Elliptic Function Based Algorithms to Prove Jacobi Theta Function Relations

Liangjie Ye

Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Altenbergerstraße 69, A-4040 Linz, Austria

Abstract

In this paper we prove identities involving the classical Jacobi theta functions of the form $\sum c(i_1,i_2,i_3,i_4)\theta_1(z|\tau)^{i_1}\theta_2(z|\tau)^{i_2}\theta_3(z|\tau)^{i_3}\theta_4(z|\tau)^{i_4}=0$ with $c(i_1,i_2,i_3,i_4)\in\mathbb{K}[\Theta]$, where $\mathbb{K}$ is a computable field and $\Theta:=\left\{\theta_1^{(2k+1)}(0|\tau):k\in\mathbb{N}\right\}\cup\left\{\theta_j^{(2k)}(0|\tau):k\in\mathbb{N}\text{ and }j=2,3,4\right\}$. We give two algorithms that solve this problem. The second algorithm is simpler and works in a restricted input class.

Key words: Jacobi theta functions, modular forms, algorithmic zero-recognition, computer algebra, automatic proving of special function identities

1. Introduction

Our ultimate goal is to develop computer-assisted treatment for identities among Jacobi theta functions, namely, to automatize the proving procedures of relations and the discovery of relations. For instance, we want to establish algorithms to prove identities like the following:

Example 1.1. (Rademacher, 1973, (93.22))

$$\theta_3^{(4)}(0|\tau)\theta_3(0|\tau)-3(\theta_3''(0|\tau))^2-2\theta_3(0|\tau)^2\theta_2(0|\tau)^4\theta_4(0|\tau)^4\equiv{}^1\,0.$$

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Email address: liangjie.ye@risc.jku.at (Liangjie Ye).

URL: https://www.risc.jku.at/home/lye (Liangjie Ye).

[1] We use the notation $f_1(z_1,z_2,\dots)\equiv f_2(z_1,z_2,\dots)$ if we want to emphasize that the equality between the functions holds for all possible choices of the arguments z_j.

Example 1.2. (DLMF, 2015, 20.7.1)

$$\theta_2(0|\tau)^2\theta_2(z|\tau)^2 - \theta_3(0|\tau)^2\theta_3(z|\tau)^2 + \theta_4(0|\tau)^2\theta_4(z|\tau)^2 \equiv 0.$$

Example 1.3. (Whittaker and Watson, 1927, p. 485)

$$\theta_2(0|\tau)\,\theta_3(0|\tau)\,\theta_4(0|\tau)\,\theta_1(2z|\tau) - 2\theta_1(z|\tau)\theta_2(z|\tau)\theta_3(z|\tau)\theta_4(z|\tau) \equiv 0.$$

Example 1.4. (Mumford, 1973, p. 17)

$$\sum_{j=1}^{4} \theta_j(x|\tau)\theta_j(y|\tau)\theta_j(u|\tau)\theta_j(v|\tau) - 2\theta_3(x_1|\tau)\theta_3(y_1|\tau)\theta_3(u_1|\tau)\theta_3(v_1|\tau) \equiv 0,$$

where $x_1 := \frac{1}{2}(x+y+u+v)$ and $y_1 := \frac{1}{2}(x+y-u-v)$, $u_1 := \frac{1}{2}(x-y+u-v)$ and $v_1 := \frac{1}{2}(x-y-u+v)$.

Example 1.5. (Hardy, 1940, p. 218) A form of the cubic modular equation is

$$\theta_3(0|\tau)\theta_3(0|3\tau) - \theta_4(0|\tau)\theta_4(0|3\tau) - \theta_2(0|\tau)\theta_2(0|3\tau) \equiv 0.$$

Example 1.6. (Borwein and Borwein, 1987, p. 112) A form of the seventh-order modular equation is

$$\sqrt{\theta_3(0|\tau)\theta_3(0|7\tau)} - \sqrt{\theta_4(0|\tau)\theta_4(0|7\tau)} - \sqrt{\theta_2(0|\tau)\theta_2(0|7\tau)} \equiv 0.$$

Example 1.7. (Berndt, 1991, p. 285) Let $\eta(\tau) := e^{\pi i\tau/12}\prod_{k=1}^{\infty}(1-e^{2\pi i\tau k})$. Then

$$\theta_3(0|\tau)^2\theta_3(0|5\tau)^2 - \theta_2(0|\tau)^2\theta_2(0|5\tau)^2 - \theta_4(0|\tau)^2\theta_4(0|5\tau)^2 \equiv 8\eta(2\tau)^2\eta(10\tau)^2.$$

Let us recall the definition of Jacobi theta functions $\theta_j(z|\tau)$ ($j = 1, \dots, 4$):

Definition 1.1. (DLMF, 2015, 20.2(i)) Let $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $q := e^{\pi i\tau}$, then

$$\begin{aligned}
\theta_1(z,q) &:= \theta_1(z|\tau) := 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin((2n+1)z),\\
\theta_2(z,q) &:= \theta_2(z|\tau) := 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos((2n+1)z),\\
\theta_3(z,q) &:= \theta_3(z|\tau) := 1+2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz),\\
\theta_4(z,q) &:= \theta_4(z|\tau) := 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz).
\end{aligned}$$

In order to make the presentation simpler, we have:

Notation. Given $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}^4$, we define

$$\theta^{\alpha}(z) := \theta^{\alpha}(z|\tau) := \theta_1(z|\tau)^{\alpha_1}\theta_2(z|\tau)^{\alpha_2}\theta_3(z|\tau)^{\alpha_3}\theta_4(z|\tau)^{\alpha_4} = \theta_1(z)^{\alpha_1}\theta_2(z)^{\alpha_2}\theta_3(z)^{\alpha_3}\theta_4(z)^{\alpha_4}.$$

As a first step towards the goal we mentioned in the beginning, in Ye (2017) we provided an algorithm to prove identities involving the derivatives of $\theta_j(z|\tau)$ ($j = 1, 2, 3, 4$), in specific, involving

$$\theta_j^{(k)} := \theta_j^{(k)}(0|\tau) := \frac{\partial^k \theta_j}{\partial z^k}(z|\tau)\bigg|_{z=0}, \quad k \in \mathbb{N} := \{0,1,2,\dots\}.$$

For example, Algorithm 5.11 of Ye (2017) can assist us to prove identities like

$$\theta_3^{(4)}\theta_3 - 3(\theta_3'')^2 - 2\theta_3^2\theta_2^4\theta_4^4 = 0$$

from (Rademacher, 1973, (93.22)),

$$\frac{\theta_\alpha^{(5)}}{\theta_1'} - 3\left(\frac{\theta_\alpha''}{\theta_\alpha}\right)^2 + 2\left(\frac{\theta_\alpha''}{\theta_\alpha} - \frac{\theta_\beta''}{\theta_\beta}\right)\left(\frac{\theta_\alpha''}{\theta_\alpha} - \frac{\theta_\gamma''}{\theta_\gamma}\right) = 0$$

from (Rademacher, 1973, (93.7)), where $\alpha = 2,3,4$, and

$$\frac{\theta_1^{(3)}}{\theta_1'} - \frac{\theta_2''}{\theta_2} - \frac{\theta_3''}{\theta_3} - \frac{\theta_4''}{\theta_4} = 0$$

from (Lawden, 1998, p. 22).

In general, this algorithm can do zero-recognition on any function in $\mathbb{K}[\Theta]$, which is the $\mathbb{K}$-algebra generated by

$$\Theta := \left\{\theta_1^{(2k+1)}(0|\tau) : k \in \mathbb{N}\right\} \cup \left\{\theta_j^{(2k)}(0|\tau) : k \in \mathbb{N} \text{ and } j = 2,3,4\right\},$$

where $\mathbb{K} \subseteq \mathbb{C}$ is an effectively computable field which contains all the complex constants we need (i.e., i, $e^{\pi i/4}$, etc.). The reason we omit $\theta_1^{(k_1)}(0|\tau)$ when $k_1 \in 2\mathbb{N}$, and omit $\theta_m^{(k_2)}(0|\tau)$ ($m = 2,3,4$) when $k_2 \in 2\mathbb{N}+1$ is that by Definition 1.1 these are equal to zero.

In this article we extend the function space $\mathbb{K}[\Theta]$ to

$$R_1 := \mathbb{K}[\Theta][\theta_1(z|\tau), \theta_2(z|\tau), \theta_3(z|\tau), \theta_4(z|\tau)],$$

by which we define the $\mathbb{K}[\Theta]$-algebra generated by $\theta_1(z|\tau), \theta_2(z|\tau), \theta_3(z|\tau)$ and $\theta_4(z|\tau)$. We solve the following problem algorithmically:

Problem 1.1. Given $f \in R_1$, decide whether $f = 0$.

The framework used to solve Problem 1.1 is the theory of elliptic functions and modular forms. In particular, we have to use an essential tool, which is Algorithm 5.11 from Ye (2017). As a result, we provide Algorithm 3.1 for solving Problem 1.1.

Example 1.8. Our algorithm will be used to prove

$$c_1\theta_3(z)^2\theta_4(z)^2 + c_2\theta_4(z)^4 + c_3\theta_3(z)^4 + c_4\theta_1(z)^2\theta_2(z)^2 \equiv 0,$$

where

$$c_1 := -16\theta_2^5\theta_3^2\theta_4^3 - 4\theta_2\theta_3^6\theta_4^3 - 4\theta_2\theta_3^2\theta_4^7 - 32\theta_3^2\theta_4^3\theta_2'' + 32\theta_2\theta_3^2\theta_4^2\theta_4'',$$
$$c_2 := 14\theta_2^5\theta_3^4\theta_4 + 2\theta_2\theta_3^8\theta_4 + 2\theta_2\theta_3^4\theta_4^5 + 16\theta_3^4\theta_4\theta_2'' - 16\theta_2\theta_3^4\theta_4'',$$
$$c_3 := 2\theta_2^5\theta_4^5 + 2\theta_2\theta_3^4\theta_4^5 + 2\theta_2\theta_4^9 + 16\theta_4^5\theta_2'' - 16\theta_2\theta_4^4\theta_4''$$

and

$$c_4 := -12\theta_2^5\theta_3^2\theta_4^2.$$

However, we observed that in the literature most identities fitting into Problem 1.1 are also in a smaller class, in which the coefficient set $\mathbb{K}[\Theta]$ is replaced by a subalgebra

$$\mathbb{K}[\widetilde{\Theta}]_h := \{p(\theta_2(0), \theta_3(0), \theta_4(0)) : p \in \mathbb{K}[x, y, z] \text{ homogeneous}\},$$

and we define

$$R_2 := \mathbb{K}[\widetilde{\Theta}]_h[\theta_1(z|\tau), \theta_2(z|\tau), \theta_3(z|\tau), \theta_4(z|\tau)].$$

Restricting $\mathbb{K}[\Theta]$ to $\mathbb{K}[\widetilde{\Theta}]_h$, we provide Algorithm 6.1 to solve the following problem algorithmically without invoking Algorithm 5.11 of Ye (2017). Algorithm 6.1 is faster than Algorithm 3.1 in our experiments. We will give some brief arguments concerning the speed comparison in the end of this paper. Moreover, working with this restricted class, we also found some classical mathematical insights, such as Proposition 5.1 and Lemma 5.4.

Problem 1.2. Given $f \in R_2$, decide whether $f = 0$.

Example 1.9. (DLMF, 2015, 20.7.1) Our algorithm will be used to prove

$$\theta_2(0)^2\theta_2(z)^2 - \theta_3(0)^2\theta_3(z)^2 + \theta_4(0)^2\theta_4(z)^2 \equiv 0.$$

The paper is organized as follows. In Section 2 we present a theorem to decompose any $f(z|\tau) \in R_1$ into a set of quasi-elliptic components of $f(z|\tau)$, and prove that $f(z|\tau) \equiv 0$ if and only if its quasi-elliptic components are all equal to zero. In Section 3 we give an Algorithm to decide if a quasi-elliptic component of any function in R_1 is equal to zero or not, thus we achieve the goal to prove or disprove $f(z|\tau) \equiv 0$. In Section 4 we derive a theorem connecting the Weierstrass elliptic function and the theta functions in a (new) way, which plays an important role for solving Problem 1.2. Working in the restricted space R_2, in Section 5 we obtain a critical lemma about the finite-orbit weight. In Section 6 we give an Algorithm to decide if any function in R_2 is equal to zero or not, thus we achieve the goal of solving Problem 1.2.

Convention. (i) Throughout the paper τ is always in the upper-half plane $\mathbb{H}$ and for $z = ce^{i\varphi}$ ($c > 0, 0 \leq \varphi < 2\pi$) we define $z^r := c^r e^{ir\varphi}$ for $r \in \frac{1}{2}\mathbb{Z}$.

(ii) For two sets A and B, we use B^A to present the set of functions $\{f : A \to B\}$.

(iii) For any $\alpha \in \mathbb{Z}^n$ we assume that $\alpha = (\alpha_1, \ldots, \alpha_n)$ and define $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

2. Decomposing $f^\Psi(z|\tau) \in R_1$

Definition 2.1. Given $M \subseteq \mathbb{N}^4$ finite, define

$$f_M : \mathbb{K}[\Theta]^M \to R_1$$
$$\psi \mapsto f_M(\psi) =: f_M^\psi$$

where

$$f_M^\psi(z|\tau) := \sum_{\alpha \in M} \psi(\alpha)\theta^\alpha(z|\tau).$$

Notation. If M is clear from the context, we write f instead of f_M, and f^ψ instead of f_M^ψ. Sometimes, for convenience, we use $f^\psi(z)$ to present $f^\psi(z|\tau)$.

As an illustration of Definition 2.1, let us look at the identity in Example 1.8. Here we have

$$M = \{(0,0,2,2),(0,0,0,4),(0,0,4,0),(2,2,0,0)\}, \tag{1}$$

$\psi((0,0,2,2)) = c_1$, $\psi((0,0,0,4)) = c_2$, $\psi((0,0,4,0)) = c_3$, $\psi((2,2,0,0)) = c_4$ and

$$f^\psi(z) = c_1\theta_3(z)^2\theta_4(z)^2 + c_2\theta_4(z)^4 + c_3\theta_3(z)^4 + c_4\theta_1(z)^2\theta_2(z)^2.$$

In order to decompose $f_M^\psi \in R_2$, we decompose the corresponding M first.

Definition 2.2. Given $M \subseteq \mathbb{N}^4$ finite, $a,b \in \{1,2\}$, and $t \in \mathbb{N}$, let

$$X_{t,a,b}(M) := \{\alpha \in M : |\alpha| = t,\ \alpha_1+\alpha_4 \equiv a+1 (\bmod 2),\ \alpha_1+\alpha_2 \equiv b+1 (\bmod 2)\},$$

and define the following partition of M:

$$X(M) := \{X_{t,a,b}(M) \neq \emptyset : t \in \mathbb{N} \text{ and } a,b \in \{1,2\}\}.$$

Example 2.1. (i) Let M be the the same as in expression (1). Then

$$X(M) = \{X_{4,1,1}(M)\} = \{M\}.$$

(ii) Let $M = \{(0,0,2,0),(0,0,0,2),(2,0,2,0),(2,1,1,0)\}$. Then

$$\begin{aligned} X(M) &= \{X_{2,1,1}(M), X_{4,1,1}(M), X_{4,2,1}(M)\} \\ &= \{\{(0,0,2,0),(0,0,0,2)\},\{(2,0,2,0)\},\{(2,1,1,0)\}\}. \end{aligned}$$

We shall note that for a given $M \subseteq \mathbb{Z}^4$ finite, $X(M)$ is unique. One can check that if $X(M) = \{M_1,\ldots,M_n\}$ then $M_i \cap M_j = \emptyset$ when $i \neq j$ and the disjoint union

$$M_1 \dot\cup \cdots \dot\cup M_n = M.$$

Definition 2.3. [set of quasi-elliptic components of ψ] Given $M \subseteq \mathbb{N}^4$ finite, let $X(M) = \{M_1,\ldots,M_n\}$. For $\psi \in R_\Theta^M$ we define

$$Q(\psi) := \{\psi_1,\ldots,\psi_n\}$$

where $\psi_j := \psi|_{M_j}$.

Definition 2.4. [set of quasi-elliptic components of f^ψ] Given $\psi \in {R_\theta}^M$, we define

$$Q(f^\psi) := \{f^{\psi_1},\ldots,f^{\psi_n}\},$$

where $\{\psi_1,\ldots,\psi_n\} = Q(\psi)$.

Example 2.2. Let $M = \{(0,0,2,0),(0,0,0,2),(2,0,2,0),(2,1,1,0)\}$ as in Example 2.1 (ii) and

$$f^\psi = f_M^\psi = c_1\theta_3(z)^2 + c_2\theta_4(z)^2 + c_3\theta_1(z)^2\theta_3(z)^2 + c_4\theta_1(z)^2\theta_2(z)\theta_3(z)$$

with the $c_j \in \mathbb{K}[\Theta]$. Then

$$Q(f^\psi) = \{f_1,f_2,f_3\},$$

where $f_1 = c_1\theta_3(z)^2 + c_2\theta_4(z)^2$, $f_2 = c_3\theta_1(z)^2\theta_3(z)^2$ and $f_3 = c_4\theta_1(z)^2\theta_2(z)\theta_3(z)$.

Corollary 2.1. If g is a quasi-elliptic component of some f^{Ψ}, then it is the quasi-elliptic component of itself.

Proof. The proof can be done by directly following Definitions 2.2 and 2.4. $\square$

Definition 2.5. If f^{Ψ} is the quasi-elliptic component of itself, we say that f^{Ψ} is quasi-elliptic.

Theorem 2.1. Let $f^{\Psi} = f^{\Psi}_M \in R_1$ and $Q(f^{\Psi}) = \{f^{\Psi_1}, \ldots, f^{\Psi_n}\}$, then

$$f^{\Psi}(z|\tau) \equiv 0 \text{ if and only if } f^{\Psi_j}(z|\tau) \equiv 0 \text{ for all } j \in \{1\ldots n\}.$$

Before we prove this theorem, we need to recall the following lemma.

Lemma 2.1. (Whittaker and Watson, 1927, P. 465) Let $N := e^{-\pi i\tau - 2iz}$. For $j \in \{1,2,3,4\}$ we have $\theta_j(z+\pi\tau|\tau) = \varepsilon_1(j)\theta_j(z|\tau)$ and $\theta_j(z+\pi|\tau) = \varepsilon_2(j)\theta_j(z|\tau)$ where $\varepsilon_1(j)$ and $\varepsilon_2(j)$ are defined in Table 2.1.

j	1	2	3	4
$\varepsilon_1(j)$	$-N$	N	N	$-N$
$\varepsilon_2(j)$	-1	-1	1	1

Table 1.

Proof of Theorem 2.1. If $f^{\Psi_j}(z|\tau) \equiv 0$ for all $j \in \{1\ldots n\}$ then $f^{\Psi}(z|\tau) \equiv 0$ is immediate.

Suppose $f^{\Psi}(z|\tau) \equiv 0$. Write $f^{\Psi}(z|\tau) := \sum_{\alpha\in M} \psi(\alpha)\theta^{\alpha}(z|\tau)$ and write M as a union of disjoint non-empty sets $X_1(M) \cup X_2(M) \cup \cdots \cup X_m(M)$ where for $t \in \{1\ldots m\}$,

$$X_t(M) := \{\alpha \in M : |\alpha| = d_t\}$$

with $d_1, \ldots, d_m$ pairwise distinct. In this proof we use $f(z)$ to present $f^{\Psi}(z|\tau)$. We can write $f(z) = \sum_{t=1}^{m} f_t(z)$ where $f_t(z) := \sum_{\alpha\in X_t(M)} \psi(\alpha)\theta^{\alpha}(z)$.

Next we write

$$0 \equiv f(z) \equiv \sum_{t=1}^{m} (f_{t,1}(z) + f_{t,2}(z)),$$

where

$$f_{t,1}(z) := \sum_{\alpha\in X_{t,1}(M)} \psi(\alpha)\theta^{\alpha}(z) \quad \text{and} \quad f_{t,2}(z) := \sum_{\alpha\in X_{t,2}(M)} \psi(\alpha)\theta^{\alpha}(z)$$

with $X_{t,1}(M) := \{\alpha \in X_t(M) : \alpha_1 + \alpha_4 \text{ even}\}$ and $X_{t,2}(M) := \{\alpha \in X_t(M) : \alpha_1 + \alpha_4 \text{ odd}\}$.

By employing Table 2.1, we obtain for $t \in \{1, \ldots, m\}$,

$$f_{t,1}(z+\pi\tau) \equiv N^{d_t} f_{t,1}(z) \quad \text{and} \quad f_{t,2}(z+\pi\tau) \equiv -N^{d_t} f_{t,1}(z).$$

Then for $k \in \{0, 1, \ldots, 2m-1\}$,

$$f_t(z+k\pi\tau) \equiv f_{t,1}(z+k\pi\tau) + f_{t,2}(z+k\pi) \equiv (N^{d_t})^k f_{t,1}(z) + (-N^{d_t})^k f_{t,2}(z).$$

Thus we have,

$$0 \equiv f(z) \equiv f(z+k\pi\tau) \equiv \sum_{t=1}^{m} (N^{d_t})^k f_{t,1}(z) + (-N^{d_t})^k f_{t,2}(z),$$

which can be written as

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ N^{d_1} & -N^{d_1} & \cdots & N^{d_m} & -N^{d_m} \\ (N^{d_1})^2 & (-N^{d_1})^2 & \cdots & (N^{d_m})^2 & (-N^{d_m})^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (N^{d_1})^{2m-1} & (-N^{d_1})^{2m-1} & \cdots & (N^{d_m})^{2m-1} & (-N^{d_m})^{2m-1} \end{pmatrix} \begin{pmatrix} f_{1,1} \\ f_{1,2} \\ \vdots \\ f_{m,1} \\ f_{m,2} \end{pmatrix} = 0 \qquad (2)$$

Since $N \neq 0$, the determinant of this Vandermonde matrix is nonzero. Therefore we can multiply both sides of (2) by the inverse of the Vandermonde matrix and obtain $f_{t,i} = 0$ for all $t \in \{1, \ldots, m\}$ and $i \in \{1, 2\}$.

Next we write

$$0 \equiv f_{t,1}(z) \equiv f_{t,1,1}(z) + f_{t,1,2}(z) \quad \text{and} \quad 0 \equiv f_{t,2}(z) \equiv f_{t,2,1}(z) + f_{t,2,2}(z) \qquad (3)$$

where for $a \in \{1, 2\}$

$$f_{t,a,1}(z) := \sum_{\alpha \in X_{t,a,1}(M)} \psi(\alpha)\theta^{\alpha}(z) \quad \text{and} \quad f_{t,a,2}(z) := \sum_{\alpha \in X_{t,a,2}(M)} \psi(\alpha)\theta^{\alpha}(z)$$

with $X_{t,a,1}(M) := \{\alpha \in X_{t,a}(M) : \alpha_1 + \alpha_2 \text{ even}\}$ and $X_{t,a,2}(M) := \{\alpha \in X_{t,a}(M) : \alpha_1 + \alpha_2 \text{ odd}\}$. Again by using Table 2.1 on the terms appearing in $f_{t,1}(z)$ and $f_{t,2}(z)$, we obtain for $a \in \{1, 2\}$,

$$0 \equiv f_{t,a}(z) \equiv f_{t,a}(z+\pi) \equiv f_{t,a,1}(z+\pi) + f_{t,a,2}(z+\pi) \equiv f_{t,a,1}(z) - f_{t,a,2}(z).$$

This together with (3) implies $f_{t,a,1} = f_{t,a,2} = 0$ for all $t \in \{1, \ldots, m\}$ and $a \in \{1, 2\}$.

In view of Definition 2.2 choose j such that $M_j = X_{t,a,b}(M)$, then

$$f^{\psi_j}(z) \equiv f_{t,a,b}(z) \equiv \sum_{\alpha \in M_j} \psi(\alpha)\theta^{\alpha}(z) \equiv 0$$

for all $j \in \{1, \ldots, n\}$ where $n = |X(M)|$. $\square$

3. Zero-recognition for $f^{\psi} \in R_1$

In this section we will use elliptic function properties to decide whether any given $f^{\psi} \in R_1$ is identically zero.

Theorem 3.1. Let $\sum_{\alpha \in Y} \psi(\alpha)\theta^{\alpha}(z|\tau) \in R_1$ be quasi-elliptic. Then for all $\alpha, \beta \in Y$, $\frac{\theta^{\alpha}(z|\tau)}{\theta^{\beta}(z|\tau)}$ is elliptic with respect to z.

Proof. Suppose that $\sum_{\alpha \in Y} \psi(\alpha)\theta^{\alpha}(z|\tau)$ is a quasi-elliptic component of some $f_M^{\psi}(z|\tau) = \sum_{\alpha \in M} \psi(\alpha)\theta^{\alpha}(z|\tau)$. By assumption Y is equal to an element in $X(M)$. Consequently $Y =$

$X_{t,a,b}(M)$ for some $t \in \{1,\ldots,m\}$ and $a,b \in \{1,2\}$. Take an arbitrary $\alpha \in X_{t,a,b}(M)$. By Table 2.1 we have $\theta^{\alpha}(z+\pi\tau) \equiv (-1)^{a+1}N^{d_t}\theta^{\alpha}(z)$ and $\theta^{\alpha}(z+\pi) \equiv (-1)^{b+1}\theta^{\alpha}(z)$, which implies that for any $\alpha,\beta \in X_{t,a,b}(M)$,

$$\frac{\theta^{\alpha}(z+\pi\tau)}{\theta^{\beta}(z+\pi\tau)} \equiv \frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)} \quad \text{and} \quad \frac{\theta^{\alpha}(z+\tau)}{\theta^{\beta}(z+\tau)} \equiv \frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)}.$$

Therefore $\frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)}$ is elliptic. □

Definition 3.1. Let $\omega_1,\omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. A period-parallelogram with periods ω_1 and ω_2 is denoted by

$$P(\omega_1,\omega_2) := \{t_1\omega_1 + t_2\omega_2 : t_1,t_2 \in [0,1[\}.$$

Note. In this paper, $\omega_1 = \pi$ and $\omega_2 = \pi\tau$.

Lemma 3.1. Let L be the lattice generated by $\omega_1,\omega_2 \in \mathbb{C}$. For every $z \in \mathbb{C}$ there exists one and only one point $z_1 \in P(\omega_1,\omega_2)$ such that $z_1 \sim_L z$.

Proof. For any fixed $z = a\omega_1 + b\omega_2 \in \mathbb{C}$ with $a,b \in \mathbb{R}$, we can always find $m,n \in \mathbb{Z}$ and $t_1,t_2 \in [0,1[$ such that $a = m+t_1$ and $b = n+t_2$. Let $z_1 := t_1\omega_1 + t_2\omega_2$, then $z_1 \in P(\omega_1,\omega_2)$ and $z_1 \sim_L z$. Assume there exists another point $z_2 \in P(\omega_1,\omega_2)$ with $z_2 \sim_L z$, then $z_2 \sim_L z_1$. Suppose $z_2 := t_3\omega_1 + t_4\omega_2$ with $t_3,t_4 \in [0,1[$. Then $t_3 - t_1 \in \mathbb{Z}$ and $t_4 - t_2 \in \mathbb{Z}$. This implies $t_3 = t_1$ and $t_4 = t_2$, i.e., $z_2 = z_1$. □

Proposition 3.1. (Whittaker and Watson, 1927, 21.12) For each $j \in \{1,2,3,4\}$, $\theta_j(z)$ has one and only one zero in $P(\pi,\pi\tau)$. The zeros of $\theta_1(z)$, $\theta_2(z)$, $\theta_3(z)$, $\theta_4(z)$ are at the points congruent respectively to 0, $\frac{\pi}{2}$, $\frac{\pi}{2}+\frac{\pi\tau}{2}$, $\frac{\pi\tau}{2}$, respectively, modulo $\{m\pi + n\pi\tau : m,n \in \mathbb{Z}\}$.

Definition 3.2. Given a meromorphic function f on $\mathbb{C}$, we define

$$\text{poles}\,(f) := \{z \in \mathbb{C} : f \text{ has a pole at } z\}$$

and

$$\text{zeros}\,(f) := \{z \in \mathbb{C} : f \text{ has a zero at } z\}.$$

A crucial tool for us to do zero-recognition is the following theorem.

Theorem 3.2. For any nonzero elliptic function f with periods ω_1 and ω_2, one has

$$\#\left(\text{poles}\,(f) \cap P(\omega_1,\omega_2)\right) = \#\left(\text{zeros}\,(f) \cap P(\omega_1,\omega_2)\right).$$

Note. poles $(f) \cap P(\omega_1,\omega_2)$ and zeros $(f) \cap P(\omega_1,\omega_2)$ are finite sets.

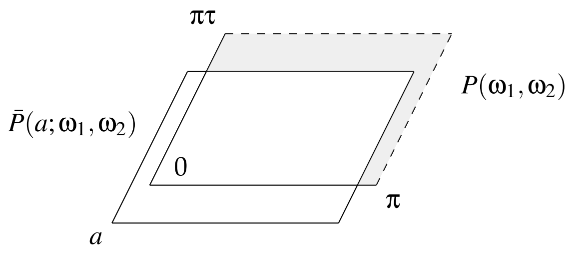
Proof of Theorem 3.2. Let $H := \{z \in P(\omega_1,\omega_2) : f \text{ has a pole or zero at } z\}$, $h_1 := \max\{t_1 : t_1\omega_1 + t_2\omega_2 \in H \text{ with } t_1,t_2 \in [0,1[\}$ and $h_2 := \max\{t_2 : t_1\omega_1 + t_2\omega_2 \in H \text{ with } t_1,t_2 \in [0,1[\}$. We define a closed period parallelogram by

$$\bar{P}(a;\omega_1,\omega_2) := \{a + b\omega_1 + c\omega_2 : b,c \in [0,1]\}$$

with

$$a := -\frac{1-h_1}{2}\omega_1 - \frac{1-h_2}{2}\omega_2.$$

The following image interprets the positions of $\bar{P}(a;\omega_1,\omega_2)$ and $P(\omega_1,\omega_2)$.



By the definition of $\bar{P}(a;\omega_1,\omega_2)$, one can easily check that for any $y \in P(\omega_1,\omega_2)$ if y is a zero (or a pole) of $f(z)$, then y is also in the interior of $\bar{P}(a;\omega_1,\omega_2)$; and $f(z)$ has poles or zeros neither in the gray area

$$\{z : z \in P(\omega_1,\omega_2) \text{ and } z \notin \bar{P}(a;\omega_1,\omega_2)\}$$

nor on the line segments where $P(\omega_1,\omega_2)$ intersects the boundary of $\bar{P}(a;\omega_1,\omega_2)$. Hence by Lemma 3.1, $f(z)$ does not have any zeros or poles on the whole boundary of $\bar{P}(a;\omega_1,\omega_2)$, and no zeros or poles in the region

$$\{z : z \in \bar{P}(a;\omega_1,\omega_2)\} \text{ and } z \notin P(\omega_1,\omega_2)\}.$$

Therefore the set of zeros and poles in $P(\omega_1,\omega_2)$ is equal to the set of zeros and poles in the interior of $\bar{P}(a;\omega_1,\omega_2)$. By a classical argument, e.g. (Chandrasekharan, 1985, p. 23, Th. 3) we complete the proof. □

Note. Usually in the literature Theorem 3.2 is stated in different ways, e.g. in (Chandrasekharan, 1985, p. 23, Th. 3), (Jones and Singerman, 2005, p. 75, Th. 3.6.4) and (Whittaker and Watson, 1927, p. 432).

Definition 3.3. Given $M \in \mathbb{N}^4$ finite, we define

$$\min(M) := \{(\beta_1,\beta_2,\beta_3,\beta_4) \in M : \beta_1 = \min\{\alpha_1 : (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in M\}\}.$$

The following theorem is the key of doing zero-recognition in R_1.

Theorem 3.3. Let $f^\Psi(z|\tau) := \sum_{\alpha\in M} \psi(\alpha)\theta^\alpha(z)$ be quasi-elliptic. For any $\beta = (\beta_1,\beta_2,\beta_3,\beta_4) \in \min(M)$, let $g_\beta(z|\tau) := \frac{f^\Psi(z|\tau)}{\theta^\beta(z)}$. Then the series expansion of $g_\beta(z|\tau)$ is of the form

$$g_\beta(z|\tau) = \sum_{j=0}^{\infty} d_j(\tau) z^j$$

with $d_j(\tau) \in \mathbb{K}(\Theta)$;[2] and if $d_j(\tau) \equiv 0$ for $j = 0,\ldots,\beta_2+\beta_3+\beta_4$ then $f^\Psi(z|\tau) \equiv 0$.

[2] $\mathbb{K}(\Theta)$ denotes the quotient field of $\mathbb{K}[\Theta]$ consisting of all quotients $P[\Theta]/Q[\Theta]$ with $P[\Theta], Q[\Theta] \in \mathbb{K}[\Theta]$.

Proof. From Definition 1.1 we know that for fixed $\tau \in \mathbb{H}$ the $\theta_j(z|\tau)$ $(j = 1, \ldots, 4)$ are analytic functions on the whole complex plane with respect to z, and for fixed $z \in \mathbb{C}$, the $\theta_j(z|\tau)$ $(j = 1, \ldots, 4)$ are analytic functions of τ for all $\tau \in \mathbb{H}$. By Proposition 3.1, only $\theta_1(z)$ has a zero at $z = 0$. Since all $\theta_1^{\beta_1}(z)$ in the denominator of $g_\beta(z|\tau)$ cancels against each $\theta^\alpha(z)$ by the choice of β, we deduce that $g_\beta(z|\tau)$ is analytic at $z = 0$. Hence we have the Taylor expansion around $z = 0$.

By Theorem 3.1, $g_\beta(z|\tau)$ is an elliptic function with respect to z. We observe that the only possible poles of $g_\beta(z|\tau)$ in $P(\pi, \pi\tau)$ are $\frac{\pi}{2}$, $\frac{\pi}{2} + \frac{\pi\tau}{2}$ and $\frac{\pi\tau}{2}$, thus $g_\beta(z|\tau)$ has at most $\beta_2 + \beta_3 + \beta_4$ poles including multiplicities in $P(\pi, \pi\tau)$. If $a_j(\tau) \equiv 0$ for $j = 0, \ldots, \beta_2 + \beta_3 + \beta_4$, then $g_\beta(z|\tau)$ has a zero at $z = 0$ with multiplicity at least $\beta_2 + \beta_3 + \beta_4 + 1$, which means the number of zeros of $g_\beta(z|\tau)$ in $P(\pi, \pi\tau)$ must be at least $\beta_2 + \beta_3 + \beta_4 + 1$. By Theorem 3.2, $g_\beta(z|\tau)$ must be zero, so $f^\Psi(z|\tau)$ is zero.
□

The algorithmic content of Theorem 3.3 is the following:

Algorithm 3.1. Given $f^\Psi \in R_1$ with $f^\Psi = f^\Psi_M$, we have the following algorithm to decide whether $f^\Psi = 0$.

Input: $f^\Psi \in R_1$.
Output: True if $f^\Psi = 0$; False if $f^\Psi \neq 0$.

Write $f^\Psi(z|\tau) = \sum_{j=1}^{n} f^{\Psi_j}(z|\tau)$ where the $f^{\Psi_j}(z|\tau)$ are the quasi-elliptic components of $f^\Psi(z|\tau)$.

Set $j := 1$. While $j \leq n$ do
    Choose $\beta = (\beta_1, \beta_2, \beta_3, \beta_4) \in M_j$ such that $\beta_1 = \min\{\alpha_1 : (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in M_j\}$;
    Let $g(z|\tau) := \frac{f^{\Psi_j}(z|\tau)}{\theta^\beta(z|\tau)}$;
    write $g(z|\tau) = \sum_{k=0}^{\infty} d_k(\tau) z^k$;
    Set $k := 0$. While $k \leq \beta_2 + \beta_3 + \beta_4$ do
        if $d_k(\tau) \equiv 0$;
            $k{+}{+}$;
        otherwise return False;
    end do;
$j{+}{+}$;
end do;
return True;

Note. In Algorithm 3.1, we use Algorithm 5.11 of Ye (2017) to check whether $d_k(\tau) \equiv 0$.

Theorem 3.4. Algorithm 3.1 is correct.

Proof. According to Definition 2.2 we can always write any $f^\Psi \in R_1$ into a sum of quasi-elliptic components of $f^\Psi = f^\Psi_M$ for some finite set $M \subseteq \mathbb{N}^4$.

Assume $f^\Psi = 0$. Then by Theorem 2.1, $f^{\Psi_j} = 0$ for all $j \in \{1, \ldots, n\}$. Hence the corresponding $g = 0$, which implies $d_k(\tau) \equiv 0$ for all $k \in \mathbb{N}$. Therefore Algorithm 3.1 returns True.

Assume $f^{\Psi} \neq 0$. By Theorem 2.1, there exists a $t \in \{1,\dots,n\}$ such that $f^{\Psi_t} \neq 0$. Then the corresponding g is nonzero. If g is a constant, then $d_0 \neq 0$ and Algorithm 3.1 returns False. Assume g is not a constant. By Theorem 3.1, $g(z|\tau)$ is an elliptic function. Since $g(z|\tau)$ has at most $\beta_2+\beta_3+\beta_4$ poles in $P(\pi,\pi\tau)$, by Theorem 3.2 we deduce that $g(z|\tau)$ has at most $\beta_2+\beta_3+\beta_4$ zeros in $P(\pi,\pi\tau)$. This means $d_0(\tau),\dots,d_{\beta_2+\beta_3+\beta_4}(\tau)$ cannot be all zero. Thus Algorithm 3.1 returns False. $\square$

Example 1.8 (continued). Prove

$$f^{\Psi}(z) := c_1\theta_3(z)^2\theta_4(z)^2 + c_2\theta_4(z)^4 + c_3\theta_3(z)^4 + c_4\theta_1(z)^2\theta_2(z)^2 \equiv 0,$$

where the c_j are chosen as in Example 1.8.

Proof. One can check by Definition 2.2 that $f(z)$ is the quasi-elliptic component of itself, so in this case $f^{\Psi}(z) = f^{\Psi_1}(z)$. Following Algorithm 3.1,

$$g(z) := \frac{f^{\Psi}(z)}{c_3\theta_4(z)^4} = c_1\frac{\theta_3(z)^2}{\theta_4(z)^2} + c_2 + c_3\frac{\theta_3(z)^4}{\theta_4(z)^4} + c_4\frac{\theta_1(z)^2\theta_2(z)^2}{\theta_4(z)^4}.$$

Then

$$g(z) = \sum_{k=0}^{\infty} d_k(\tau)z^k$$

with $d_0(\tau) = c_4\theta_1^2\theta_2^2 + c_3\theta_3^4 + c_1\theta_3^2\theta_4^2 + c_2\theta_4^4$ and $d_k(\tau)$ for $k=1,\dots,4$ of a form similar to $d_0(\tau)$. By Algorithm 5.11 in Ye (2017) we can prove that $d_0 = \cdots = d_4 = 0$. Thus by Algorithm 3.1 we have $g = 0$. $\square$

Note. This identity contains only one quasi-elliptic component, and in general the identities we found in the literature are stated in their simplest form. Consequently, to produce an identity with more than one quasi-elliptic component, we need to take one identity containing one quasi-elliptic component (multiplied by an element of R_1) and add to it another identity containing one quasi-elliptic component (multiplied by an element of R_1).

4. Theta functions and Weierstrass $\wp$ function

We are going to derive some connections between theta functions and the $\wp$ function. By applying them, we will obtain a faster algorithm on the restricted class R_2.

Definition 4.1. [elliptic theta-quotients]

$$J := \{\theta^{\alpha}(z) : \alpha \in \mathbb{Z}^4 \text{ such that } \theta^{\alpha}(z) \text{ is elliptic}\}.$$

Lemma 4.1. J forms a multiplicative group which is generated by

$$j_1 := \frac{\theta_2(z)^2}{\theta_1(z)^2},\ j_2 := \frac{\theta_3(z)^2}{\theta_1(z)^2} \text{ and } j_3 := \frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3}.$$

In particular, for a given $p(z) = \theta_1(z)^{\alpha_1}\theta_2(z)^{\alpha_2}\theta_3(z)^{\alpha_3}\theta_4(z)^{\alpha_4} \in J$, the presentation in terms of the generators is

$$p = j_1^{\frac{\alpha_2-\alpha_4}{2}} j_2^{\frac{\alpha_3-\alpha_4}{2}} j_3^{\alpha_4}.$$

Proof. By the help of Table 2.1, one can verify that $j_1, j_2, j_3 \in J$ and that J is a multiplicative group. Suppose $p(z) = \theta_1(z)^{\alpha_1}\theta_2(z)^{\alpha_2}\theta_3(z)^{\alpha_3}\theta_4(z)^{\alpha_4}$, then $p(z) = p(z+\pi\tau)$ and $p(z) = p(z+\pi)$, because every element in J is elliptic. On the other hand, by Table 2.1 we have

$$p(z+\pi\tau) = (-1)^{\alpha_1+\alpha_4}N^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}p(z) \text{ and } p(z+\pi) = (-1)^{\alpha_1+\alpha_2}p(z).$$

Hence $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 0$, $\alpha_1+\alpha_4$ is even and $\alpha_1+\alpha_2$ is even. This implies that if α_2 is even then α_3 and α_4 must be even, and if α_2 is odd then α_3 and α_4 are also odd. Therefore $\frac{\alpha_2-\alpha_4}{2}$, $\frac{\alpha_3-\alpha_4}{2}$ and α_4 are all integers. Moreover,

$$\begin{aligned} j_1^{\frac{\alpha_2-\alpha_4}{2}} j_2^{\frac{\alpha_3-\alpha_4}{2}} j_3^{\alpha_4} &= \theta_1^{-\alpha_2+\alpha_4-\alpha_3+\alpha_4-3\alpha_4}\theta_2^{\alpha_2-\alpha_4+\alpha_4}\theta_3^{\alpha_3-\alpha_4+\alpha_4}\theta_4^{\alpha_4} \\ &= \theta_1^{-\alpha_2-\alpha_3-\alpha_4}\theta_2^{\alpha_2}\theta_3^{\alpha_3}\theta_4^{\alpha_4} \\ &= \theta_1^{\alpha_1}\theta_2^{\alpha_2}\theta_3^{\alpha_3}\theta_4^{\alpha_4} \\ &= p. \end{aligned}$$

□

Proposition 4.1. (Freitag and Busam, 2005, p. 266, Prop. V.2.11) The Weierstrass $\wp$ function has a Laurent expansion

$$\wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)E_{2k+2}z^{2k},$$

where $E_{2k+2} := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m\omega_1+n\omega_2)^{-2k-2}$ is an Eisenstein series.

Theorem 4.1. The generators j_1, j_2 and j_3 of J satisfy

$$\left.\begin{aligned} j_1 &= \tfrac{\theta_2(0)^2}{\theta_1(0)^2}(\wp(z)-e_1); \\ j_2 &= \tfrac{\theta_3(0)^2}{\theta_1(0)^2}(\wp(z)-e_3); \end{aligned}\right\}\ 3$$

$$j_3 = -\frac{1}{2\theta_1'(0)^2}\wp'(z),$$

where $\wp(z) := \wp(z,\pi,\pi\tau)$ is the Weierstrass elliptic function with periods π and $\pi\tau$, $e_1 := \frac{1}{3}(\theta_3(0)^4+\theta_4(0)^4)$ and $e_3 := \frac{1}{3}(\theta_2(0)^4-\theta_4(0)^4)$.

Proof. Since $\frac{\theta_2(z)^2}{\theta_1(z)^2}$ is elliptic with a double pole at $z=0$ and is an even function, we can expand it as

$$\begin{aligned} \frac{\theta_2(z)^2}{\theta_1(z)^2} &= \frac{1}{z^2}\left(\frac{\theta_2(0)^2}{\theta_1'(0)^2} + \left(-\frac{\theta_2(0)^2\theta_1^{(3)}(0)}{3\theta_1'(0)^3} + \frac{\theta_2(0)\theta_2'{}'(0)}{\theta_1'(0)^2}\right)z^2 + \cdots\right) \\ &= \frac{\theta_2(0)^2}{\theta_1'(0)^2}\frac{1}{z^2} + \left(-\frac{\theta_2(0)^2\theta_1^{(3)}(0)}{3\theta_1'(0)^3} + \frac{\theta_2(0)\theta_2'{}'(0)}{\theta_1'(0)^2}\right) + \cdots. \end{aligned}$$

[3] See p. 102 of Köcher and Krieg (1985).

Following Proposition 4.1,

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} c_n z^{2n},$$

where $c_n = (2n+1)E_{2n+2}$ and $\omega = m\pi + n\pi\tau$ $(m,n \in \mathbb{Z})$. Thus $\wp(z) - \frac{\theta_1'(0)^2}{\theta_2(0)^2}\frac{\theta_2(z)^2}{\theta_1(z)^2}$ has no pole, which implies that it has to be constant, i.e.,

$$\wp(z) - \frac{\theta_1'(0)^2}{\theta_2(0)^2}\frac{\theta_2(z)^2}{\theta_1(z)^2} = \frac{\theta_1^{(3)}(0)}{3\theta_1'(0)} - \frac{\theta_2''(0)}{\theta_2(0)} = \frac{1}{3}(\theta_3(0)^4 + \theta_4(0)^4) = e_1,$$

where the second last equality is proven using Algorithm 5.11 in Ye (2017). Thus

$$\frac{\theta_2(z)^2}{\theta_1(z)^2} = \frac{\theta_2(0)^2}{\theta_1'(0)^2}(\wp(z) - e_1). \tag{4}$$

Analogously, we have

$$\wp(z) - \frac{\theta_1'^2}{\theta_3^2}\frac{\theta_3(z)^2}{\theta_1(z)^2} = \frac{\theta_1^{(3)}}{3\theta_1'} - \frac{\theta_3''}{\theta_3} = \frac{1}{3}(\theta_2^4 - \theta_4^4) = e_2,$$

where the second last equality is proven using Algorithm 5.11 of Ye (2017), and thus

$$\frac{\theta_3(z)^2}{\theta_1(z)^2} = \frac{\theta_3^2}{\theta_1'^2}(\wp(z) - e_3). \tag{5}$$

One can verify that $j_3 = \frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3} \in J$ is an odd elliptic function, and we have the series expansion

$$\frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3} = a_{-3}z^{-3} + a_{-1}z^{-1} + a_1 z + \cdots,$$

where

$$a_{-3} := \frac{\theta_2\theta_3\theta_4}{\theta_1'^3},$$

$$a_{-1} = \frac{1}{2\theta_1'^5}(\theta_3\theta_4\theta_1'^2\theta_2'' + \theta_2\theta_4\theta_1'^2\theta_3'' + \theta_2\theta_3\theta_1'^2\theta_4'' + \theta_2\theta_3\theta_1'\theta_1^{(3)}),$$

and a_1 is also in $\mathbb{K}(\Theta)$ but irrelevant to this proof. We have checked with Algorithm 5.11 of Ye (2017) that a_{-1} is zero. From Proposition 4.1 we derive

$$\wp'(z;\omega_1,\omega_2) \equiv -\frac{2}{z^3} + \sum_{k=1}^{\infty} 2k(2k+1)E_{2k+2}z^{2k-1}. \tag{6}$$

Therefore

$$\frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3} + \frac{1}{2}\frac{\theta_2\theta_3\theta_4}{\theta_1'^3}\wp'(z) \tag{7}$$

has no poles, which has to be constant. We take $z = 0$ and it turns out that the expression (7) is equal to zero. Thus

$$\frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3} = -\frac{1}{2}\frac{\theta_2\theta_3\theta_4}{\theta_1'^3}\wp'(z) = -\frac{1}{2\theta_1'^2}\wp'(z),$$

where the last equality follows from the famous identity

$$\theta_1' \equiv \theta_2\theta_3\theta_4,$$

which can be also proven by Algorithm 5.11 of Ye (2017). $\square$

Remark. Replacing z by $\frac{\pi}{2}$ in (4) and using $\theta_2\left(\frac{\pi}{2}\right)=0$, we obtain $\wp\left(\frac{\pi}{2}\right)=e_1$; substituting z by $\frac{\pi+\pi\tau}{2}$ in (5) and using $\theta_3\left(\frac{\pi+\pi\tau}{2}\right)=0$ gives $\wp\left(\frac{\pi+\pi\tau}{2}\right)=e_3$. It can be verified that $\frac{\theta_3(z)^2}{\theta_1(z)^2}$ is also elliptic, and similarly we have

$$\frac{\theta_4(z)^2}{\theta_1(z)^2}=\frac{\theta_4^2}{\theta_1'^2}(\wp(z)-e_2)$$

where $e_2:=-\frac{1}{3}(\theta_2^4+\theta_3^4)$. Moreover, by $\theta_4(\frac{\pi\tau}{2})=0$ we obtain $\wp\left(\frac{\pi\tau}{2}\right)=e_2$.

5. The finite-orbit-weight

In this section we will show the particularity of R_2, in terms of the finite-orbit weight, which will be used in the next section as a crucial property.

Definition 5.1. Let $M(\mathbb{H}):=\{g: g \text{ meromorphic on } \mathbb{H}\}$. Define a group action

$$\begin{aligned}\mathrm{SL}_2(\mathbb{Z})\times M(\mathbb{H}) &\longrightarrow M(\mathbb{H})\\ (\rho,g)&\mapsto g|_k\rho\end{aligned}$$

where $g|_k\rho(\tau):=(c\tau+d)^{-k}g\left(\frac{a\tau+b}{c\tau+d}\right)$ for $\rho:=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$ and $\tau\in\mathbb{H}$. For each $k\in\mathbb{Z}$ we define the k-orbit of g by $G_k(g):=\{g|_k\rho:\rho\in\mathrm{SL}_2(\mathbb{Z})\}$.

Proposition 5.1. For a nonzero $g\in M(\mathbb{H})$ and $k\in\mathbb{Z}$, if $|G_k(g)|$ is finite then k is unique with this property.

Proof. Let k and t be integers such that $G_k(g)$ and $G_t(g)$ are both finite orbit sets. We need to prove that $k=t$. Let $s:=k-t$. Take any $g|_t\rho\in G_t(g)$ with $\rho=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Then

$$\begin{aligned}g|_t\rho(\tau)&=(c\tau+d)^{-t}g\left(\frac{a\tau+b}{c\tau+d}\right)\\ &=(c\tau+d)^{s}(c\tau+d)^{-k}g\left(\frac{a\tau+b}{c\tau+d}\right)\\ &=(c\tau+d)^{s}\cdot g|_k\rho(\tau).\end{aligned}$$

Hence we can rewrite the set $G_t(g)$ as

$$\begin{aligned}G_t(g)&=\left\{(c\tau+d)^s\cdot g|_k\rho:\rho=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})\right\}\\ &=\left\{(c\tau+d)^s\cdot g_{a,b,c,d}:\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})\text{ and } g_{a,b,c,d}\in G_k(g)\right\},\end{aligned}$$

where $g_{a,b,c,d} := g|_k\rho$ with $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Assume $s \neq 0$ and $G_k(g) = \{a_1, \ldots, a_n\}$, and define the map

$$\gamma \colon \mathrm{SL}_2(\mathbb{Z}) \to G_k(g)$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g_{a,b,c,d}.$$

Let $A_j := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : g_{a,b,c,d} = a_j \right\}$. By Definition 5.1, the map γ is surjective, thus $A_j \neq \emptyset$. Then we can write $\mathrm{SL}_2(\mathbb{Z}) = \bigcup_{j=1}^{n} A_j$ where $A_i \cap A_j = \emptyset$ if $i \neq j$. Let

$$B_j := \left\{ (c,d) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A_j \right\}.$$

For every pair $(c,d) \in \mathbb{Z}^2$ with $\gcd(c,d) = 1$, there must exist some pairs $(a,b) \in \mathbb{Z}^2$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Hence there exists $r \in \{1, \ldots, n\}$ such that B_r is infinite; otherwise $\mathrm{SL}_2(\mathbb{Z}) \neq \bigcup_{j=1}^{n} A_j$. We also have

$$\left\{ (c\tau + d)^s a_r : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A_r \right\} \subseteq \left\{ (c\tau + d)^s g_{a,b,c,d} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\} = G_t(g),$$

which implies

$$N := \left| \left\{ (c\tau + d)^s a_r : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A_r \right\} \right| \leq |G_t(g)|. \tag{8}$$

On the other hand

$$N = \left| \{ (c\tau + d)^s : (c,d) \in B_r \} \right|, \tag{9}$$

and the right hand side of (9) is equal to infinity because $c_1\tau + d_1 \neq c_2\tau + d_2$ when $(c_1,d_1) \neq (c_2,d_2)$, and because the set B_r is infinite. Thus N is equal to infinity, and by (8), $|G_t(g)| = \infty$, which contradicts the assumption that $G_t(g)$ is a finite orbit set. Therefore $s = 0$. $\square$

Definition 5.2. Given $g \in M(\mathbb{H})$ nonzero and $k \in \mathbb{Z}$ such that $|G_k(g)|$ is finite, we define the finite-orbit-weight of g by

$$W(g) := k.$$

By using Definitions 5.1 and 5.2 one can verify the following:

Proposition 5.2. Given $g_1, \ldots, g_n \in M(\mathbb{H})$ with $W(g_j) = k_j$. Then
(1) $W(g_1 \cdots g_n) = k_1 + \cdots + k_n$,
(2) If $k_1 = \cdots = k_n = k$ and $g_1 + \cdots + g_n \neq 0$, then $W(g_1 + \cdots + g_n) = k$.

Lemma 5.1. (Serre, 1973, p. 78, Thm. 2) The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note. According to this lemma, $\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \rangle$, hence

$$G_k(g) = \{g|_k \rho : \rho \in \langle S, T \rangle\}.$$

Thus in our working frame, to compute $G_k(g)$, we compute $\{g|_k \rho : \rho \in \langle S, T \rangle\}\}$.

Lemma 5.2. (Whittaker and Watson, 1927, p. 475) For the action of S on $\theta_j(z|\tau)$ ($j = 1, \ldots, 4$) we have

$$\theta_1\left(z\middle| -\frac{1}{\tau}\right) \equiv -i(-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_1(z\tau|\tau); \quad \theta_2\left(z\middle| -\frac{1}{\tau}\right) \equiv (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_4(z\tau|\tau);$$
$$\theta_3\left(z\middle| -\frac{1}{\tau}\right) \equiv (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_3(z\tau|\tau); \quad \theta_4\left(z\middle| -\frac{1}{\tau}\right) \equiv (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_2(z\tau|\tau).$$

Directly from Definition 1.1 one can deduce the following.

Lemma 5.3. For the action of T on $\theta_j(\tau)$ ($j = 1, \ldots, 4$) we have

$$\theta_1(z|\tau+1) \equiv e^{\frac{\pi i}{4}} \theta_1(z|\tau); \quad \theta_2(z|\tau+1) \equiv e^{\frac{\pi i}{4}} \theta_2(z|\tau);$$
$$\theta_3(z|\tau+1) \equiv \theta_4(z|\tau); \quad \theta_4(z|\tau+1) \equiv \theta_3(z|\tau).$$

Now we show the special property of functions in R_2.

Lemma 5.4. Let $f^{\psi}(z|\tau) = \sum_{\alpha \in M} \psi(\alpha)\theta^{\alpha}(z) \in R_2$ and $\beta \in \min(M)$. Suppose the series expansion of $\frac{f^{\psi}(z|\tau)}{\psi(\beta)\theta^{\beta}(z)}$ around $z = 0$ is of the form $\sum_{n=0}^{\infty} d_n(\tau) z^n$ with $d_n(\tau) \in \mathbb{K}(\Theta)$. Then $W(d_n) = n$ when $d_n \neq 0$.

Proof. By Theorem 3.3, $\frac{f^{\psi}(z|\tau)}{\psi(\beta)\theta^{\beta}(z)}$ has a taylor expansion around $z = 0$. As in the beginning of this section we assume that $f^{\psi}(z|\tau)$ is a quasi-elliptic component of itself, hence by Theorem 3.1, $\frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)}$ is elliptic for every $\alpha \in M$.

In view of $\frac{f^{\psi}(z|\tau)}{\psi(\beta)\theta^{\beta}(z)} = \sum_{\alpha \in M} \frac{\psi(\alpha)\theta^{\alpha}(z)}{\psi(\beta)\theta^{\beta}(z)}$, we are going to show that the assertion is true for every $\frac{\psi(\alpha)\theta^{\alpha}(z)}{\psi(\beta)\theta^{\beta}(z)}$, then we show the assertion is true for $\frac{f^{\psi}(z|\tau)}{\psi(\beta)\theta^{\beta}(z)}$. For any fixed $\alpha \in M$, by Lemma 4.1 and Theorem 4.1 there exist integers a, b, c, such that

$$\frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)} = \left(-\frac{1}{2}\right)^{c} \cdot \frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}} p(z) \tag{10}$$

where $p(z) := (\wp(z) - e_1)^a (\wp(z) - e_3)^b \wp'(z)^c$.

Applying Lemmas 5.2 and 5.3 one can verify that $W(\theta_2(0)^2) = 1$ and

$$G_1(\theta_2(0)^2) = \{\pm\theta_2(0)^2, \pm i\theta_2(0)^2, \pm\theta_3(0)^2, \pm i\theta_3(0)^2, \pm\theta_4(0)^2, \pm i\theta_4(0)^2\}.$$

Similarly we have $W(\theta_3(0)^2) = 1$ and $W(\theta_1'(0)^2) = 3$. Then by Proposition 5.2 (1) we obtain

$$\begin{aligned} W\left(\frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}\right) &= W(\theta_2(0)^{2a}\theta_3(0)^{2b}) - W(\theta_1'(0)^{2a+2b+2c}) \\ &= W(\theta_2(0)^{2a}) + W(\theta_3(0)^{2b}) - W(\theta_1'(0)^{2a+2b+2c}) \\ &= a + b - 3a - 3b - 3c \\ &= -2a - 2b - 3c. \end{aligned}$$

Next we compute $W([z^n]p(z))$, where by $[z^n]p(z)$ we mean the coefficient of z^n in the series expansion of $p(z)$ around $z = 0$. Let us first consider

$$p_1(z) := z^{2a+2b+3c} p(z) = z^{2a}(\wp(z) - e_1)^a z^{2b}(\wp(z) - e_3)^b z^{3c} \wp'(z)^c. \tag{11}$$

Let $g_1(z) := z^2(\wp - e_1)$. By Proposition 4.1 we have

$$g_1(z) = 1 - e_1 z^2 + \sum_{m=1}^{\infty} (2m+1) E_{2m+2} z^{2m+2}$$

where $E_{2m+2} := \sum_{\omega \in L, \omega \neq 0} \omega^{-(2m+2)}$ is an Eisenstein series and L is the lattice generated by π and $\pi\tau$. One can easily verify by using Definition 5.1 that $W(1) = 0$. Again using Lemma 5.2 and Lemma 5.3 one can verify that $W(e_1) = 2$. In addition, according to (Serre, 1973, p. 83) for $m \geq 1$,

$$W(E_{2m+2}) = W\left(\sum_{\omega \in L, \omega \neq 0} \omega^{-(2m+2)}\right) = 2m + 2.$$

Therefore, for any $n \geq 0$, if $[z^n]g_1(z) \neq 0$ then

$$W([z^n]g_1(z)) = n. \tag{12}$$

Next we do a case distinction on the power of $g_1(z)$ in (11).

Case 1: $a \geq 0$. Then

$$W\left([z^n]g_1(z)^a\right) = W\left(\sum_{n_1+n_2+\cdots+n_a=n} [z^{n_1}]g_1(z) \cdots [z^{n_a}]g_1(z)\right).$$

By (12) and by Proposition 5.2.1, for any combination $n_1, \ldots, n_a$ such that $n_1 + \cdots + n_a = n$ we have

$$\begin{aligned} W([z^{n_1}]g_1(z) \cdots [z^{n_a}]g_1(z)) &= W([z^{n_1}]g_1(z)) + \cdots + W([z^{n_a}]g_1(z)) \\ &= n_1 + \cdots + n_a \\ &= n. \end{aligned}$$

Hence if $a \geq 0$, we find that

$$W\left([z^n]g_1(z)^a\right) = n \quad \text{when } [z^n]g_1(z)^a \neq 0.$$

Case 2: $a < 0$. Then

$$W\left([z^n]g_1(z)^a\right) = W\left([z^n]\left(\frac{1}{g_1(z)}\right)^{-a}\right)$$
$$= W\left(\sum_{n_1+n_2+\cdots+n_{-a}=n}[z^{n_1}]\left(\frac{1}{g_1(z)}\right)\cdots[z^{n_{-a}}]\left(\frac{1}{g_1(z)}\right)\right).$$

Assuming $g_1(z) = \sum_{j=0}^{\infty} v_j z^j$ we have $\frac{1}{g_1(z)} = \sum_{j=0}^{\infty} u_j z^j$, noting that $v_0 = u_0 = 1$. We have proven that for all $n \geq 0$, $W(v_n) = n$ when $v_n \neq 0$. Now we prove that $W(u_n) = n$ when $u_n \neq 0$ by induction on n. When $n = 0$ we have $W(u_0) = W(v_0) = 0$. Assume for $n \leq N$, $W(u_n) = n$. Let $n = N+1$. Using $\sum_{j=0}^{\infty} v_j z^j \cdot \sum_{j=0}^{\infty} u_j z^j = 1$ we obtain

$$u_{N+1} = -\frac{v_1 u_N + v_2 u_{N-1} + \cdots + v_N u_1 + v_{N+1} u_0}{v_0} = -v_1 u_N - v_2 u_{N-1} - \cdots - v_N u_1 - v_{N+1}.$$

By Proposition 5.2.2, if $u_{N+1} \neq 0$, then

$$W(u_{N+1}) = W(-v_1 u_N - v_2 u_{N-1} - \cdots - v_N u_1 - v_{N+1}) = N+1. \tag{13}$$

Hence $W(u_n) = n$ when $u_n \neq 0$. For any combination $n_1, \ldots, n_{-a}$ that $n_1 + \cdots + n_{-a} = n$ we have

$$W\left([z^{n_1}]\left(\frac{1}{g_1(z)}\right)\cdots[z^{n_a}]\left(\frac{1}{g_1(z)}\right)\right) = W\left([z^{n_1}]\left(\frac{1}{g_1(z)}\right)\right) + \cdots + W\left([z^{n_a}]\left(\frac{1}{g_1(z)}\right)\right)$$
$$= n_1 + \cdots + n_{-a}$$
$$= n.$$

Again by Proposition 5.2.2 and by (13), for any $a < 0$ we find that

$$W\left([z^n]g_1(z)^a\right) = n \quad \text{when } [z^n]g_1(z)^a \neq 0.$$

Analogously we deduce that for $b, c \in \mathbb{Z}$,

$$W\left([z^n]z^{2b}(\wp - e_3)^b\right) = n \quad \text{and} \quad W([z^n]z^{3c}\wp'(z)^c) = n$$

whenever the function to which W is applied is nonzero. Consequently we deduce that when $[z^n]p_1(z) \neq 0$,

$$W\left([z^n]p_1(z)\right) = W\left([z^n]z^{2a}(\wp(z) - e_1)^a z^{2b}(\wp(z) - e_3)^b z^{3c}\wp'(z)^c\right)$$
$$= W\left(\sum_{n_1+n_2+n_3=n}[z^{n_1}](\wp(z) - e_1)^a \cdot [z^{n_1}]z^{2b}(\wp(z) - e_3)^b \cdot [z^{n_3}]z^{3c}\wp'(z)^c\right)$$
$$= n_1 + n_2 + n_3$$
$$= n,$$

where the second last equality follows from Proposition 5.2.1. This implies when $[z^n]p(z) \neq 0$,

$$W\left([z^n]p(z)\right) = W\left([z^{n+2a+2b+3c}]p_1(z)\right) = n + 2a + 2b + 3c.$$

Therefore if $[z^n]\frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)} \neq 0$, identity (10) implies

$$\begin{aligned} W\left([z^n]\frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)}\right) &= W\left((-\frac{1}{2})^c \cdot \frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}\, p(z)\right) \\ &= W\left(\frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}\right) + W\left([z^{2n}]p(z)\right) \\ &= -2a-2b-3c+n+2a+2b+3c \\ &= n. \end{aligned}$$

Moreover, since both $\psi(\alpha)$ and $\psi(\beta)$, by definition of β, are homogeneous polynomials in $\mathbb{K}[\widetilde{\Theta}]_h$ with the same degree, one can check, by using Lemmas 5.2 and 5.3, that $W\left(\frac{\psi(\alpha)}{\psi(\beta)}\right) = 0$ for all $\alpha \in M$. Hence

$$W\left([z^n]\frac{\psi(\alpha)\theta^{\alpha}(z)}{\psi(\beta)\theta^{\beta}(z)}\right) = 0+n = n \text{ when } [z^n]\frac{\psi(\alpha)\theta^{\alpha}(z)}{\psi(\beta)\theta^{\beta}(z)} \neq 0$$

and

$$W(d_n) = \sum_{\alpha\in M}[z^n]\frac{\psi(\alpha)\theta^{\alpha}(z)}{\psi(\beta)\theta^{\beta}(z)} = n \text{ when } d_n \neq 0.$$

□

6. Zero-recognition for $f^{\Psi} \in R_2$

Let us recall Definition 4.1. By Lemma 4.1 and Theorem 4.1, for any $\frac{\theta^{\alpha}}{\theta^{\beta}} \in J$ with $\alpha = (\alpha_1,\dots,\alpha_4)$ and $\beta = (\beta_1,\dots,\beta_4)$, we can write

$$\frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)} = \left(-\frac{1}{2}\right)^c \cdot \frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}\, p(z), \tag{14}$$

where $p(z) := (\wp(z)-e_1)^a(\wp(z)-e_3)^b\,\wp'(z)^c$. The $p(z)$ has the following property.

Proposition 6.1. Let $p(z)$ be the same as above and let g_n denote the coefficient of z^n in the series expansion of $p(z)$ around $z=0$. Then when $g_n \neq 0$ we have

$$|G_{w_n}(g_n)| \leq 3 = |G_2(e_1)|,$$

where w_n is the finite-orbit-weight of g_n.

Proof. From the proof of Lemma 5.4 we observe that g_n is a polynomial in e_1, e_3 and E_{2s+2} with some $s \geq 1$, and

$$W(g_n) = n' := n+2a_j+2b_j+3c_j$$

when $g_n \neq 0$. Let $p_1,\dots,p_t$ be the components of g_n, where each such component is a (finite) power product $e_1^{k_1}e_2^{k_2}E_4^{\ell_1}E_6^{\ell_2}\cdots$ with a coefficient in $\mathbb{K}$. One has

$$|G_{n'}(g_n)| = |\{(p_1+\cdots+p_t)|_{n'}\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|. \tag{15}$$

Additionally, from the proof of Lemma 5.4, the p_i in (15) are of the form

$$e_1^{k_1}e_3^{k_2}\prod_{s\in M_i}E_{2s+2}^{\ell_s}$$

where $k_1,k_2,\ell_s\in\mathbb{N}$, $M_i\subseteq\mathbb{N}$ and $2k_1+2k_2+\sum_{s\in M_i}(2s+2)\ell_s=n'$. It can be verified by using Lemma 5.2 and Lemma 5.3 that $G_2(e_1)=G_2(e_3)=\{e_1,e_2,e_3\}$, thus $W(e_1)=W(e_3)=3$.

By (Serre, 1973, p. 83) when $m\geq 1$, E_{2s+2} is a modular form of weight $2s+2$, which means

$$E_{2s+2}|_{2s+2}\rho=E_{2s+2} \text{ for all } \rho\in \mathrm{SL}_2(\mathbb{Z}).$$

Consequently,

$$E_{2s+2}^{\ell_s}|_{(2s+2)\ell_s}\rho=E_{2s+2}^{\ell_s} \quad \text{for all } \rho\in \mathrm{SL}_2(\mathbb{Z}).$$

By Proposition 5.2 (1),

$$W\left(\prod_{s\in M_i}E_{2s+2}^{\ell_s}\right)=\sum_{s\in M_i}(2s+2)\ell_s.$$

By Proposition 5.2 (2) we obtain $W(p_i)=2k_1+2k_2+\sum_{s\in M_i}(2s+2)\ell_s=n'$ for all $i\in\{1,\dots,t\}$. Hence we continue (15) by

$$\begin{aligned}|G_{n'}(g_n)|&=\{(p_1+\cdots+p_t)|_{n'}\rho:\rho\in \mathrm{SL}_2(\mathbb{Z})\}|\\&\leq|\{\{p_1|_{n'}\rho,\dots,p_t|_{n'}\rho\}:\rho\in \mathrm{SL}_2(\mathbb{Z})\}|\\&=|\{\{(e_1^{k_{1,1}}e_3^{k_{1,2}}\gamma_1)|_{n'}\rho,\dots,(e_1^{k_{t,1}}e_3^{k_{t,2}}\gamma_t)|_{n'}\rho\}:\rho\in \mathrm{SL}_2(\mathbb{Z})\}|\\&=|\{\{(e_1^{k_{1,1}}e_3^{k_{1,2}})|_{2(k_{1,1}+k_{1,2})}\rho,\dots,(e_1^{k_{t,1}}e_3^{k_{t,2}})|_{2(k_{1,1}+k_{1,2})}\rho\}:\rho\in \mathrm{SL}_2(\mathbb{Z})\}|,\end{aligned}\tag{16}$$

where the γ_i are the corresponding $\prod_{m\in M_i}E_{2s+2}^{\ell_s}$ of p_i. On the other hand, for $k\in\mathbb{N}$,

$$G_{2k}(e_1^k)=\{e_1^k|_{2k}\rho:\rho\in \mathrm{SL}_2(\mathbb{Z})\}=\{\underbrace{e_1|_2\rho\cdots e_1|_2\rho}_{k}:\rho\in \mathrm{SL}_2(\mathbb{Z})\}=\{e_1^k,e_2^k,e_3^k\}$$

and analogously $G_{2k}(e_2^k)=G_{2k}(e_3^k)=\{e_1^k,e_2^k,e_3^k\}$. Then

$$\{e_1^{k_1}e_3^{k_2}|_{2(k_1+k_2)}\rho:\rho\in \mathrm{SL}_2(\mathbb{Z})\}=\{e_1^{k_1}|_{2k_1}\rho\cdot e_2^{k_2}|_{2k_2}\rho:\rho\in \mathrm{SL}_2(\mathbb{Z})\}=\{e_2^{k_1}e_1^{k_2},e_3^{k_1}e_2^{k_2},e_1^{k_1}e_3^{k_2}\},$$

which means there are only three possibilities when applying an arbitrary $\rho\in \mathrm{SL}_2(\mathbb{Z})$ on every $e_1^{k_{i,1}}e_3^{k_{i,2}}$ of (16). Note that the powers $k_{i,j}$ are irrelevant, i.e., we can choose three representatives ρ_1, ρ_2 and ρ_3 such that for all $i\in\{1,\dots,t\}$,

$$(e_1^{k_{i,1}}e_3^{k_{i,2}})|_{2(k_{i,1}+k_{i,2})}\rho_1=e_2^{k_{i,1}}e_1^{k_{i,2}},$$

$$(e_1^{k_{i,1}}e_3^{k_{i,2}})|_{2(k_{i,1}+k_{i,2})}\rho_2=e_3^{k_{i,1}}e_2^{k_{i,2}}$$

and

$$(e_1^{k_{i,1}}e_3^{k_{i,2}})|_{2(k_{i,1}+k_{i,2})}\rho_3=e_1^{k_{i,1}}e_3^{k_{i,2}}.$$

Hence the right hand side of (16) is equal to

$$\left\{\{e_2^{k_{1,1}}e_1^{k_{1,2}},\dots,e_2^{k_{t,1}}e_1^{k_{t,2}}\},\{e_3^{k_{1,1}}e_2^{k_{1,2}},\dots,e_3^{k_{t,1}}e_2^{k_{t,2}}\},\{e_1^{k_{1,1}}e_3^{k_{1,2}},\dots,e_1^{k_{t,1}}e_3^{k_{t,2}}\}\right\}.$$

Thus $|G_{n'}(g_n)|\leq 3$ when $g_n\neq 0$. □

Lemma 6.1. Let $f^\Psi(z|\tau)=\sum_{\alpha\in M}\psi(\alpha)\theta^\alpha(z|\tau)\in R_2$ and $\beta=(\beta_1,\beta_2,\beta_3,\beta_4)\in\min(M)$. Suppose

$$\frac{f^\Psi(z|\tau)}{\psi(\beta)\theta^\beta(z)}=\sum_{n=0}^{\infty}d_n(\tau)z^n \text{ with } d_n(\tau)\in\mathbb{K}(\Theta).$$

Let $M = \{y^{(1)}, \dots, y^{(m)}\}$ with $y^{(j)} = (y_1^{(j)}, y_2^{(j)}, y_3^{(j)}, y_4^{(j)})$. For $1 \le j \le m$ let

$$\begin{aligned} a_j :=& \frac{y_2^{(j)} - y_4^{(j)} - \beta_2 + \beta_4}{2}, \\ b_j :=& \frac{y_3^{(j)} - y_4^{(j)} - \beta_3 + \beta_4}{2}, \\ c_j :=& y_4^{(j)} - \beta_4, \\ r_j :=& y_1^{(j)} - \beta_1, \end{aligned}$$

and

$$t_j := \frac{\psi(y^{(j)})\theta_2(0)^{2a_j}\theta_3(0)^{2b_j}}{\psi(\beta)\theta_1'(0)^{2a_j+2b_j+2c_j}}.$$

For all $n \ge 0$, if $d_n \ne 0$ then

$$|G_n(d_n)| \le |\{\{t_1|_{r_1}\rho, \dots, t_m|_{r_m}\rho, e_1|_2\rho\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|,$$

Proof. First of all we write

$$\frac{f^{\psi}(z|\tau)}{\psi(\beta)\theta^{\beta}(z)} = h_1 + \dots + h_m$$

with $h_j := \frac{\psi(y^{(j)})\theta^{y^{(j)}}(z)}{\psi(\beta)\theta^{\beta}(z)}$. From the proof of Lemma 5.4 we see that for all $j \in \{1, \dots, m\}$,

$$W\left([z^n]h_j(z)\right) = n$$

and

$$W\left(\frac{\psi(y^{(j)})\theta_2(0)^{2a_j}\theta_3(0)^{2b_j}}{\psi(\beta)\theta_1'(0)^{2a_j+2b_j+2c_j}}\right) = -2a_j - 2b_j - 3c_j.$$

Then by Proposition 6.1 and expression (14) we deduce

$$\left|G_n\left([z^n]h_j(z)\right)\right| \le \left|\left\{\{t_j|_{r_j}\rho, e_1|_2\rho\right\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\right\}\right|,$$

where $r_j := -2a_j - 2b_j - 3c_j = y_1^{(j)} - \beta_1$ following from the definition of a_j, b_j, c_j and $t_j := \frac{\psi(y^{(j)})\theta_2(0)^{2a_j}\theta_3(0)^{2b_j}}{\psi(\beta)\theta_1'(0)^{2a_j+2b_j+2c_j}}$. Consequently, when $d_n \ne 0$ we have $W(d_n) = n$ by Lemma 5.4 and

$$\begin{aligned} |G_n(d_n)| &= \left|\left\{[z^n](h_1(z) + \dots + h_m(z))\Big|_n \rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\right\}\right| \\ &\le \left|\left\{\left\{[z^n]h_1(z)\Big|_n \rho, \dots, [z^n]h_m(z)\Big|_n \rho\right\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\right\}\right| \\ &\le |\{\{t_1|_{r_1}\rho, \dots, t_m|_{r_m}\rho, e_1|_2\rho\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|. \end{aligned}$$

□

Definition 6.1. (Freitag and Busam, 2005, p. 326) Let $q = e^{\pi i \tau}$ with $\tau \in \mathbb{H}$. Given $k \in \mathbb{N}$, a modular form of weight k is an analytic function g on $\mathbb{H}$ such that

$$g\left(\frac{a\tau + b}{c\tau + d}\right) \equiv (c\tau + d)^k g(\tau) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and $g(\tau)$ can be written as a Taylor series in powers of q with complex coefficients; i.e.,

$$g(\tau) \equiv \sum_{t=0}^{\infty} a_t e^{\pi i \tau j} \equiv \sum_{j=0}^{\infty} a_j q^j.$$

By valence formula (Freitag and Busam, 2005, Th. VI.2.3), one can deduce the following

Lemma 6.2. Let $q := e^{\pi i \tau}$ and g be a modular form of weight k with a q-expansion $g(q) = \sum_{j=0}^{\infty} v_j q^j$.

$$\text{If } v_j = 0 \text{ for } j \leq \lfloor \frac{k}{6} \rfloor, \text{ then } g = 0.$$

Theorem 6.1. Let $q := e^{\pi i \tau}$, $t_1, \dots, t_m$, $r_1, \dots, r_m$ and d_n be the same as in Lemma 6.1, and let

$$\ell := |\{\{t_1|_{r_1}\rho, \dots, t_m|_{r_m}\rho, e_1|_2\rho\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|.$$

For $n \geq 0$ suppose d_n has a q-expansion

$$\sum_{j=0}^{\infty} v_{n,j} q^j.$$

Then

$$d_n = 0 \text{ if and only if } v_{n,j} = 0 \text{ for } j \leq \lfloor \frac{n\ell}{6} \rfloor.$$

Proof. If $d_n(\tau) \equiv \sum_{j=0}^{\infty} v_{n,j} q^j \equiv 0$, it immediately implies that all v_j are zero.

Assume $v_{n,j} = 0$ for $j \leq \lfloor \frac{n\ell}{6} \rfloor$. If $d_n \neq 0$, by Lemma 5.4 we have $W(d_n) = n$ and by Lemma 6.1, $|G_n(d_n)| \leq \ell$. Suppose $G_n(d_n) = \{s_1, \dots, s_{\ell_n}\}$ and $\ell_n \leq \ell$. Then for every $i \in \{1, \dots, \ell_n\}$, there exists a unique $j \in \{1, \dots, \ell_n\}$ such that $s_i|_n S = s_j$; and there exists a unique $k \in \{1, \dots, \ell_n\}$ such that $s_i|_n T = s_k$. Then

$$\left(\prod_{i=1}^{\ell_n} s_i\right)\Big|_{n\ell_n} S = \prod_{j=1}^{\ell_n} s_j \quad \text{and} \quad \left(\prod_{i=1}^{\ell_n} s_i\right)\Big|_{n\ell_n} T = \prod_{j=1}^{\ell_n} s_j.$$

This yields

$$\left(\prod_{i=1}^{\ell_n} s_i\right)\Big|_{n\ell_n} \rho = \prod_{j=1}^{\ell_n} s_j \quad \text{for all } \rho \in \mathrm{SL}_2(\mathbb{Z}).$$

Moreover, we have proven in Lemma 4.10 of Ye (2017) that $\prod_{j=1}^{\ell_n} s_j$ is a Taylor series in q. Thus $\prod_{j=1}^{\ell_n} s_j$ is a modular form of weight $n\ell_n$.

Since $\ell_n \leq \ell$ we have $v_{n,j} = 0$ for $j \leq \lfloor \frac{n\ell_n}{6} \rfloor$. By Lemma 6.2, $\prod_{j=1}^{\ell_n} s_j = 0$. Because of the fact that for any meromorphic functions h and g on $\mathbb{H}$, if $(h|_n\rho)(\tau) = g(\tau)$ then $h(\tau) \equiv 0$ if and only if $g(\tau) \equiv 0$, we deduce that s_j must be zero for all $j \in \{1, \dots, \ell_n\}$, otherwise $s_j \neq 0$ for all $j \in \{1,\ldots,\ell_n\}$ which contradicts $\prod_{j=1}^{\ell_n} s_i = 0$. As $d_n \in G_n(d_n) = \{s_1,\ldots,s_{\ell_n}\}$, we deduce that $d_n = 0$, which contradicts the earlier assumption $d_n \neq 0$. Therefore $d_n = 0$. □

Algorithm 6.1. Let $q = e^{\pi i \tau}$ and $f^{\Psi}(z|\tau) = \sum_{\alpha \in M} \psi(\alpha)\theta^{\alpha}(z|\tau) \in R_2$. We have the following algorithm to prove or disprove $f^{\Psi}(z|\tau) \equiv 0$.

Input: $f^{\Psi} \in R_2$.
Output: True if $f^{\Psi} = 0$; False if $f^{\Psi} \neq 0$.

Write $f^{\Psi}(z|\tau) = \sum_{j=1}^{n} f^{\Psi_j}(z|\tau)$ where the $f^{\Psi_j}(z|\tau) := \sum_{\alpha \in M_j} \psi(\alpha)\theta^{\alpha}(z|\tau)$ are the quasi-elliptic components of $f^{\Psi}(z|\tau)$.

Set $i := 1$. While $i \leq n$ do
Let $m := |M_i|$ and $\{y^{(1)},\ldots,y^{(m)}\} := M_i$;
Choose $\beta \in \min(M_i)$;
For $j \in \{1,\ldots,m\}$,
compute $a_j := \frac{y_2^{(j)} - y_4^{(j)} - \beta_2 + \beta_4}{2}$, $b_j := \frac{y_3^{(j)} - y_4^{(j)} - \beta_3 + \beta_4}{2}$, $c_j := y_4^{(j)} - \beta_4$;
compute $r_j := y_1^{(j)} - \beta_1$;
let $t_j := \frac{\psi(y^{(j)})\theta_2(0)^{2a_j}\theta_3(0)^{2b_j}}{\psi(\beta)\theta_1'(0)^{2a_j+2b_j+2c_j}}$;
Compute $\ell := |\{\{t_1|_{r_1}\rho,\ldots,t_m|_{r_m}\rho, e_1|_2\rho\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|$;
Let $g(z) := \frac{f^{\Psi_i}(z)}{\theta^{\beta}(z)}$;
Compute $g(z) = \sum_{k=0}^{\infty} d_k(\tau) z^k$;
Set $k := 0$. While $k \leq \beta_2 + \beta_3 + \beta_4$ do
if $d_k(\tau) \equiv O(q^{\frac{k\ell}{6}+1})$;
$k++$;
otherwise return False;
end do;
$j++$;
end do;
return True;

Theorem 6.2. Algorithm 6.1 is correct.

Proof. By Lemma 6.1, $d_k(\tau) \equiv 0$ if and only if $d_k(\tau) \equiv O(q^{\frac{k\ell}{6}+1})$. Since the only difference between Algorithm 3.1 and Algorithm 6.1 is the way in which we check $d_k(\tau) \equiv 0$, it follows that Algorithm 6.1 is correct. □

Example 1.9.(DLMF, 2015, 20.7.1) Prove

$$\theta_2(0)^2\theta_2(z)^2 - \theta_3(0)^2\theta_3(z)^2 + \theta_4(0)^2\theta_4(z)^2 \equiv 0.$$

Proof. Let $\beta := (0,0,0,2)$ and

$$g(z) := \frac{\theta_2(0)^2\theta_2(z)^2}{\theta_4(0)^2\theta_4(z)^2} - \frac{\theta_3(0)^2\theta_3(z)^2}{\theta_4(0)^2\theta_4(z)^2} + 1.$$

Since $g(z)$ is an even function we obtain

$$g(z) = \sum_{k=0}^{\infty} d_{2k}(\tau) z^{2k}$$

with

$$d_0(\tau) = \frac{\theta_2(0)^4 - \theta_3(0)^4 + \theta_4(0)^4}{\theta_4(0)^4}$$

and

$$d_2(\tau) = \frac{\theta_2(0)^3\theta_4(0)\theta_2''(0) - \theta_3(0)^3\theta_4(0)\theta_3''(0) - \theta_2(0)^4\theta_4''(0) + \theta_3(0)^4\theta_4''(0)}{\theta_4(0)^5};$$

and $d_{2k}(\tau)$ ($k > 1$) are irrelevant to this proof. According to Algorithm 6.1 we need to show that $d_0(\tau) = O(q)$ and $d_2(\tau) = O(q^{\frac{\ell}{3}+1})$ where

$$\ell = \left| \left\{ 1, \frac{\theta_2(0)^4\theta_3(0)^2}{\theta_1'(0)^2}\Big|_0 \rho, \frac{\theta_2(0)^2\theta_3(0)^4}{\theta_1'(0)^2}\Big|_0 \rho, e_1|_2\rho : \rho \in \mathrm{SL}_2(\mathbb{Z}) \right\} \right|.$$

By implementing Algorithm 6.1 in Mathematica, we obtain that $\ell = 6$, and $d_0(\tau) = O(q)$ and $d_2(\tau) = O(q^3)$. □

Remark. Example 1.9 can also be solved by Algorithm 3.1.

The following proposition shows that there is a further decomposition step that can be done before doing zero-recognition.

Proposition 6.2. Given $f^{\Psi}(z) = \sum_{\alpha \in M} \psi(\alpha)\theta^{\alpha}(z) \in R_1$, then $f(z) \equiv 0$ if and only if

$$\sum_{\alpha \in N_i} \psi(\alpha)\theta^{\alpha}(z) \equiv 0$$

for $i = 1,2$, where $N_1 := \{(\alpha_1, \dots, \alpha_4) \in M : \alpha_1 \text{ is odd}\}$ and $N_2 := \{(\alpha_1, \dots, \alpha_4) \in M : \alpha_1 \text{ is even}\}$.

Proof. If $\sum_{\alpha \in N_i} \psi(\alpha)\theta^{\alpha}(z) \equiv 0$ for $i = 1,2$ then $f(z) \equiv 0$ is immediate. If $f(z) \equiv 0$, let

$$f_i(z) \equiv \sum_{\alpha \in N_i} \psi(\alpha)\theta^{\alpha}(z).$$

By Definition 1.1, $\theta_1(z)$ is an odd function while the other three are even functions, hence

$$0 \equiv f^{\Psi}(z) \equiv f^{\Psi}(-z) \equiv -f_1(z) + f_2(z).$$

This together with $f^{\Psi}(z) \equiv f_1(z) + f_2(z)$ implies $f_1(z) \equiv 0$ and $f_2(z) \equiv 0$. □

Speed comparison. The only difference between Algorithms 3.1 and 6.1 is the way of dealing with in the series expansion $\sum_{k=0}^{\infty} d_k(\tau) z^k$; namely, to check if certain $d_k(\tau)$ are zero.

In Algorithm 6.1 we do this by computing the orbit

$$\{t_1|_{r_1}\rho, \ldots, t_m|_{r_m}\rho, e_1|_2\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\},$$

which is needed for the orbit length ℓ. The t_j do not contain any of $\theta_j^{(k)}$ $(k \geq 1)$, except for θ_1'. All of $\theta_2, \theta_3, \theta_4$ and θ_1' have very simple modular transformations. [4] In contrast, Algorithm 3.1 uses Algorithm 5.11 of Ye (2017) and it directly computes the leading term orbits of certain $d_k(\tau)$, which contains $\theta_j^{(k)}$ $(k \geq 1)$ with sophisticated modular transformations. [5] In addition, the coefficients $d_k(\tau)$ become more and more complicated when the degree of z grows. Thus Algorithm 3.1 needs more time on the orbit computation than Algorithm 6.1, especially when the identity we want to prove contains a large input. More details can be found in Section 6.3 of Ye (2016).

7. Conclusion

In the literature there are not many high degree identities found in R_1 and R_2. The one with the highest degree we were able to find in R_1 is

$$\theta_1(z)^4 + \theta_3(z)^4 \equiv \theta_2(z)^4 + \theta_4(z)^4$$

from (Whittaker and Watson, 1927, p. 462), whilst we have a way of producing all relations in R_1, which can be found in Chapter 6 of Ye (2016). Moreover, we are preparing a paper that determines the generators of the ideal containing all relations in R_2.

On the other hand, based on this article, algorithmically dealing with other types of identities becomes possible. For instance, we have algorithms to prove

$$\theta_2\,\theta_3\,\theta_4\,\theta_1(2z, q) - 2\theta_1(z)\theta_2(z)\theta_3(z)\theta_4(z) \equiv 0,$$

a identity from (Whittaker and Watson, 1927, p. 485) and

$$\sum_{j=1}^{4} \theta_j(x)\theta_j(y)\theta_j(u)\theta_j(v) - 2\theta_3(x_1)\theta_3(y_1)\theta_3(u_1)\theta_3(v_1) \equiv 0,$$

a identity from (Mumford, 1973, p. 17), where $x_1 := \frac{1}{2}(x+y+u+v)$ and $y_1 := \frac{1}{2}(x+y-u-v)$, $u_1 := \frac{1}{2}(x-y+u-v)$ and $v_1 := \frac{1}{2}(x-y-u+v)$.

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[4] See Lemmas 5.2 and 5.3.

[5] See Corollary 3.5 of Ye (2017).

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