B_2. Using the same technique adopted in Lemma 2.18 we see that if $\mathrm{Lam}\big(\widetilde{B}\big)$ were positive, then we could scale the points in $Z_{\mathbb{C}}^{B_1}$ by arbitrary scalars $\eta \in \mathbb{C} \setminus \{0\}$, obtaining an infinite Laman number. This would contradict the pseudo-Laman hypothesis, so the statement is proved. $\square$

Lemma 2.23. *If a pseudo-Laman bigraph $B = (G, H)$ without self-loops untangles via $\bar{e} \in \mathcal{E}$ into bigraphs B_1 and B_2 such that $\bar{e}$ is a bridge in G, then either B_1 or B_2 is not pseudo-Laman.*

Proof. Suppose that both B_1 and B_2 are pseudo-Laman; we show that this leads to a contradiction. Using the hypothesis one sees that $\dim\bigl(G'_1 \setminus \{\bar{e}\}\bigr) = \dim(G'_1) - 1$; moreover, if $\bar{e}$ is also a bridge, then $\dim\bigl(H'_2 \setminus \{\bar{e}\}\bigr) = \dim(H'_2) - 1$, otherwise we have $\dim\bigl(H'_2 \setminus \{\bar{e}\}\bigr) = \dim(H'_2)$. Since $\dim(G) = \dim(G'_1) + \dim(G'_2)$ and $\dim(H) = \dim(H'_1) + \dim(H'_2)$, the pseudo-Lamanity of B_1 and B_2 implies

$$\dim(G'_1) - 1 + \dim(G'_2) + \dim(H'_1) + \dim(H'_2) \;\geq\; |\mathcal{E}_1| + |\mathcal{E}_2| + 2,$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are the biedges of B_1 and B_2, respectively (the inequality $\geq$ takes into account the fact that $\bar{e}$ may or may not be a bridge). Since $|\mathcal{E}| = |\mathcal{E}_1| + |\mathcal{E}_2| + 1$, the previous equation in turn implies

$$\dim(G) + \dim(H) \;\geq\; |\mathcal{E}| + 2,$$

contradicting the hypothesis that B is pseudo-Laman. ☐

3. Bidistances and quotients

Tropical geometry is a technique that allows to transform systems of polynomial equations into systems of piecewise linear equations. This is possible if one works over the field of Puiseux series: an algebraic relation between Puiseux series implies a piecewise linear relation between their orders (which are rational numbers). One hopes that the piecewise linear system is easier to solve; if so, one has candidates for the orders of solutions of the initial system, and sometimes this is enough to obtain the desired information. This technique has been successfully used, amongst others, by Mikhalkin [Mik05] to count the number of algebraic curves with some prescribed properties.

We use a similar idea for computing the Laman number of a pseudo-Laman bigraph. For each such bigraph B, we need to know the number of solutions of the system defining $Z^B_{\mathbb{C}}$. This number coincides with the number of solutions of a "perturbed" system over the Puiseux field (Lemma 3.3). The orders occurring in each solution of this new system satisfy piecewise linear conditions, specified in Definition 3.7. We prove (Lemmas 3.20, 3.21 and 3.22) that the solutions determining the same piecewise linear system are in bijection with the complex solutions of another system of equations of the same type, i.e. the system $Z^{B'}_{\mathbb{C}}$ of another pseudo-Laman bigraph B'. This leads to a first recursion scheme, expressed in Theorem 3.24.

Notation. Denote by $\mathbb{K}$ the field $\mathbb{C}\{\{s\}\}$ of Puiseux series with coefficients in $\mathbb{C}$. Recall that $\mathbb{K}$ is of characteristic zero and is algebraically closed. The field $\mathbb{K}$ comes with a valuation $\nu\colon \mathbb{K} \setminus \{0\} \longrightarrow \mathbb{Q}$ associating to an element $\sum_{i=k}^{+\infty} c_i \, s^{i/n}$ the rational number k/n, where $k \in \mathbb{Z}$, $c_k \neq 0$. Recall that $\nu(a \cdot b) = \nu(a) + \nu(b)$ and $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$.

Definition 3.1. Let $B = (G, H)$ be a bigraph. Define $f_{B,\mathbb{K}}$ to be the map obtained as the extension of scalars, via the natural inclusion $\mathbb{C} \longrightarrow \mathbb{K}$, of the rational map f_B

associated to B (see Definition 2.7). This means that, with the same notation as in Definition 2.6, we define

$$\mathbb{P}_{\mathbb{K}}^{\dim(G)-1} := \mathbb{P}\big(\mathbb{K}^V \,/\, L_G \otimes_{\mathbb{C}} \mathbb{K}\big),$$
$$\mathbb{P}_{\mathbb{K}}^{\dim(H)-1} := \mathbb{P}\big(\mathbb{K}^W \,/\, L_H \otimes_{\mathbb{C}} \mathbb{K}\big),$$

and then $f_{B,\mathbb{K}} \colon \mathbb{P}_{\mathbb{K}}^{\dim(G)-1} \times \mathbb{P}_{\mathbb{K}}^{\dim(H)-1} \dashrightarrow \mathbb{P}_{\mathbb{K}}^{|\mathcal{E}|-1}$ is given by the same equations as f_B.

Remark 3.2. By construction we have that $\deg(f_B)$ is defined if and only if $\deg(f_{B,\mathbb{K}})$ is defined, and in that case they coincide. In fact, f_B is dominant if and only if $f_{B,\mathbb{K}}$ is so. In this case, let $Y_{\mathbb{C}}$ be the open subset where f_B is defined. Because $f_B \colon Y_{\mathbb{C}} \longrightarrow \mathbb{P}_{\mathbb{C}}^{|\mathcal{E}|-1}$ is a dominant morphism between complex varieties, there exists an open subset $\mathcal{U}_{\mathbb{C}} \subseteq \mathbb{P}_{\mathbb{C}}^{|\mathcal{E}|-1}$ such that the fiber of f_B over any point of $\mathcal{U}_{\mathbb{C}}$ consists of $\deg(f_B)$ distinct points. Since $f_{B,\mathbb{K}}$ is the extension of scalars of f_B, it follows that the fiber of $f_{B,\mathbb{K}}$ over any point in $\mathcal{U}_{\mathbb{K}} := \mathcal{U}_{\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} \mathrm{Spec}(\mathbb{K})$ consists of $\deg(f_B)$ distinct points. In fact, for every $q_{\mathbb{K}} \in \mathcal{U}_{\mathbb{K}}$ we have $f_{B,\mathbb{K}}^{-1}(q_{\mathbb{K}}) \cong f_B^{-1}(q_{\mathbb{C}}) \times_{\mathrm{Spec}(\mathbb{C})} \mathrm{Spec}(\mathbb{K})$, where $q_{\mathbb{C}}$ is the image of $q_{\mathbb{K}}$ under the natural morphism $\mathcal{U}_{\mathbb{K}} \longrightarrow \mathcal{U}_{\mathbb{C}}$. So the cardinality of $f_{B,\mathbb{K}}^{-1}(q_{\mathbb{K}})$ is the same as the cardinality of $f_B^{-1}(q_{\mathbb{C}})$. We conclude then that $\deg(f_{B,\mathbb{K}}) = \deg(f_B)$

The fact that the map $f_{B,\mathbb{K}}$ is defined over $\mathbb{C}$, and not over $\mathbb{K}$, gives us a lot of freedom as far as the valuation of the general points over which we consider the fiber is concerned. Lemma 3.3 makes this statement more precise.

Lemma 3.3. *Let B be a bigraph such that $\mathrm{Lam}(B) > 0$. Fix a vector $\mathrm{wt} = \big(\mathrm{wt}(e)\big)_{e \in \mathcal{E}} \in \mathbb{Q}^{\mathcal{E}}$. Then $\deg(f_{B,\mathbb{K}})$ coincides with the cardinality of the fiber of $f_{B,\mathbb{K}}$ over any point $p \in \mathbb{P}_{\mathbb{K}}^{|\mathcal{E}|-1}$ of the form $p = \big(\lambda_e\, s^{\mathrm{wt}(e)}\big)_{e \in \mathcal{E}}$, where $(\lambda_e)_{e \in \mathcal{E}}$ is a general point in $\mathbb{C}^{\mathcal{E}}$.*

Proof. Consider the rational map $f_B \colon \mathbb{P}_{\mathbb{C}}^{\dim(G)-1} \times \mathbb{P}_{\mathbb{C}}^{\dim(H)-1} \dashrightarrow \mathbb{P}_{\mathbb{C}}^{|\mathcal{E}|-1}$, which is dominant by hypothesis. To prove the statement it is enough to show that a point p as in the hypothesis lies in the set $\mathcal{U}_{\mathbb{K}}$ defined in Remark 3.2. Suppose by contradiction that $p \notin \mathcal{U}_{\mathbb{K}}$. Since $\mathcal{U}_{\mathbb{K}}$ is Zariski open, it is defined by a disjunction of polynomial inequalities with coefficients in $\mathbb{C}$. Let $g \neq 0$ be one of these inequalities: by assumption $g(p) = 0$, but this implies that $\tilde{g}\big((\lambda_e)_{e \in \mathcal{E}}\big) = 0$ for some non-zero polynomial $\tilde{g}$ over $\mathbb{C}$, contradicting the generality of $(\lambda_e)_{e \in \mathcal{E}}$. $\square$

Let B be a bigraph such that $\mathrm{Lam}(B) > 0$. Fix a vector $\mathrm{wt} = \big(\mathrm{wt}(e)\big)_{e \in \mathcal{E}} \in \mathbb{Q}^{\mathcal{E}}$ and a biedge $\bar{e} \in \mathcal{E}$. Arguing as in Remark 2.12, we see that it is enough to consider fibers of $f_{B,\mathbb{K}}$ over points p of the form $p_{\bar{e}} = 1$, while $p_e = \lambda_e\, s^{\mathrm{wt}(e)}$ for a general point $(\lambda_e)_{e \in \mathcal{E} \setminus \{\bar{e}\}}$ in $\mathbb{C}^{\mathcal{E} \setminus \{\bar{e}\}}$. This is why we formulate the following assumption, which is used throughout this section.

Assumption 3.4. Let B be a pseudo-Laman bigraph with biedges $\mathcal{E}$ such that $\mathrm{Lam}(B) > 0$. Notice that by Proposition 2.10 this implies in particular that B has no self-loops. Fix a biedge $\bar{e} \in \mathcal{E}$, fix $\mathrm{wt} \in \mathbb{Q}^{\mathcal{E} \setminus \{\bar{e}\}}$ and let $(\lambda_e)_{e \in \mathcal{E} \setminus \{\bar{e}\}}$ be a general point in $\mathbb{C}^{\mathcal{E}\setminus\{\bar{e}\}}$. Let $p \in \mathbb{P}_{\mathbb{K}}^{|\mathcal{E}|-1}$ be such that $p_{\bar{e}} = 1$ and $p_e = \lambda_e \, s^{\mathrm{wt}(e)}$ for all biedges $e \in \mathcal{E} \setminus \{\bar{e}\}$.

Remark 3.5. Let $B = (G, H)$ be a bigraph with biedges $\mathcal{E}$ and use Assumption 3.4. Following Lemma 2.16 one can prove that $f_{B,\mathbb{K}}^{-1}(p)$ is isomorphic to

$$Z_{\mathbb{K}}^{B} := \operatorname{Spec} \left(\begin{array}{ll} x_{\bar{u}\bar{v}} = y_{\bar{t}\bar{w}} = 1 & \bar{u} \prec \bar{v},\, \bar{t} \prec \bar{w} \\ x_{uv}\, y_{tw} = \lambda_e \, s^{\mathrm{wt}(e)} & \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\},\, u \prec v,\, t \prec w \\ \sum_{\mathscr{C}} x_{uv} = 0 & \text{for all cycles } \mathscr{C} \text{ in } G \\ \sum_{\mathscr{D}} y_{tw} = 0 & \text{for all cycles } \mathscr{D} \text{ in } H \end{array} \right) \subseteq \mathbb{K}^{P} \times \mathbb{K}^{Q},$$

where the notation is as in Definition 2.14.

Example 3.6. Continuing Example 2.15, if we fix the vector wt to be $(1)_{\mathcal{E}\setminus\{\bar{e}\}}$, then the scheme $Z_{\mathbb{K}}^{B}$ is defined by the equations

$$\begin{aligned} x_{23} &= 1, & y_{23} &= 1, \\ x_{12}\, y_{12} &= \lambda_{\mathbf{r}}\, s, & x_{31}\, y_{31} &= \lambda_{\mathbf{g}}\, s, \\ x_{24}\, y_{24} &= \lambda_{\mathbf{o}}\, s, & x_{43}\, y_{43} &= \lambda_{\mathbf{b}}\, s, \end{aligned}$$

and by the same equations as in Example 2.15 coming from the cycles of the bigraph.

Once a point p of the form $\left(\lambda_e \, s^{\mathrm{wt}(e)}\right)_{e \in \mathcal{E}}$ is fixed in the codomain of the map $f_{B,\mathbb{K}}$, then for every point $q \in f_{B,\mathbb{K}}^{-1}(p)$ we can consider the vector of the valuations of its coordinates. In this way in Definition 3.7 we associate to each such point q a discrete object, which we later call *bidistance* (see Definition 3.9). We then partition the set $f_{B,\mathbb{K}}^{-1}(p)$ according to the bidistances that are determined by its points.

Definition 3.7. Let $B = (G, H)$ be a bigraph with biedges $\mathcal{E}$ and use Assumption 3.4. Fix $q \in f_{B,\mathbb{K}}^{-1}(p)$. Then $q = \big([(x_v)_{v \in V}], [(y_w)_{w \in W}]\big)$ and by construction

$$\frac{x_u - x_v}{x_{\bar{u}} - x_{\bar{v}}} \cdot \frac{y_t - y_w}{y_{\bar{t}} - y_{\bar{w}}} = \lambda_e \, s^{\mathrm{wt}(e)} \quad \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\}, \text{ where}$$

$$\begin{array}{ll} \{\bar{u}, \bar{v}\} = \tau_G(\bar{e}), \quad \bar{u} \prec \bar{v}, \\ \{\bar{t}, \bar{w}\} = \tau_H(\bar{e}), \quad \bar{t} \prec \bar{w}, \end{array} \quad \text{and} \quad \begin{array}{ll} \{u, v\} = \tau_G(e), \quad u \prec v, \\ \{t, w\} = \tau_H(e), \quad t \prec w. \end{array}$$

Define

$$\begin{aligned} d_V(u, v) &:= \nu \left(\frac{x_u - x_v}{x_{\bar{u}} - x_{\bar{v}}} \right) \quad \text{for all } (u, v) \in P, \\ d_W(t, w) &:= \nu \left(\frac{y_t - y_w}{y_{\bar{t}} - y_{\bar{w}}} \right) \quad \text{for all } (t, w) \in Q. \end{aligned}$$

In this way we get two functions $d_V \colon P \longrightarrow \mathbb{Q}$ and $d_W \colon Q \longrightarrow \mathbb{Q}$, where P and Q are as in Definition 2.13. Notice that the definition of P and Q prevents the argument of the valuation in the definition of d_V and d_W to be zero. Moreover, both d_V and d_W depend on q, but not on the representatives $(x_v)_{v \in V}$ and $(y_w)_{w \in W}$.

Lemma 3.8. *With notation and assumption as in Definition 3.7, the functions $d_V \colon P \longrightarrow \mathbb{Q}$ and $d_W \colon Q \longrightarrow \mathbb{Q}$ satisfy:*

- ▷ *$d_V(u, v) = d_V(v, u)$ for all $(u, v) \in P$, and similarly for d_W;*

▷ *$d_V(u, v) + d_W(t, w) = \mathrm{wt}(e)$ for all $e \in \mathcal{E} \setminus \{\bar{e}\}$, where $\{u, v\} = \tau_G(e)$ and $\{t, w\} = \tau_H(e)$;*
▷ *$d_V(\bar{u}, \bar{v}) = d_W(\bar{t}, \bar{w}) = 0$, where $\{\bar{u}, \bar{v}\} = \tau_G(\bar{e})$ and $\{\bar{t}, \bar{w}\} = \tau_H(\bar{e})$;*
▷ *for every cycle $\mathscr{C}$ in G, the minimum of the values of d_V on the pairs of vertices (u, v) appearing in $\mathscr{C}$ is attained at least twice, and similarly for d_W.*

Proof. The statement follows from the definitions and the properties of the valuation. Notice that the values $d_V(\bar{u}, \bar{v})$ and $d_W(\bar{t}, \bar{w})$ are defined because $\bar{e}$ is not a self-loop, since there are no self-loops in B. In fact, otherwise we would have $\mathrm{Lam}(B) = 0$ by Proposition 2.10. □

Definition 3.9. Let B be a bigraph with biedges $\mathcal{E}$ without self-loops, let $\bar{e}$ be a fixed biedge, and let $\mathrm{wt} \in \mathbb{Q}^{\mathcal{E} \setminus \{\bar{e}\}}$. A *bidistance* d on B compatible with wt is a pair (d_V, d_W) of functions $d_V : P \longrightarrow \mathbb{Q}$ and $d_W : Q \longrightarrow \mathbb{Q}$ such that the conditions of Lemma 3.8 are satisfied.

From now on, since only a single weight vector wt is involved at a time, we do not repeat the clause "compatible with wt" when talking about a bidistance.

Remark 3.10. Let B be a bigraph and use Assumption 3.4. Then any $q \in f^{-1}_{B,\mathbb{K}}(p)$ defines a bidistance d on B, and via the isomorphism provided by Remark 3.5 also any point in $Z^B_{\mathbb{K}}$ defines a bidistance.

As mentioned before, we are going to count the number of points in a general fiber of $f_{B,\mathbb{K}}$ that determine a fixed bidistance. We do so by computing the Laman number of a "smaller" bigraph, obtained via a quotient operation as explained in Definition 3.11.

Definition 3.11. Let $B = (G, H)$ be a bigraph with set of biedges $\mathcal{E}$ without self-loops, and fix a bidistance $d = (d_V, d_W)$ on B. Define a new bigraph B_d as follows. For every $\alpha \in \mathrm{im}(d_V)$, define the graphs $G_{\geq \alpha}$ and $G_{> \alpha}$ to be the subgraphs of G determined by all edges with endpoints u and v such that $d_V(u, v) \geq \alpha$ and $d_V(u, v) > \alpha$, respectively. Similarly, for every $\beta \in \mathrm{im}(d_W)$, define $H_{\geq \beta}$ and $H_{> \beta}$. Let

$$G_{d_V} := \dot{\bigcup_{\alpha \in \mathrm{im}(d_V)}} G_{\geq \alpha} \big/ G_{> \alpha} \qquad \text{and} \qquad H_{d_W} := \dot{\bigcup_{\beta \in \mathrm{im}(d_W)}} H_{\geq \beta} \big/ H_{> \beta} \, .$$

Here by $G_{\geq \alpha} \big/ G_{> \alpha}$ and $H_{\geq \beta} \big/ H_{> \beta}$ we mean the quotients of graphs as described in Definition 2.5, followed by removing singleton components without edges. The union symbol $\dot{\bigcup}$ indicates the disjoint union of graphs.

There is a natural bijection between edges of G and edges of G_{d_V}, sending each edge e in G to the corresponding edge in the quotient $G_{\geq d_V(\tau_G(e))} \big/ G_{> d_V(\tau_G(e))}$. We define B_d to be the bigraph (G_{d_V}, H_{d_W}) with set of biedges $\mathcal{E}$ inherited from B. Moreover, we fix any total order on the vertices in B_d.

Remark 3.12. Notice that in Definition 3.11 we did not use any of the properties of bidistances. This means that the definition of B_d makes sense also for bigraphs B and pairs of functions $d_V : P \longrightarrow \mathbb{Q}$ and $d_W : Q \longrightarrow \mathbb{Q}$. This is going to be important and useful in the second part of Section 4.

Lemma 3.13 shows that the condition of being pseudo-Laman is preserved under the quotient operation.

Lemma 3.13. *If $B = (G, H)$ is a pseudo-Laman bigraph without self-loops and d is a bidistance on B, then the quotient graph B_d is also pseudo-Laman.*

Proof. We first prove that for any graph $G = (V, E)$ and for any subgraph $G' \subseteq G$ the following equation holds:

$$\dim(G) \;=\; \dim(G') + \dim(G \,/\, G').$$

Let $G = \dot{\bigcup}_{i=1}^{k} G_i$ be the decomposition of G into connected components. We write $G' = \dot{\bigcup}_{i=1}^{k} G'_i$, where G'_i is the part of G' belonging to G_i. Let V_i and V'_i be the set of vertices of G_i and G'_i, respectively. Now, G'_i itself may be disconnected, so let n_i be the number of connected components of G'_i. Contraction of edges of G_i does not introduce new components, thus $G_i \,/\, G'_i$ consist of one connected component. Moreover, each connected component of G'_i will correspond to one vertex in $G_i \,/\, G'_i$. It follows that

$$\dim(G_i) = |V_i| - 1, \quad \dim(G'_i) = |V'_i| - n_i, \quad \dim(G_i/G'_i) = (|V_i| - |V'_i| + n_i) - 1,$$

such that $\dim(G_i) = \dim(G'_i) + \dim(G_i/G'_i)$ for all i. Now our claim follows, since $\dim\ \dot{\bigcup}_{i=1}^{k} G_i = \sum_{i=1}^{k} \dim(G_i)$.

If $B = (G, H)$ is a bigraph and d is a bidistance on it then

$$\dim(G) = \dim(G_{d_V}) \qquad \text{and} \qquad \dim(H) = \dim(H_{d_W}).$$

We prove only the first equality, since the second follows analogously. Let $\overline{\alpha}$ be the minimum value attained by d_V. In this case $G = G_{\geq \overline{\alpha}}$ and therefore

$$\dim(G) \;=\; \dim(G_{\geq \overline{\alpha}}) \;=\; \dim(G_{> \overline{\alpha}}) + \dim(G_{\geq \overline{\alpha}} \,/\, G_{> \overline{\alpha}}).$$

By repeating this argument considering one by one all values in $\operatorname{im}(d_V)$ in increasing order, we prove the claimed equality. The proof is concluded by noticing that by construction the number of biedges of B_d equals the number of biedges of B. □

Example 3.14. Continuing Example 3.6, we fix the following bidistance $d = (d_V, d_W)$, see Figure 7 for illustration.

$$\begin{aligned}
d_V(1,2) &= 0, & d_W(1,2) &= 1,\\
d_V(1,3) &= 1, & d_W(1,3) &= 0,\\
d_V(2,3) &= 0, & d_W(2,3) &= 0,\\
d_V(2,4) &= 1, & d_W(2,4) &= 0,\\
d_V(3,4) &= 0, & d_W(3,4) &= 1.
\end{aligned}$$

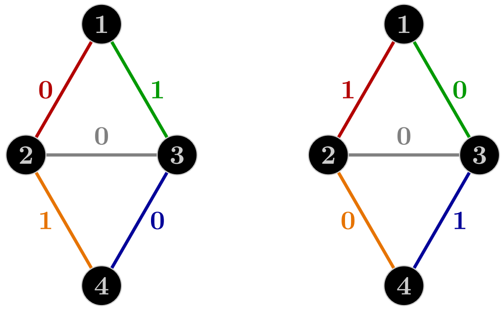


FIGURE 7. A bigraph on which a bidistance has been fixed.

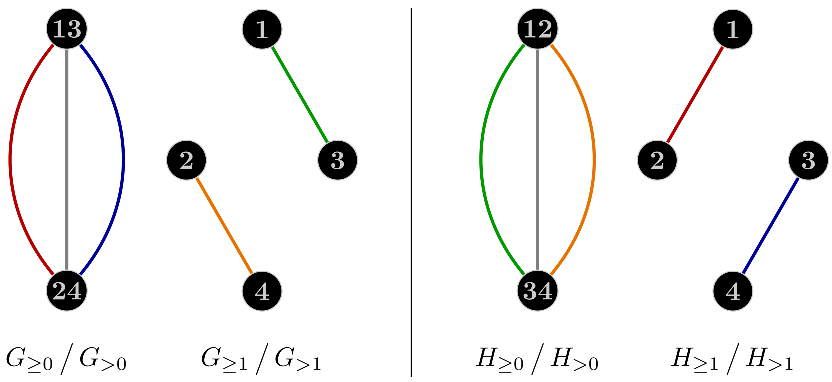


FIGURE 8. The bigraph B_d, where B and d are as in Example 3.14.

The resulting bigraph B_d is shown in Figure 8. The scheme $Z_{\mathbb{C}}^{B_d}$ associated to B_d is defined by the following equations:

$$
\begin{aligned}
x_{13|24} &= 1, & y_{12|34} &= 1,\\
x_{13|24}\, y_{1|2} &= \lambda_{\mathbf{r}}, & x_{1|3}\, y_{12|34} &= \lambda_{\mathbf{g}},\\
x_{2|4}\, y_{12|34} &= \lambda_{\mathbf{o}}, & x_{13|24}\, y_{3|4} &= \lambda_{\mathbf{b}},
\end{aligned}
$$
$$
x_{13|24} + x_{24|13} = x_{1|3} + x_{3|1} = x_{2|4} + x_{4|2} = 0,
$$
$$
y_{12|34} + y_{34|12} = y_{1|2} + y_{2|1} = y_{3|4} + y_{4|3} = 0.
$$

To link points in a general fiber of $f_{B,\mathbb{K}}$ defining a given bidistance d with the Laman number of B_d, we introduce in Definition 3.16 a family of varieties $\widetilde{A}_{\mathbb{C}}^{d}$, parametrized by a parameter σ. This family has the property that a general element is isomorphic to $Z_{\mathbb{K}}^{B}$, while a special element is isomorphic to $Z_{\mathbb{C}}^{B_d}$. To prove this, we establish in Lemmas 3.21 and 3.22 the following two bijections:

$$\left\{ \begin{array}{c} \text{points in } Z^B_{\mathbb{K}} \text{ that determine} \\ \text{the bidistance } d \end{array} \right\}$$

$$\updownarrow \text{ Lemma 3.22}$$

$$\left\{ \text{ points in } A^d_{\mathbb{C}} \right\}$$

$$\updownarrow \text{ Lemma 3.21}$$

$$\left\{ \text{ points in } Z^{B_d}_{\mathbb{C}} \right\}$$

Remark 3.15. Notice that, for a fixed bigraph B with biedges $\mathcal{E}$ and a fixed choice of a vector $\mathrm{wt} \in \mathbb{Q}^{\mathcal{E} \setminus \{\bar{e}\}}$ and of a bidistance (d_V, d_W), we can suppose that all entries of the vector wt and all values $d_V(u,v)$ and $d_W(t,w)$ are integer numbers. Indeed, consider the subgroup of $\mathbb{Q}$ generated by the rational numbers $\{\mathrm{wt}(e)\}_{e \in \mathcal{E} \setminus \{\bar{e}\}}$, $\mathrm{im}(d_V)$ and $\mathrm{im}(d_W)$: such a group is of the form $\frac{m}{n}\mathbb{Z}$, and so we can apply to our setting the automorphism of $\mathbb{K}$ sending s to $s^{m/n}$, proving the claim.

Definition 3.16. Let B be a bigraph with biedges $\mathcal{E}$ and use Assumption 3.4. Given a bidistance d on B, we can suppose by Remark 3.15 that wt, d_V and d_W take integer values. We define the scheme $\widetilde{A}^d_{\mathbb{C}}$ in $\mathbb{C}^P \times \mathbb{C}^Q \times \mathbb{C}$, where we take coordinates $(\tilde{x}_{uv})_{(u,v) \in P}$, $(\tilde{y}_{tw})_{(t,w) \in Q}$ and σ:

$$\widetilde{A}^d_{\mathbb{C}} := \operatorname{Spec} \begin{pmatrix} \tilde{x}_{\bar{u}\bar{v}} = \tilde{y}_{\bar{t}\bar{w}} = 1 & \bar{u} \prec \bar{v},\ \bar{t} \prec \bar{w} \\ \tilde{x}_{uv}\, \tilde{y}_{tw} = \lambda_e & \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\},\ u \prec v,\ t \prec w \\ \sum_{\mathscr{C}} \tilde{x}_{uv}\, \sigma^{d_V(u,v) - m(\mathscr{C})} = 0 & \text{for all cycles } \mathscr{C} \text{ in } G \\ \sum_{\mathscr{D}} \tilde{y}_{tw}\, \sigma^{d_W(t,w) - m(\mathscr{D})} = 0 & \text{for all cycles } \mathscr{D} \text{ in } H \end{pmatrix},$$

where $m(\mathscr{C})$ denotes the minimum value attained by the function d_V on the cycle $\mathscr{C}$, and similarly for d_W. Since the differences $d_V(u,v) - m(\mathscr{C})$ and $d_W(t,w) - m(\mathscr{D})$ are non-negative integers, we see that all equations are indeed polynomial in $\tilde{x}$, $\tilde{y}$ and σ. Moreover, we define

$$A^d_{\mathbb{C}} := \widetilde{A}^d_{\mathbb{C}} \cap \{\sigma = 0\},$$

where $\{\sigma = 0\}$ denotes the hyperplane defined by the equation $\sigma = 0$.

Example 3.17. Continuing Example 3.14, the scheme $\widetilde{A}^d_{\mathbb{C}}$ is defined by the equations:

$$\tilde{x}_{23} = \tilde{y}_{23} = 1,$$

$$\begin{aligned} &\tilde{x}_{12}\,\tilde{y}_{12} = \lambda_{\mathbf{r}}, && \tilde{x}_{12} + \tilde{x}_{21} = \tilde{x}_{13} + \tilde{x}_{31} = \tilde{x}_{23} + \tilde{x}_{32} = \tilde{x}_{24} + \tilde{x}_{42} = \tilde{x}_{34} + \tilde{x}_{43} = 0, \\ &\tilde{x}_{13}\,\tilde{y}_{13} = \lambda_{\mathbf{g}}, && \tilde{y}_{12} + \tilde{y}_{21} = \tilde{y}_{13} + \tilde{y}_{31} = \tilde{y}_{23} + \tilde{y}_{32} = \tilde{y}_{24} + \tilde{y}_{42} = \tilde{y}_{34} + \tilde{y}_{43} = 0, \\ &\tilde{x}_{24}\,\tilde{y}_{24} = \lambda_{\mathbf{o}}, && \tilde{x}_{12} + \tilde{x}_{23} + \tilde{x}_{31}\sigma = \tilde{y}_{12}\sigma + \tilde{y}_{23} + \tilde{y}_{31} = 0, \\ &\tilde{x}_{34}\,\tilde{y}_{34} = \lambda_{\mathbf{b}}, && \tilde{x}_{24}\sigma + \tilde{x}_{43} + \tilde{x}_{32} = \tilde{y}_{24} + \tilde{y}_{43}\sigma + \tilde{y}_{32} = 0. \end{aligned}$$

In Lemma 3.21 we use a special set of generators for the ideals defining $A^d_{\mathbb{C}}$ and $Z^{B_d}_{\mathbb{C}}$. To describe this set of generators we need the concept of spanning forest for a bigraph, introduced in Definition 3.18.

Definition 3.18. Let $B = (G, H)$ be a bigraph. A *spanning forest* $\mathcal{F}$ for B is a pair $(\mathcal{F}_G, \mathcal{F}_H)$ of spanning forests for G and H respectively. A spanning forest for a graph is a tuple of spanning trees, one for each connected component of the graph.

In particular, we need spanning forests with an additional property, specified in Remark 3.19.

Remark 3.19. Let $B = (G, H)$ be a bigraph. We consider spanning forests $\mathcal{F}_G$ and $\mathcal{F}_H$ for G and H, respectively, such that for any edge in G with vertices u, v, the value $d_V(u, v)$ is equal to the minimum of the values of d_V in the unique path in $\mathcal{F}_G$ connecting u and v, and similarly for $\mathcal{F}_H$.

The construction of such forests can be achieved by iteratively removing non-bridges (see Definition 2.17) with endpoints u and v such that $d_V(u, v)$ is minimal within the non-bridges of the graph in the current iteration. This construction can be proven to be correct using the loop invariant $\delta\colon (V \times V) \setminus \Delta \longrightarrow \mathbb{Q}$, where $\Delta = \{(v, v) : v \in V\}$ and $\delta(u, v)$ is defined as follows. We consider all paths $v_0 = u, v_1, \ldots, v_n = v$ from u to v and take the minimum of the values $\big\{d_V(v_i, v_{i+1}) : i \in \{0, \ldots, n\}\big\}$ for each of them. Then $\delta(v, u)$ is the maximum of all these values. Note that if u and v are connected by an edge in the graph, then $\delta(u, v) = d_V(u, v)$. The map δ is indeed a loop invariant since, if we are about to delete an edge with endpoints u and v, then this edge has to be a non-bridge of minimal $d_V(u, v)$. Hence, there is a cycle containing both u and v and the endpoints of another edge with the same d_V-value, since the minimum d_V-value occurs at least twice in every cycle. Therefore, there is still a path from u to v with the same minimum, so the set of minima in the definition of δ does not change at all. In a similar way one argues that all other values of δ are not changed either.

The forests constructed in this way share a useful property (which is employed in Section 4), namely if in B_d we consider the set of edges corresponding to $\mathcal{F}_G$ and $\mathcal{F}_H$, then such a set forms a spanning forest for B_d.

Lemma 3.20 is an auxiliary result describing how to obtain a special system of generators for $Z^B_{\mathbb{C}}$ once a spanning forest for B is fixed.

Lemma 3.20. *Let B be a bigraph. Use Assumption 3.4 and define $Z^B_{\mathbb{C}}$ according to Definition 2.14. Then $Z^B_{\mathbb{C}}$ is a complete intersection, and every choice of a spanning forest for B determines a set of* $\operatorname{codim}\big(Z^B_{\mathbb{C}}\big)$ *generators for the ideal of $Z^B_{\mathbb{C}}$.*

Proof. Notice that the dimension of the ambient affine space of $Z^B_{\mathbb{C}}$ is $|P| + |Q|$, where P and Q are as in Definition 2.13. Moreover $Z^B_{\mathbb{C}}$ is zero-dimensional, since $\mathrm{Lam}(B)$ is defined. We are going to exhibit a system consisting of $|P|+|Q|$ equations defining $Z^B_{\mathbb{C}}$.

Let $\mathcal{F} = (\mathcal{F}_G, \mathcal{F}_H)$ be a spanning forest for the bigraph $B = (G, H)$ with biedges $\mathcal{E}$. For every $(u, v) \in P$ such that u and v are not connected by an edge of $\mathcal{F}_G$, we consider the following equation:

$$x_{uv} - \sum_{i=0}^{n-1} x_{u_i u_{i+1}} = 0,$$

where $(u_0 = u, \dots, u_n = v)$ is the unique path in $\mathcal{F}_G$ from u to v. Similarly we construct equations for each $(t, w) \in Q$ for which t and w are not connected by an edge of $\mathcal{F}_H$. It is easy to see that these equations generate the same ideal as the equations coming from all cycles in G and in H. The number of such equations is

$$\begin{aligned} &|P| - |\{\text{edges of } \mathcal{F}_G\}| + |Q| - |\{\text{edges of } \mathcal{F}_G\}| \\ =\; &|P| - \dim(G) + |Q| - \dim(H). \end{aligned}$$

The above equations, together with

$$\begin{cases} x_{\bar{u}\bar{v}} = y_{\bar{t}\bar{w}} = 1, & \bar{u} \prec \bar{v},\ \bar{t} \prec \bar{w}, \\ x_{uv}\, y_{tw} = \lambda_e, & \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\},\ u \prec v,\ t \prec w, \end{cases}$$

define $Z^B_{\mathbb{C}}$. Therefore, the total number of equations is

$$|\mathcal{E}| + 1 + \big(|P| - \dim(G)\big) + \big(|Q| - \dim(H)\big),$$

which equals $|P| + |Q|$ since B is pseudo-Laman and has no self-loops by Assumption 3.4. In particular, $Z^B_{\mathbb{C}}$ is a complete intersection. $\square$

Lemma 3.21. *Let B be a bigraph. Use Assumption 3.4 and fix a bidistance d on B. Suppose that B_d (as in Definition 3.11) satisfies $\mathrm{Lam}(B_d) > 0$. Then the scheme $\widetilde{A}^d_{\mathbb{C}}$ (as in Definition 3.16) can be defined by $|P| + |Q|$ equations. Furthermore, the scheme $A^d_{\mathbb{C}}$ is isomorphic to $Z^{B_d}_{\mathbb{C}}$, so in particular it consists of $\mathrm{Lam}(B_d)$ distinct points and is defined by $\mathrm{codim}\big(A^d_{\mathbb{C}}\big)$ equations.*

Proof. Let $B = (G, H)$ with biedges $\mathcal{E}$ as in the statement. As in Lemma 3.20, we can give a smallest set of equations for $\widetilde{A}^d_{\mathbb{C}}$ depending on a choice of a spanning forest. By a special choice of the spanning forest, namely by choosing the forests $\mathcal{F}_G$ and $\mathcal{F}_H$ for G and H, respectively, as described in Remark 3.19, we may achieve that the equations are of the form

$$\tilde{x}_{uv} - \sum_{i=0}^{n-1} \tilde{x}_{u_i u_{i+1}} \sigma^{d_V(u_i, u_{i+1}) - d_V(u,v)} \;=\; 0$$

together with

$$\begin{cases} \tilde{x}_{\bar{u}\bar{v}} = \tilde{y}_{\bar{t}\bar{w}} = 1, & \bar{u} \prec \bar{v},\ \bar{t} \prec \bar{w}, \\ \tilde{x}_{uv}\, \tilde{y}_{tw} = \lambda_e, & \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\},\ u \prec v,\ t \prec w. \end{cases}$$

The number of these equations is hence $|P| + |Q|$. We obtain a set of equations for $A^d_{\mathbb{C}}$ by setting $\sigma = 0$ in the previous ones. Note that we could not have obtained this kind of equations if we started from an arbitrary spanning forest for B.

Let P_d and Q_d be the sets as in Definition 2.13 starting from B_d. The elements of P_d are of the form $\big([u]_{>\alpha}, [v]_{>\alpha}\big)$, where $[u]_{>\alpha}$ is the class of the vertex $u \in V$ in the set of vertices of $G_{\geq \alpha} \big/ G_{>\alpha}$, and α is a value in the image of d_V. In the following we simply write $[u]$ and $[v]$ for such classes (and similarly for Q_d).

Define two morphisms $\varphi\colon Z^{B_d}_{\mathbb{C}} \longrightarrow A^d_{\mathbb{C}}$ and $\psi\colon A^d_{\mathbb{C}} \longrightarrow Z^{B_d}_{\mathbb{C}}$ as follows. For a point $q = \big((x_{[u][v]})_{([u],[v]) \in P_d}, (y_{[t][w]})_{([t],[w]) \in Q_d}\big)$ in $Z^{B_d}_{\mathbb{C}}$, define $\varphi(q)$ to be the point whose x_{uv}-coordinate is $x_{[u][v]}$ and whose y_{tw}-coordinate is $y_{[t][w]}$. For a point $\tilde{q} = \big((\tilde{x}_{uv})_{(u,v)\in P}, (\tilde{y}_{tw})_{(t,w)\in Q}\big)$ in $A^d_{\mathbb{C}}$, define $\psi(\tilde{q})$ to be the point whose $x_{[u][v]}$-coordinate equals $\tilde{x}_{uv}$ and whose $y_{[t][w]}$-coordinate equals $\tilde{y}_{tw}$. We must show that both maps are well-defined, and from this by construction we see that they are isomorphisms. To show that φ is well-defined, we must prove that $\varphi(q) \in A^d_{\mathbb{C}}$. Notice that the coordinates of $\varphi(q)$ satisfy the equations determined by the biedges of B because by construction the coordinates of q do so. Consider now an equation of $A^d_{\mathbb{C}}$ obtained by setting $\sigma = 0$ in an equation of $\widetilde{A}^d_{\mathbb{C}}$ determined by a cycle $\mathscr{C}$ in G (analogous considerations can be done for cycles in H). Let α be the minimum value attained by d_V along the cycle $\mathscr{C}$. By construction, such an equation is of the form $\sum_{\mathscr{C}_\alpha} \tilde{x}_{uv} = 0$, where the subscript $\mathscr{C}_\alpha$ indicates that the sum is taken over the pairs (u,v) in P appearing in the cycle $\mathscr{C}$ and satisfying $d_V(u,v) = \alpha$. On the other hand, such a cycle determines a cycle in $G_{\geq\alpha} \big/ G_{>\alpha}$, which defines an equation of the same form, namely $\sum_{\mathscr{C}_\alpha} x_{[u][v]} = 0$, satisfied by the coordinates of q. Hence $\varphi(q) \in A^d_{\mathbb{C}}$.

To show that ψ is well-defined we must first prove that if $[u] = [u']$ and $[v] = [v']$ for two pairs $(u,v), (u',v') \in P$ such that $d_V(u,v) = d_V(u',v')$, then the coordinates of the point $\tilde{q}$ satisfy $\tilde{x}_{uv} = \tilde{x}_{u'v'}$. This is true since by hypothesis there is a cycle in G involving two edges between u and v and u' and v', respectively, such that every other edge in the cycle has endpoints whose d_V-value is strictly greater than $d_V(u,v)$. The definition of $\widetilde{A}^d_{\mathbb{C}}$ implies that such an equation holds for points in $A^d_{\mathbb{C}}$. Secondly, we should prove that $\psi(\tilde{q}) \in Z^{B_d}_{\mathbb{C}}$, and here we argue as in the previous paragraph. □

Lemma 3.22. *With the notation as in Lemma 3.21, so in particular a bidistance d is fixed, there is a bijection between $A^d_{\mathbb{C}}$ and the set of points in $Z^B_{\mathbb{K}}$ that determine the bidistance d.*

Proof. We know from Remark 3.5 and from Lemma 3.21 that both $A^d_{\mathbb{C}}$ and $Z^B_{\mathbb{K}}$ consist of finitely many points. Let $q \in Z^B_{\mathbb{K}}$ be a point determining the bidistance d: this means that $q = \big((x_{uv})_{(u,v)\in P}, (y_{tw})_{(t,w)\in Q}\big)$ with $\nu(x_{uv}) = d_V(u,v)$ and $\nu(y_{tw}) = d_W(t,w)$. We can write $x_{uv} = \widehat{x}_{uv}\, s^{d_V(u,v)}$ and $y_{tw} = \widehat{y}_{tw}\, s^{d_W(t,w)}$ where the elements $\widehat{x}_{uv}$ and $\widehat{y}_{tw}$ have zero valuation. Therefore $\widehat{q} = \big((\widehat{x}_{uv})_{(u,v)\in P}, (\widehat{y}_{tw})_{(t,w)\in Q}\big)$ is a point of $s^d \cdot Z^B_{\mathbb{K}}$, which is the scheme in $\mathbb{K}^P \times \mathbb{K}^Q$ defined by the equations

$$\begin{cases} \widehat{x}_{\bar{u}\bar{v}} = \widehat{y}_{\bar{t}\bar{w}} = 1, & \bar{u} \prec \bar{v},\ \bar{t} \prec \bar{w}, \\ \widehat{x}_{uv}\, \widehat{y}_{tw} = \lambda_e, & \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\},\ u \prec v,\ t \prec w, \\ \sum_{\mathscr{C}} \widehat{x}_{uv}\, s^{d_V(u,v) - m(\mathscr{C})} = 0, & \text{for all cycles } \mathscr{C} \text{ in } G, \\ \sum_{\mathscr{D}} \widehat{y}_{tw}\, s^{d_W(t,w) - m(\mathscr{D})} = 0, & \text{for all cycles } \mathscr{D} \text{ in } H, \end{cases}$$

where the notation is as in Definition 3.16. Since all coordinates of $\widehat{q}$ have valuation equal to zero, we can define $\tilde{x}_{uv} := \widehat{x}_{uv} \mod (s)$, obtaining $\tilde{x}_{uv} \in \mathbb{C}$, and similarly for $\tilde{y}_{tw}$. It follows that the point $\tilde{q} = \big((\tilde{x}_{uv})_{(u,v)\in P}, (\tilde{y}_{tw})_{(t,w)\in Q}\big)$ satisfies the equations of $A^d_{\mathbb{C}}$. In this way we obtain a map from the set of points in $Z^B_{\mathbb{K}}$ that determine the bidistance d to $A^d_{\mathbb{C}}$.

Let now $\tilde{q}$ be a point of $A^d_{\mathbb{C}}$. From Lemma 3.21 we know that $A^d_{\mathbb{C}}$ is a complete intersection and that it is defined by $\operatorname{codim}\bigl(A^d_{\mathbb{C}}\bigr)$ equations $g_i = 0$ of the form

$$g_i\bigl((x_{uv})_{(u,v)\in P}, (y_{tw})_{(t,w)\in Q}\bigr) \;=\; \tilde{g}_i\bigl((x_{uv})_{(u,v)\in P}, (y_{tw})_{(t,w)\in Q}, 0\bigr),$$

where the equations $\tilde{g}_i = 0$ define $\widetilde{A}^d_{\mathbb{C}}$. Since $A^d_{\mathbb{C}}$ is smooth by Lemma 3.21, we know that the Jacobian determinant

$$\det\left(\left\{\frac{\partial g_i}{\partial \tilde{x}_{uv}}\right\}, \left\{\frac{\partial g_i}{\partial \tilde{y}_{tw}}\right\}\right)\bigg|_{\tilde{q}}$$

evaluated at $\tilde{q}$ is non-zero. Then, by the implicit function theorem for formal power series (see [Bou03, A.IV.37, Corollary]) applied to the system of equations $\tilde{g}_i = 0$, there exists a unique point $\widehat{q} \in \mathbb{C}[\![\sigma]\!]^P \times \mathbb{C}[\![\sigma]\!]^Q$ such that $\tilde{g}_i(\widehat{q}, \sigma) = 0$ and the constant terms of the coordinates of $\widehat{q}$ equal the coordinates of $\tilde{q}$. The point $\widehat{q}$ determines in turn a point in $s^d \cdot Z^B_{\mathbb{K}}$ whose coordinates have valuation equal to zero, and therefore a point in $Z^B_{\mathbb{K}}$ whose coordinates have valuation prescribed by d. Hence, we get a map from $A^d_{\mathbb{C}}$ to the set of points in $Z^B_{\mathbb{K}}$ determining the bidistance d.

Suppose now that there were two points q and q' in $Z^B_{\mathbb{K}}$ determining the bidistance d and specializing to the same point in $A^d_{\mathbb{C}}$. After applying a suitable automorphism of $\mathbb{K}$ of the form $s \mapsto s^{m/n}$, we can suppose that both the points in $s^d \cdot Z^B_{\mathbb{K}}$ corresponding to q and q' were given by power series in s. This would contradict the uniqueness of the power series solution provided by the implicit function theorem. Therefore, the two maps we have just specified provide the desired bijection. □

Remark 3.23. If B is a pseudo-Laman bigraph with $\operatorname{Lam}(B) = 0$ and d is a bidistance on B, then the definition of $Z^B_{\mathbb{K}}$ makes still sense, as well as the definitions of $\widetilde{A}^d_{\mathbb{C}}$, $A^d_{\mathbb{C}}$ and $Z^{B_d}_{\mathbb{C}}$. In this case, the scheme $Z^B_{\mathbb{K}}$ is nothing but the empty set, and the proof of Lemma 3.21 can still be used to prove that the schemes $A^d_{\mathbb{C}}$ and $Z^{B_d}_{\mathbb{C}}$ are isomorphic. To conclude that $Z^{B_d}_{\mathbb{C}}$ is also the empty set one can argue as in Lemma 3.22 showing that if $Z^{B_d}_{\mathbb{C}}$ were not empty, then one could construct a point in $Z^B_{\mathbb{K}}$, contradicting the hypothesis.

We can summarize the findings of this section in the following theorem.

Theorem 3.24. *Let B be a pseudo-Laman bigraph with biedges $\mathcal{E}$ without self-loops. Fix a biedge $\bar{e} \in \mathcal{E}$, and fix $\mathrm{wt} \in \mathbb{Q}^{\mathcal{E}\setminus\{\bar{e}\}}$. Then we have*

$$\operatorname{Lam}(B) \;=\; \sum_d \operatorname{Lam}(B_d),$$

where d runs over all bidistances on B compatible with wt.

Proof. When $\operatorname{Lam}(B) > 0$, the statement follows directly from the combination of Lemma 3.21 and Lemma 3.22. The case $\operatorname{Lam}(B) = 0$ is covered by Remark 3.23. □

4. Two formulas for the Laman number

In this section we explore two diametrically opposed approaches to obtain a formula for the Laman number of a bigraph from Theorem 3.24; the two approaches differ in the way we choose a weight vector wt, which so far could take any value.

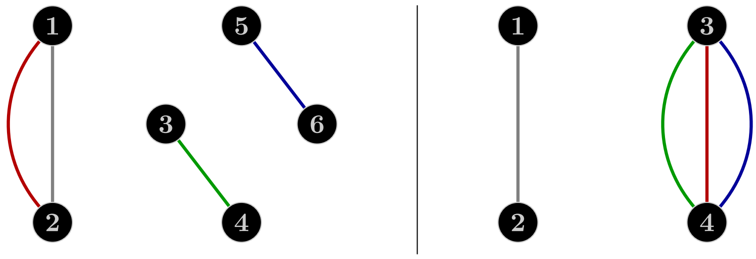


FIGURE 9. An example of an elementary bigraph.

In the first case we are going to select a general weight vector: in this way, for all bidistances d compatible with wt, the bigraphs B_d have a very particular shape (Lemma 4.6), and it turns out that their Laman number is 1 (Proposition 4.4); hence the determination of the Laman number is translated into the computation of all bidistances compatible with wt (Theorem 4.7). In the second case, we instead fix a very special weight vector, namely the vector $(1, \ldots, 1)$: with this choice it is easy to determine which bidistances are compatible with wt (Lemma 4.10); the bigraphs B_d that one obtains are, however, significantly more complicated than in the previous case, but it is possible to use this approach recursively (Theorem 4.15), translating any situation to a limited number of simple base cases (Proposition 2.10).

We start considering general weight vectors by introducing the family of bigraphs that are building blocks for the formula for the Laman number in this case.

Definition 4.1. We say that a bigraph is *elementary* if each connected component of its two graphs has exactly two vertices and no self-loops.

Let $B = (G, H)$ be an elementary bigraph with biedges $\mathcal{E}$, let $\bar{e}$ be a fixed biedge and let $\mathrm{wt} \in \mathbb{Q}^{\mathcal{E}\setminus\{\bar{e}\}}$ be a weight vector. Any bidistance on B compatible with wt is uniquely determined by the value of the pair of vertices in each component, and it satisfies the equations $d_V + d_W = \mathrm{wt}$ together with the conditions $d_V(\bar{u}, \bar{v}) = 0$ and $d_W(\bar{t}, \bar{w}) = 0$, where $\tau_G(\bar{e}) = \{\bar{u}, \bar{v}\}$ and $\tau_H(\bar{e}) = \{\bar{t}, \bar{w}\}$. Since there are no non-trivial cycles in an elementary bigraph, except the ones of the form (u, v, u) for two vertices u and v in the same component, the bidistances compatible with wt are essentially the solutions of a linear system of equations of the form

$$M \cdot \begin{pmatrix} d_V \\ d_W \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathrm{wt} \end{pmatrix}, \qquad \text{where } \begin{pmatrix} d_V \\ d_W \end{pmatrix} \in \mathbb{Q}^{|P|} \times \mathbb{Q}^{|Q|},$$

with P and Q as in Definition 2.13. The columns of M correspond to the two vertices of each component of G or H.

By exploiting that each bidistance satisfies $d_V(u, v) = d_V(v, u)$ and $d_W(t, w) = d_W(w, t)$, we restrict the linear system to one whose domain is $\mathbb{Q}^{|P|/2} \times \mathbb{Q}^{|Q|/2}$. We denote by M_B the matrix representing this system.

The rows of M_B correspond to biedges; each row has two 1's, except for the two rows corresponding to biedge $\bar{e}$, which have only one entry equal to 1; all other entries are 0. For example, the linear system for the bigraph B from Figure 9, is

$$\begin{array}{r} \text{gray} \to \\ \text{gray} \to \\ \textbf{red} \to \\ \textbf{green} \to \\ \textbf{blue} \to \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} d_V(1,2) \\ d_W(3,4) \\ d_V(3,4) \\ d_V(5,6) \\ d_W(3,4) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathrm{wt}_{\mathbf{r}} \\ \mathrm{wt}_{\mathbf{g}} \\ \mathrm{wt}_{\mathbf{b}} \end{pmatrix}.$$

Since B is elementary, we have

$$|P|/2 + |Q|/2 = \dim(G) + \dim(H).$$

Hence, if B is pseudo-Laman, then M_B is a square matrix.

Lemma 4.2. *Let B be an elementary pseudo-Laman bigraph. The determinant of the matrix M_B is either -1, 0, or 1.*

Proof. The matrix M_B consists of two column blocks; each row has at most one entry equal to 1 in each block, and all other entries are zero. We can suppose that no row in M_B is composed by only zeros, because otherwise $\det(M_B) = 0$ and we are done. We first consider those rows containing exactly one 1. Performing Laplace expansion along such a row, we get $\det(M_B) = \pm\det(M)$ for some smaller matrix M. If we can repeat this until we get $M = (1)$, then we have that $\det(M_B) = \pm 1$. Therefore, the assertion is reduced to the following claim:

Claim. Let m and n be positive integers and let M be an $(m+n)\times(m+n)$ matrix such that each row of M has exactly one 1 in its first m entries, and exactly one 1 in its last n entries; all other entries of M are 0. Then $\det(M) = 0$.

If one of the first m columns of M contains only zeros, then $\det(M) = 0$ Otherwise, we can permute the rows of M as to obtain the $m \times m$ identity matrix in the upper-left corner. Afterward, we use the first m rows to eliminate all 1's in the lower-left $n \times m$ block; the determinant may only change its sign by these operations. As a result, we obtain a block matrix of the form $\left(\begin{smallmatrix} I_m & * \\ 0 & N \end{smallmatrix}\right)$, and hence $\det(M) = \pm\det(N)$. It is easy to see that the $n\times n$ matrix N has the following form: each row contains either only 0's or precisely one 1 and one -1. Hence, by adding up all columns of N, we obtain the zero vector, witnessing that $\det(N) = 0$. $\square$

Remark 4.3. If B is an elementary bigraph, and d is a bidistance, then B_d is equal to B. Therefore, the recursion formula in Theorem 3.24 is not useful for computing its Laman number.

Proposition 4.4. *Let B be an elementary pseudo-Laman bigraph with biedges $\mathcal{E}$. Let $\bar{e}$ be a fixed biedge and let $\mathrm{wt} \in \mathbb{Q}^{\mathcal{E}\setminus\{\bar{e}\}}$ be a general weight vector. Then:*

- ▷ *if the determinant of M_B is zero, then $\mathrm{Lam}(B) = 0$;*
- ▷ *if the determinant of M_B equals ± 1, then $\mathrm{Lam}(B) = 1$.*

Proof. By Lemma 4.2, the determinant of M_B is either -1, 0 or 1.

If the determinant is zero, then there are no bidistances compatible with wt, since the latter is a assumed to be a general point in $\mathbb{Q}^{\mathcal{E}\setminus\{\bar{e}\}}$. By Theorem 3.24, the Laman number of B is 0.

If the determinant is ± 1 we conclude that there is exactly one compatible bidistance. In fact, the scheme $Z^B_{\mathbb{C}}$ is defined by equations of the form "monomial equal general constant". The exponents in the monomials form a matrix that coincides with M_B. The number of solutions is equal to $|\det(M_B)|$, as a special case of the Bernstein-Khovanskii-Kushnirenko formula (see [CLO05, Chapter 7, Theorem 5.4]). □

Now we need a notion analogous to M_B for an arbitrary bigraph $B = (G, H)$ with biedges $\mathcal{E}$. Let $\bar{e}$ be a fixed biedge and let $\mathrm{wt} \in \mathbb{Q}^{\mathcal{E} \setminus \{\bar{e}\}}$ be a general weight vector. Let d be a bidistance on B. We fix spanning forests $\mathcal{F}_G$ and $\mathcal{F}_H$ for G and H, respectively, as in Remark 3.19. We define the matrix $M_{\mathcal{F}_G,\mathcal{F}_H}$ to be the one representing the linear system $d_V + d_W = \mathrm{wt}$ where the values of d_V and d_W are restricted to the endpoints of the edges of the forests, i.e.

$$M_{\mathcal{F}_G,\mathcal{F}_H} \cdot \begin{pmatrix} \big(d_V(u,v)\big)_{\substack{\{u,v\} \text{ in } \mathcal{F}_G \\ \text{with } u \prec v}} \\ \big(d_W(t,w)\big)_{\substack{\{t,w\} \text{ in } \mathcal{F}_H \\ \text{with } t \prec w}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathrm{wt} \end{pmatrix}.$$

The number of columns of this matrix is therefore $\dim(G)+\dim(H)$. The number of rows is $|\mathcal{E}|+1$, so as in the elementary case if B is pseudo-Laman, then $M_{\mathcal{F}_G,\mathcal{F}_H}$ is a square matrix. Let us have a closer look at how the matrix $M_{\mathcal{F}_G,\mathcal{F}_H}$ is constructed. Its rows can be labeled by biedges: each of them appears exactly once, except for the fixed biedge $\bar{e}$, which appears twice since we have to take into account the two equations $d_V(\bar{u}, \bar{v}) = 0$ and $d_W(\bar{t}, \bar{w}) = 0$. If $e \in \mathcal{E} \setminus \{\bar{e}\}$ is a biedge and $\{u, v\} = \tau_G(e)$ and $\{t, w\} = \tau_H(e)$, then the corresponding row of $M_{\mathcal{F}_G,\mathcal{F}_H}$ is formed as follows:

- ▷ if both $\{u, v\}$ and $\{t, w\}$ are endpoints of edges in $\mathcal{F}_G$ and $\mathcal{F}_H$, then both $d_V(u, v)$ and $d_W(t, w)$ appear in the vector $\binom{d_V}{d_W}$, and so the row of $M_{\mathcal{F}_G,\mathcal{F}_H}$ is constructed so that its product with $\binom{d_V}{d_W}$ gives $d_V(u, v) + d_W(t, w)$;
- ▷ suppose $\{u, v\}$ are endpoints of edges in $\mathcal{F}_G$, but $\{t, w\}$ do not have this property; by construction, there exist $\{\tilde{t}, \tilde{w}\}$ that are endpoints of an edge in $\mathcal{F}_H$, and such that $d_W(t, w) = d_W(\tilde{t}, \tilde{w})$; therefore the equation $d_V(u, v)+ d_W(t, w) = \mathrm{wt}_e$ is equivalent to $d_V(u, v) + d_W(\tilde{t}, \tilde{w}) = \mathrm{wt}_e$; taking this into account, in this case we define the row of $M_{\mathcal{F}_G,\mathcal{F}_H}$ corresponding to e to be such that its product with $\binom{d_V}{d_W}$ gives $d_V(u, v) + d_W(\tilde{t}, \tilde{w})$;
- ▷ the case when $\{u, v\}$ are not endpoints of edges in $\mathcal{F}_G$ is treated similarly as the previous one.

In an analogous way one constructs the two rows corresponding to $\bar{e}$.

Example 4.5. Consider the bigraph that is shown in Figure 10, together with the given bidistance and spanning trees indicated by the thicker edges in the figure. In this case the linear system for $M_{\mathcal{F}_G,\mathcal{F}_H}$ is

$$\begin{array}{r} \text{gray} \rightarrow \\ \text{gray} \rightarrow \\ \textbf{red} \rightarrow \\ \textbf{green} \rightarrow \\ \textbf{orange} \rightarrow \\ \textbf{blue} \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_V(2,3) \\ d_W(2,3) \\ d_V(1,3) \\ d_V(2,4) \\ d_W(1,2) \\ d_W(3,4) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathrm{wt}_{\mathbf{r}} \\ \mathrm{wt}_{\mathbf{g}} \\ \mathrm{wt}_{\mathbf{o}} \\ \mathrm{wt}_{\mathbf{b}} \end{pmatrix}.$$

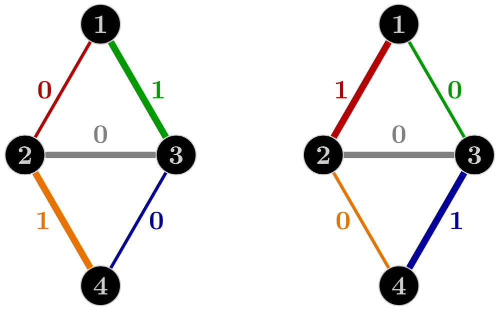
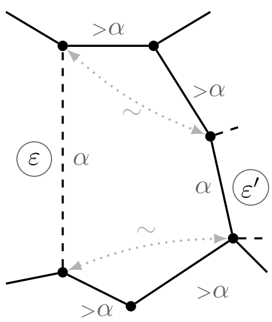


FIGURE 10. Bigraph and bidistance from Figure 7, on which now a spanning forest (denoted by thicker edges) has been fixed, that fulfills the requirements of Remark 3.19.

Lemma 4.6. *Let $B = (G, H)$ be a pseudo-Laman bigraph with biedges $\mathcal{E}$. Let $\bar{e}$ be a fixed biedge and let $\mathrm{wt} \in \mathbb{Q}^{\mathcal{E}\setminus\{\bar{e}\}}$ be a general weight vector. Let d be a bidistance on B. Then, B_d is elementary and $\mathrm{Lam}(B_d) = 1$.*

Proof. Let $\mathcal{F}_G$ and $\mathcal{F}_H$ be spanning forests for G and H, respectively, as in Remark 3.19. We claim that, for a general choice of the vector wt, the values of d_V and d_W on the edges of $\mathcal{F}_G$ and $\mathcal{F}_H$ are all different. In fact, consider the set $\mathscr{A}$ of all matrices $M_{\mathcal{F}'_G,\mathcal{F}'_H}$, where $\mathcal{F}'_G$ and $\mathcal{F}'_H$ vary over all spanning forests such that $d_V(u, v)$ is equal to the minimum of the values in the unique path in $\mathcal{F}_G$ connecting u and v, and similarly for d_W. The set $\mathscr{A}$ is finite since the matrices $M_{\mathcal{F}'_G,\mathcal{F}'_H}$ have entries in $\{0, 1\}$. Let $M_{\mathcal{F}'_G,\mathcal{F}'_H}$ be a matrix in $\mathscr{A}$ and let $D_{\mathcal{F}'_G,\mathcal{F}'_H}$ be its domain. Consider one of the hyperplanes in $D_{\mathcal{F}'_G,\mathcal{F}'_H}$ where two coordinates are equal; its image under $M_{\mathcal{F}'_G,\mathcal{F}'_H}$ is at most a hyperplane in $\mathbb{Q}^{\mathcal{E}\setminus\{\bar{e}\}}$. We identify $\mathbb{Q}^{\mathcal{E}\setminus\{\bar{e}\}}$ with the subspace of $\mathbb{Q}^{|\mathcal{E}|+1}$ of all vectors where the first two coordinates are zero. A general weight vector wt can be supposed not to lie on the union of images of all such hyperplanes through all elements of $\mathscr{A}$. This shows our claim. Moreover, a general weight vector wt does not lie in the image of any singular matrix $M_{\mathcal{F}'_G,\mathcal{F}'_H}$.

Each edge ε in G not in $\mathcal{F}_G$ appears in a unique cycle $\mathscr{C}_\varepsilon$, whose other edges are in $\mathcal{F}_G$. If $\alpha \in \mathbb{Q}$ is in the image of d_V, and ε is an edge of G whose endpoints have d_V-value α, then there is only one edge ε' in the cycle $\mathscr{C}_\varepsilon$ with the same value α, and all other edges in the cycle have greater value because of the construction of the forests and the fact that all the values of d_V and d_W are different on their edges.



Therefore, the endpoints of ε coincide with the endpoints of ε' in the quotient graph $G_{\geq\alpha} \big/ G_{>\alpha}$. It follows that the endpoints of all edges in $G_{\geq\alpha} \big/ G_{>\alpha}$ coincide, and so the bigraph B_d is elementary (the fact that B_d has no self-loops is actually ensured by the next paragraph, in which we clarify that its Laman number is positive).

We are left to prove that $\mathrm{Lam}(B_d) = 1$. In view of Proposition 4.4, it is enough to prove that M_{B_d} is nonsingular. We show this by proving that $M_{B_d} = M_{\mathcal{F}_G,\mathcal{F}_H}$ for any spanning forest $\big(\mathcal{F}_G, \mathcal{F}_H\big)$, recalling that since wt is general we can always suppose that $M_{\mathcal{F}_G,\mathcal{F}_H}$ is nonsingular. The rows of both matrices are labeled by the same set of biedges; recall that B_d inherits its biedges from B. Columns of $M_{\mathcal{F}_G,\mathcal{F}_H}$ correspond to edges of $\mathcal{F}_G$ or $\mathcal{F}_H$, and columns of M_{B_d} correspond to components of B_d. By Remark 3.19, the spanning forest $\big(\mathcal{F}_G, \mathcal{F}_H\big)$ induces a spanning forest for B_d, and since the latter is elementary, a spanning forest is nothing but an edge for each component. Hence also the columns of the two matrices are in bijection. By considering the definitions, one can convince oneself that entries of the two matrices corresponding under these two bijections are equal. □

Theorem 4.7. *Let B be a pseudo-Laman bigraph with biedges $\mathcal{E}$. Suppose that* $\mathrm{Lam}(B) > 0$*, let $\bar{e}$ be a fixed biedge and let* $\mathrm{wt} \in \mathbb{Q}^{\mathcal{E}\setminus\{\bar{e}\}}$ *be a general weight vector. Then* $\mathrm{Lam}(B)$ *is equal to the number of bidistances on B that are compatible with* wt.

Proof. This is an immediate consequence of Theorem 3.24, Lemma 4.6, and Proposition 4.4. □

Remark 4.8. Theorem 4.7 reduces the computation of the Laman number to a piecewise linear problem. We could not find an efficient algorithm for solving this problem, and this is why we are going to introduce a second formula, on which the algorithm we propose is based. Our reason for including Theorem 4.7 is that we hope that there are colleagues who have more experience with this type of problems, and who could compute Laman numbers more efficiently.

We now start with the second approach, where we fix a weight vector of the form $(1, \ldots, 1)$. First we show in Lemmas 4.9 and 4.10 that the bidistances that are compatible with such a vector can take only values in $\{0, 1\}$.

Lemma 4.9. *Let B be a pseudo-Laman bigraph with biedges $\mathcal{E}$. Suppose that* $\mathrm{Lam}(B) > 0$*. Pick $\bar{e} \in \mathcal{E}$ and fix* $\mathrm{wt} \in \mathbb{Z}^{\mathcal{E}\setminus\{\bar{e}\}}$*. Let $d = (d_V, d_W)$ be a bidistance for B and suppose that* $\mathrm{Lam}(B_d) \in \mathbb{N}\setminus\{0\}$*. Then both d_V and d_W take values in $\mathbb{Z}$.*

Proof. Suppose by contradiction that the images of d_V and d_W are not contained in $\mathbb{Z}$. We are going to construct an infinite family $\{d^\kappa : \kappa \in (0,1] \cap \mathbb{Q}\}$ of different bidistances for B that satisfies $B_{d^\kappa} = B_d$ for every $\kappa \in (0,1] \cap \mathbb{Q}$. Lemma 3.21 together with Lemma 3.22 imply then that $Z^B_{\mathbb{K}}$ consists of infinitely many points, contradicting the hypothesis.

For every $\kappa \in (0,1] \cap \mathbb{Q}$, define

$$
\begin{aligned}
d^\kappa_V &:= \kappa \cdot d_V + (1-\kappa) \cdot \lceil d_V \rceil\,, \\
d^\kappa_W &:= \kappa \cdot d_W + (1-\kappa) \cdot \lfloor d_W \rfloor\,,
\end{aligned}
$$

where $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ denote the ceiling and the floor functions, respectively. Since $\operatorname{im}(d_V)\cup\operatorname{im}(d_W)\not\subseteq\mathbb{Z}$, the family

$$\left\{d^\kappa=(d_V^\kappa,d_W^\kappa)\,:\,\kappa\in(0,1]\cap\mathbb{Q}\right\}$$

has infinitely many elements.

We show that each d^κ is a bidistance for B. Since by hypothesis

$$d_V(u,v)+d_W(t,w)=\operatorname{wt}(e)\in\mathbb{Z}\quad\text{for all } e\in\mathcal{E}\setminus\{\bar e\},$$

it follows that $\lceil d_V\rceil+\lfloor d_W\rfloor=d_V+d_W$. Hence, for all $\kappa\in(0,1]\cap\mathbb{Q}$

$$d_V^\kappa(u,v)+d_W^\kappa(t,w)=\operatorname{wt}(e)\quad\text{for all } e\in\mathcal{E}\setminus\{\bar e\}.$$

By construction it follows that $d_V^\kappa(\bar u,\bar v)=d_W^\kappa(\bar t,\bar w)=0$, where $\{\bar u,\bar v\}=\tau_G(\bar e)$ and $\{\bar t,\bar w\}=\tau_H(\bar e)$. Next, note that the two functions

$$\begin{aligned} x&\longmapsto\kappa\cdot x+(1-\kappa)\cdot\lceil x\rceil\,,\\ x&\longmapsto\kappa\cdot x+(1-\kappa)\cdot\lfloor x\rfloor \end{aligned}$$

are strictly increasing for every $\kappa\in(0,1]\cap\mathbb{Q}$. This implies that also the last property stated in Lemma 3.8 is preserved, and moreover, that $B_{d^\kappa}=B_d$. ☐

Lemma 4.10. *Let B be a pseudo-Laman bigraph with biedges $\mathcal{E}$. Suppose that* $\operatorname{Lam}(B)>0$. *Pick $\bar e\in\mathcal{E}$ and fix* $\operatorname{wt}=(1)_{\mathcal{E}\setminus\{\bar e\}}$. *Let $d=(d_V,d_W)$ be a bidistance for B and suppose that* $\operatorname{Lam}(B_d)\in\mathbb{N}\setminus\{0\}$. *Then both d_V and d_W take values in* $\{0,1\}$.

Proof. Suppose by contradiction that the claim does not hold. Then (after possibly swapping the roles of d_V and d_W) we can suppose that $d_V(u,v)>1$ for some u and v that are vertices of an edge. We are going to construct an infinite family $\{d^\kappa\,:\,\kappa\in\mathbb{N}\}$ of bidistances for B such that $B_{d^\kappa}=B_d$. Then the same argument as in the proof of Lemma 4.9 gives a contradiction.

Let α be the maximum of the values in $\operatorname{im}(d_V)$. By construction and by Lemma 4.9, we have $\alpha\geq 2$. This implies that the minimal value attained by d_W is negative. For any $\kappa\in\mathbb{N}$ define

$$d_V^\kappa(u,v):=\begin{cases}d_V(u,v)+\kappa, & \text{if } d_V(u,v)=\alpha,\\ d_V(u,v), & \text{otherwise;}\end{cases}$$

$$d_W^\kappa(t,w):=\begin{cases}d_W(t,w)-\kappa, & \text{if } d_W(t,w)=1-\alpha,\\ d_W(t,w), & \text{otherwise.}\end{cases}$$

The family $\{d^\kappa=(d_V^\kappa,d_W^\kappa)\,:\,\kappa\in\mathbb{N}\}$ consists of infinitely many elements. From the construction it follows that each d^κ is a bidistance and that $B_{d^\kappa}=B_d$. ☐

Using Proposition 2.21 the special shape of the bidistances compatible with the weight vector $(1,\ldots,1)$ allows to split the problem of computing the Laman number of a bigraph of the form B_d into the computation of the Laman numbers of two smaller bigraphs.

Lemma 4.11. *Let $B = (G, H)$ be a bigraph with biedges $\mathcal{E}$ and fix a biedge $\bar{e} \in \mathcal{E}$. Fix a bidistance $d = (d_V, d_W)$. If B is pseudo-Laman with* $\mathrm{Lam}(B) > 0$ *such that*

- ▷ *the bidistance d is compatible with* $\mathrm{wt} = (1)_{\mathcal{E}\setminus\{\bar{e}\}}$,
- ▷ *$\bar{e}$ is neither a bridge in G nor a bridge in H,*
- ▷ *neither d_V nor d_W is the zero map,*

then the quotient bigraph B_d untangles via $\bar{e} \in \mathcal{E}$ into bigraphs $B_{d,1}$ and $B_{d,2}$.

Proof. Recall from Definition 3.11 that B and B_d have the same set of biedges. We define two subsets $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{E}$ as the biedges in B_d corresponding to the following two sets of biedges in B:

$$\left\{ e \in \mathcal{E} \ : \begin{array}{ll} d_V(u, v) = 0, & \text{where } \{u, v\} = \tau_G(e) \text{ and} \\ d_W(t, w) = 1, & \text{where } \{t, w\} = \tau_H(e) \end{array} \right\},$$

$$\left\{ e \in \mathcal{E} \ : \begin{array}{ll} d_V(u, v) = 1, & \text{where } \{u, v\} = \tau_G(e) \text{ and} \\ d_W(t, w) = 0, & \text{where } \{t, w\} = \tau_H(e) \end{array} \right\}.$$

By hypothesis and using Lemma 4.10 we have that both $\mathcal{E}_1$ and $\mathcal{E}_2$ are non-empty, and that $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{\bar{e}\}$ is a partition. In G_{d_V} and H_{d_W}, edges with different values of d_V and d_W, respectively, are in different components. Hence the statement is proved. □

In order to state the final formula we introduce some notation; we then express the bigraphs obtained from Lemma 4.11 in terms of this new notation.

Definition 4.12. Let $B = \big(G, H\big)$ be a bigraph, where $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$. Given $\mathcal{M} \subseteq \mathcal{E}$, we define two bigraphs ${}^{\mathcal{M}}B$ and $B^{\mathcal{M}}$, with the same set of biedges $\mathcal{E}' = \mathcal{E} \setminus \mathcal{M}$, as follows

- ▷ Set ${}^{\mathcal{M}}B = \big(G \,/\, \mathcal{M},\ H \setminus \mathcal{M}\big)$.
- ▷ Set $B^{\mathcal{M}} = \big(G \setminus \mathcal{M},\ H \,/\, \mathcal{M}\big)$.

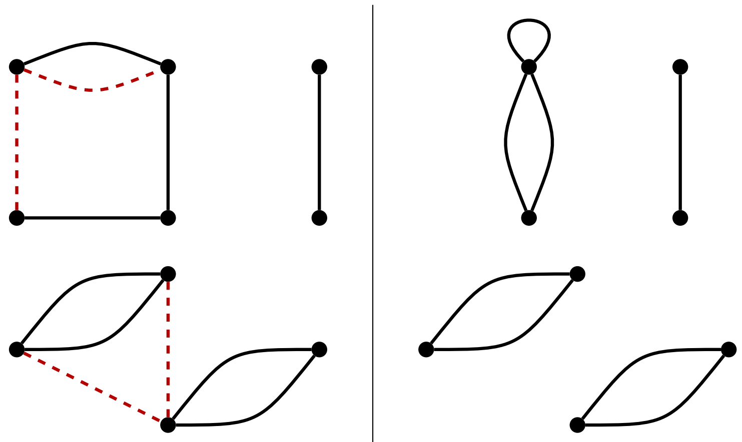
For both constructions we fix a total order on the vertices of the resulting bigraphs.

We can re-interpret Lemma 4.11 in the light of Definition 4.12 by saying that if $d = (d_V, d_W)$ is a bidistance such that both d_V and d_W take values in $\{0, 1\}$, and $\mathcal{M}$ and $\mathcal{N}$ are defined as in Lemma 4.11, then B_d untangles via $\bar{e} \in \mathcal{E}$ into ${}^{\mathcal{M}}B$ and $B^{\mathcal{N}}$. This allows us to specialize Theorem 3.24 to a recursion formula.

By what we just said and by unraveling the notions introduced in Definition 4.12 and taking into account Lemma 4.11 and Lemma 2.19 we get the following characterization.

Proposition 4.13. *Let $B = (G, H)$ be a pseudo-Laman bigraph with biedges $\mathcal{E}$ without self-loops. Pick $\bar{e} \in \mathcal{E}$, and fix* $\mathrm{wt} = (1)_{\mathcal{E}\setminus\{\bar{e}\}}$. *Let $d = (d_V, d_W)$ be a bidistance for B such that both d_V and d_W take values in $\{0, 1\}$.*

- ▷ *If d_V is the zero map, then* $\mathrm{Lam}(B_d) = \mathrm{Lam}\left({}^{\{\bar{e}\}}B\right)$.
- ▷ *If d_W is the zero map, then* $\mathrm{Lam}(B_d) = \mathrm{Lam}\left(B^{\{\bar{e}\}}\right)$.
- ▷ *If neither d_V nor d_W is the zero map, and $\bar{e}$ is neither a bridge in G_{d_V} nor a bridge in H_{d_W}, then* $\mathrm{Lam}(B_d) = \mathrm{Lam}\left({}^{\mathcal{M}}B\right) \cdot \mathrm{Lam}\left(B^{\mathcal{N}}\right)$, *where $\mathcal{M} \subseteq \mathcal{E}$*



(a) A bigraph $B = (G, H)$ and a subset $\mathcal{M}$ of the set of biedges, in dashed red.

(b) The bigraph ${}^{\mathcal{M}}B$.

FIGURE 11. Example of the construction in Definition 4.12.

is the set of biedges e such that d_W is zero on $\tau_H(e)$, and $\mathcal{N} \subseteq \mathcal{E}$ is the set of biedges e such that d_V is zero on $\tau_G(e)$.

Proof. If d_V is the zero map, then $G_{d_V} = G$ and H_{d_W} is the disjoint union of $H/\{\bar{e}\}$ and a single edge corresponding to $\{\bar{e}\}$. If $\bar{e}$ is not a bridge in G, then it is also not a bridge in $G_{d_V} = G$. So, by Lemma 2.19 we have the equality $\mathrm{Lam}(B_d) = \mathrm{Lam}\left({}^{\{\bar{e}\}}B\right)$. Suppose now that $\bar{e}$ is a bridge in G; then by Lemma 2.18 we have $\mathrm{Lam}(B_d) = 0$. It is therefore enough to prove that $\mathrm{Lam}\left({}^{\{\bar{e}\}}B\right) = 0$. We show this by proving that ${}^{\{\bar{e}\}}B$ is not pseudo-Laman. We have

$$\dim\bigl(H/\{\bar{e}\}\bigr) + \underbrace{\dim\bigl(\{\bar{e}\}\bigr)}_{=1} = \dim(H),$$

as shown in the proof of Lemma 3.13, and

$$\begin{aligned}\dim\bigl(H/\{\bar{e}\}\bigr) + \dim\bigl(G\setminus\{\bar{e}\}\bigr) &= \dim(H) - 1 + \dim(G) + 1 \\ &= |\mathcal{E}| + 1 \neq |\mathcal{E}|.\end{aligned}$$

Here $\dim\bigl(G\setminus\{\bar{e}\}\bigr) = \dim(G)+1$ because removing a bridge increases the dimension by 1. This concludes the first case; the second is proved analogously. $\square$

Lemma 4.14. *Let $B = (G, H)$ be a pseudo-Laman bigraph with biedges $\mathcal{E}$ without self-loops. Pick $\bar{e} \in \mathcal{E}$, and fix $\mathrm{wt} = (1)_{\mathcal{E}\setminus\{\bar{e}\}}$. Suppose that $d = (d_V, d_W)$ is a pair of functions $d_V \colon P \longrightarrow \{0,1\}$ and $d_W \colon Q \longrightarrow \{0,1\}$ that satisfy the first three conditions of Lemma 3.8, but not the last one. Then ${}^{\mathcal{M}}B$ or $B^{\mathcal{N}}$ has a self-loop, where the sets $\mathcal{M}$ and $\mathcal{N}$ are as in Proposition 4.13.*

Proof. By assumption, d is not a bidistance, and it must happen that there exists a cycle in G or in H such that d_V or d_W attains its minimum only once. Let us suppose that there is a cycle $\mathscr{C}$ in G such that d_V attains its minimum only on the pair (u,v), which is part of $\mathscr{C}$. If we set $\alpha = d_V(u,v)$, then we get that $G_{\geq\alpha} \big/ G_{>\alpha}$ has a self-loop. Since by definition $G_{\geq\alpha} \big/ G_{>\alpha}$ is a union of components of the graphs in either ${}^{\mathcal{M}}B$ or $B^{\mathcal{N}}$, the proof is completed. □

Proposition 2.10 gives the two base cases for the computation of the Laman number of a bigraph: if the bigraph has a self-loop, then its Laman number is zero, and if the bigraph is constituted of two copies of a single edge, then its Laman number is one. They are going to be used in combination with the formula in Theorem 4.15 to obtain a recursive algorithm.

We are now able to state the second formula for the computation of the Laman number of a bigraph. Notice that in this case we do not need to determine the set of bidistances that are compatible with the weight vector $(1,\dots,1)$.

Theorem 4.15. *Let $B=(G,H)$ be a pseudo-Laman bigraph with biedges $\mathcal{E}$ without self-loops. Let $\bar{e}$ be a fixed biedge, then*

$$\mathrm{Lam}(B) = \mathrm{Lam}\big({}^{\{\bar{e}\}}B\big) + \mathrm{Lam}\big(B^{\{\bar{e}\}}\big) + \sum_{(\mathcal{M},\mathcal{N})} \mathrm{Lam}\big({}^{\mathcal{M}}B\big)\cdot\mathrm{Lam}\big(B^{\mathcal{N}}\big), \tag{4}$$

where each pair $(\mathcal{M},\mathcal{N}) \subseteq \mathcal{E}^2$ satisfies:

- ▷ $\mathcal{M}\cup\mathcal{N}=\mathcal{E}$*;*
- ▷ $\mathcal{M}\cap\mathcal{N}=\{\bar{e}\}$*;*
- ▷ $|\mathcal{M}|\geq 2$ *and* $|\mathcal{N}|\geq 2$*;*
- ▷ *both* ${}^{\mathcal{M}}B$ *and* $B^{\mathcal{N}}$ *are pseudo-Laman.*

Proof. From Theorem 3.24 we know that

$$\mathrm{Lam}(B) = \sum_{d} \mathrm{Lam}(B_d),$$

where d runs over all bidistances on B compatible with $\mathrm{wt} = (1)_{\mathcal{E}\setminus\{\bar{e}\}}$. We distinguish two cases.

In the first case, we let $\mathrm{Lam}(B) > 0$. Let $d=(d_V,d_W)$ be a pair of functions as in Lemma 4.14, and let $\mathcal{M},\mathcal{N}\subseteq\mathcal{E}$ be the two sets of biedges defined by d. If d is not a bidistance, then by Lemma 4.14 either ${}^{\mathcal{M}}B$ or $B^{\mathcal{N}}$ has a self-loop, and so by Proposition 2.10 the contribution $\mathrm{Lam}\big({}^{\mathcal{M}}B\big)\cdot\mathrm{Lam}\big(B^{\mathcal{N}}\big)$ is zero. If d is a bidistance and $\bar{e}$ is neither a bridge in G_{d_V} nor a bridge in H_{d_W}, then by Proposition 4.13 the contribution of $\mathrm{Lam}(B_d)$ appears on the right-hand side of Equation (4). If instead $\bar{e}$ is a bridge in G_{d_V} or in H_{d_W}, then by Lemma 2.22 we conclude that $\mathrm{Lam}(B_d)=0$; at the same time by Lemma 2.23 either ${}^{\mathcal{M}}B$ or $B^{\mathcal{N}}$ is not pseudo-Laman so there is no contribution to the right-hand side of Equation (4).

It remains to settle the case $\mathrm{Lam}(B)=0$. In this case, $\mathrm{Lam}(B_d)=0$ for all bidistances compatible with wt. We have to prove that the right hand side of Equation (4) is zero, too. By Proposition 4.13, if d_V is the zero map then $\mathrm{Lam}(B_d) = \mathrm{Lam}\big({}^{\{\bar{e}\}}B\big)$. Hence, the first summand of the right-hand side of Equation (4) is zero. For the second summand, the situation is similar. For the other summands, let us fix $\mathcal{M}$ and $\mathcal{N}$ as in the hypothesis. Define $d_V(\bar{u},\bar{v}) = d_W(\bar{t},\bar{w}) = 0$, and else define $d_V(u,v)=1$ if there is an edge e in $\mathcal{M}$ such that $\tau_G(e)=(u,v)$ and $d_V(u,v)=0$ if there is no such edge; similarly for d_W. If $d=(d_V,d_W)$ is not a bidistance, then by

Table 1. Number of Laman graphs with n vertices; this sequence of numbers is A227117 in the OEIS [Slo]. There the sequence originally ended with $n = 8$, whose value was erroneously given as 609; we corrected and complemented this OEIS entry accordingly.

n	2	3	4	5	6	7	8	9	10	11	12
#	1	1	1	3	13	70	608	7222	110132	2039273	44176717

Lemma 4.14 one of the bigraphs ${}^{\mathcal{M}}B$ or $B^{\mathcal{N}}$ has a self-loop and by Proposition 2.10 the summand is zero. If $d = (d_V, d_W)$ is a bidistance, then by Proposition 4.13, we have $\operatorname{Lam}(B_d) = \operatorname{Lam}\big({}^{\mathcal{M}}B\big) \cdot \operatorname{Lam}\big(B^{\mathcal{N}}\big)$ and hence the summand is zero as well. $\square$

5. Computational results

Theorem 4.15, together with Proposition 2.10, translates naturally into a recursive algorithm, which has exponential complexity since it has to loop over all subsets of $\mathcal{E} \setminus \{\bar{e}\}$. We have implemented this algorithm [CGG$^+$16] in the computer algebra system Mathematica and in `C++`. Despite its exponential runtime, it is a tremendous improvement over the naive approach, which is to determine the number of solutions via a Gröbner basis computation. For example, to compute the Laman number 880 of the Laman graph with 10 vertices in Figure 12, our recursive algorithm took 1.7s in Mathematica and 0.18s with `C++`, while the Gröbner basis approach took about 2353s in Mathematica and 45s using the FGb library in Maple [Fau10]. Note also that the latter is feasible in practice only after replacing the parameters $\lambda(e)$ by random integers, which turns it into a probabilistic algorithm. Moreover, for speed-up, we compute the Gröbner basis only modulo a prime number so that the occurrence of large rational numbers is avoided. In contrast, our combinatorial algorithm computes the Laman number with certainty. As a consistency check, we computed the Laman numbers of all 118,051 Laman graphs with at most 10 vertices, using both approaches, and found that the results match perfectly.

For this purpose we had to compile lists of Laman graphs. In principle this is a simple task, by applying the two Henneberg rules in all possible ways. In practice, this task becomes demanding since one has to identify and eliminate duplicates, which leads to the graph isomorphism problem. Using our implementation we constructed all Laman graphs up to 12 vertices, see Table 1.

Recently, there has been large interest [BS04, ETV09, ETV13, ST10, JO12] in the maximal Laman number that a Laman graph with n vertices can have. By applying our algorithm to all Laman graphs with n vertices, we determined the maximal Laman number for $6 \leq n \leq 12$, which previously was only known for $n = 6$ and $n = 7$; the results are given in Table 2. For $n = 12$ this was a quite demanding task: computing the Laman numbers of more than 44 million graphs with 12 vertices took 56 processor days using our fast `C++` implementation.

TABLE 2. Minimal and maximal Laman number among all Laman graphs with n vertices; the minimum is 2^{n-2} and it is achieved, for example, on Laman graphs obtained by applying only the first Henneberg move (see Theorem 1.10).

n	6	7	8	9	10	11	12
min	16	32	64	128	256	512	1024
max	24	56	136	344	880	2288	6180

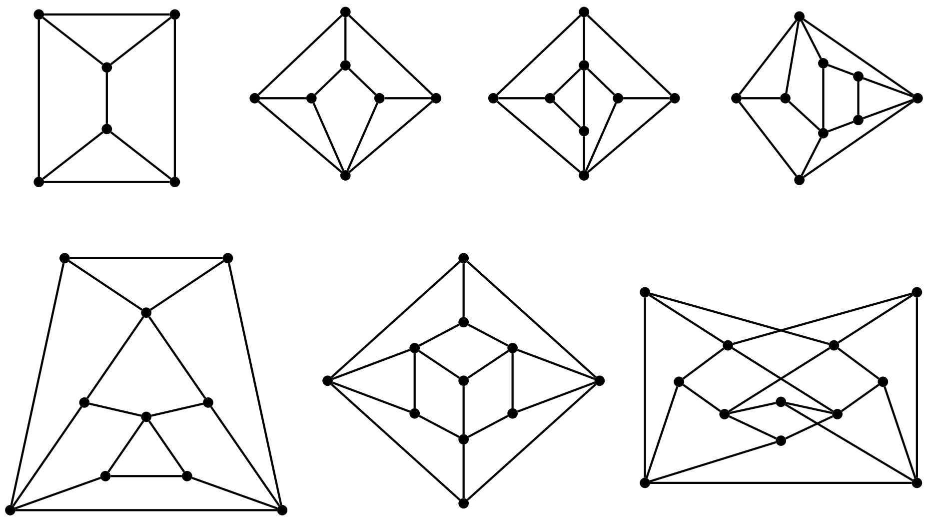


FIGURE 12. Laman graphs with $6 \leq n \leq 12$ vertices; for each n the graph with the largest Laman number is shown.

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