Computation of All Rational Solutions of First-Order Algebraic ODEs

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In this paper, we consider the class of first-order algebraic ordinary differential equations (AODEs), and study their rational solutions in three different approaches. A combinatorial approach gives a degree bound for rational solutions of a class of AODEs which do not have movable poles. Algebraic considerations yield an algorithm for computing rational solutions of quasi-linear AODEs. And finally ideas from algebraic geometry combine these results to an algroithm for finding all rational solutions of a class of first-order AODEs which covers all examples from the collection of Kamke. In particular, parametrizations of algebraic curves play an important role for a transformation of a parametrizable first-order AODE to a quasi-linear differential equation.

1 Introduction

Algebraic differential equations have been studied a lot and there is a bunch of solution methods for special classes of such ODEs. In this paper we are particularly interested in rational solutions. A degree bound of Eremenko [2] for such solutions can be used for an ansatz. Implicit solutions can be found by computing Gröbner bases (see Hubert [12]). Using algebraic geometry Feng and Gao [3, 4, 1] provide polynomial time algorithms for solving first-order autonomous AODEs. The main idea in these algorithms is to regard the AODE as an algebraic equation defining an algebraic curve. Then rational parametrization of the curve can be used to solve the ODE.

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These ideas were generalized to non-autonomous AODEs of first order [20, 22, 21, 26] and higher order [11]. A similar approach can also be used for solving one-dimensional systems of ODEs [16]. Parametrizations of general algebraic varieties yield a further generalization to partial differential equations can be found in [6, 7]. The method, as well as its specialization to ODEs [10, 5], is not restricted to finding rational solutions. However, theory is mainly developed for this case.

So far still no general solution algorithm exists. Though, there exist solution algorithms for special classes of AODEs, e. g. autonomous (c. f. [3]) and linear ones. In this paper we give an algorithm for computing all rational solutions of a special class of first-order AODEs. We consider the class of first-order AODEs, $F(x, y, y') = 0$, for which rational parametrizations of the corresponding algebraic curve exist over $\overline{\mathbb{K}(x)}$ such that x also appears rationally in the parametrization. These parametrizations allow to transform an arbitrary first-order AODE into another ODE. We show that there is a correspondence between the original AODE and this so called associated ODE. Depending on the form associated ODE we propose two major approaches. A combinatorial idea gives a degree bound for rational solutions of a class of AODEs which do not have movable poles. Algebraic considerations yield an algorithm for computing rational solutions of quasi-linear AODEs. Together with the geometric computations an algorithm for finding all rational solutions of strongly parametrizable AODEs is given.

In Section 2 we present the necessary notations and definitions. Section 3 describes the combinatorial ideas for finding a degree bound for rational solutions of a class of AODEs which do not have movable poles. In Section 4 we show how algebraic considerations can be used to find rational solutions of quasi-linear AODEs. Finally in Section 5 we combine the results with algebraic geometric ideas and give an algorithm for finding all rational solutions of a strongly parametrizable AODE.

2 Preliminaries

By $\mathbb{K}$, we denote an algebraically closed field, e. g. in practise we might think of algebraic numbers $\overline{\mathbb{Q}}$ or the field of complex numbers $\mathbb{C}$. Furthermore, we assume that $\mathbb{K}$ is equipped with a trivial derivation. The trivial derivation on $\mathbb{K}$ is naturally extended to a derivation on the field $\mathbb{K}(x)$ of rational functions for which the derivation of x is equal to 1. We always denote the derivation by $\frac{d}{dx}$, or $'$ for short. In this section, we recall some basic notations on first-order AODEs and their rational solutions.

A first-order AODE is a differential equation of the form

$$F(x, y, y') = 0\,, \tag{1}$$

where $F \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$. Without loss of generality, we always assume that F is an irreducible polynomial. A rational solution of the differential equation (1) is a rational function $y(x) \in \mathbb{K}(x)$ such that $F(x, y(x), y'(x)) = 0$. A first-order AODE may have no rational solution at all, it might have only finitely many, or a one-parameter class of rational solutions.

To study global properties of rational solutions, we sometimes pass through algebraic function fields. Let K be an algebraic function field over $\mathbb{K}$ of transcendence degree one. A $\mathbb{K}-$*valuation ring* in K, or briefly, if the ground field is clear, a *valuation ring*, is a ring $\mathcal{O} \subsetneq K$ such that $\mathbb{K} \subset \mathcal{O}$ and for every $x \in K$, either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. The valuation ring $\mathcal{O}$ admits a unique maximal ideal, say P, which is the set of all non-invertible elements. Moreover, P is principle, and every ideal of $\mathcal{O}$ forms a power of P. A generator of P, say t, is called a *local uniformizer*. For every $x \in K^*$, there exists a unique pair $(u, n) \in (\mathcal{O}^*, \mathbb{Z})$ such that $x = u \cdot t^n$, where $\mathcal{O}^*$ is the group of units. Notice that the power n does not depend on how the local parameter t is chosen. We call n the valuation of x at P. We define the map $\nu_P : K^* \to \mathbb{Z}$ by $\nu_P(x) = n$.

As a convention, we can extend ν_P to K by setting $\nu_P(0) := \infty$. It is easy to check that ν_P satisfies the following properties for all $x, y \in K$:

(i) $\nu_P(xy) = \nu_P(x) + \nu_P(y)$

(ii) $\min\{\nu_P(x), \nu_P(y)\} \leq \nu_P(x + y) \leq \max\{\nu_P(x), \nu_P(y)\}$

The map ν_P is called a valuation of K (with respect to P).

By $\mathbb{P}_{K/\mathbb{K}}$, or briefly $\mathbb{P}_K$ if the ground field $\mathbb{K}$ is clear, we denote the set of all ideals which are maximal in some valuation ring of K over $\mathbb{K}$. Elements of $\mathbb{P}_K$ are called prime divisors of K. For each prime divisor P, we denote by ν_P the corresponding valuation. Let $x \in K$. If $\nu_P(x) > 0$, we say x has a zero of order $\nu_P(x)$ at P. In case $\nu_P(x) < 0$, we say x has a pole of order $-\nu_P(x)$ at P.

Let $P \in \mathbb{P}_K$ be a prime divisor, $\mathcal{O}_P$ and ν_P the corresponding valuation ring and valuation respectively. Since $\mathbb{K}$ is algebraically closed, the residue field $\mathcal{O}_P/P$ is isomorphic to $\mathbb{K}$. Therefore, there is a natural ring projection $\mathcal{O} \to \mathcal{O}/P \equiv \mathbb{K}$. We usually denote the image of $x \in \mathcal{O}$ in $\mathbb{K}$ by $x(P)$.

We want to collect sufficient information such that an element of K is uniquely determined (up to multiplication by a non-zero constant). The object which gathers all such information about poles and zeroes is called a *divisor*. Formally, a divisor of K is an element of the free abelian group $\bigoplus_{P \in \mathbb{P}_K} \mathbb{Z}P$. For divisors δ, δ', we say $\delta \geq \delta'$ if all coefficients of $\delta - \delta'$ are non-negative. The relation $\geq$ defines a partial order on the set of divisors of K. For a divisor $\delta = n_1P_1 + n_2P_2 + \ldots + n_rP_r$, we define $\deg \delta := n_1 + \ldots + n_r$ as the degree of δ. We also denote, by $\operatorname{supp}_{\mathbb{P}_K}(\delta)$ the set of prime divisors having non-zero coefficient in δ.

Among divisors, principle divisors are natural examples. For each $x \in K^*$, the principle divisor of x in K is denoted by $[x] := \sum_{P \in \mathbb{P}_K} \nu_P(x)P$. Notice that, the sum is always finite.

For some technical purposes, we sometimes split the negative and positive part of $[x]$, denoted by

$$[x]^- = -\sum_{\substack{P\in\mathbb{P}_K\\ \nu_P(x)<0}} \nu_P(x)P\,, \qquad\qquad [x]^+ = \sum_{\substack{P\in\mathbb{P}_K\\ \nu_P(x)>0}} \nu_P(x)P\,,$$

respectively. Therefore $[x] = [x]^+ - [x]^-$, and $\deg[x]^+ = \deg[x]^-$.

The simplest algebraic function field over $\mathbb{K}$ is the field $\mathbb{K}(x)$ of rational functions in x. In this case, there is a one-to-one correspondence between prime divisors of $\mathbb{K}(x)$ and the set $\mathbb{K}\cup\{\infty\}$. Therefore, we might identify $\mathbb{P}_{\mathbb{K}(x)}$ with $\mathbb{K}\cup\{\infty\}$. By abuse of notation we denote by ν_{x_0}, for $x_0 \in \mathbb{K}\cup\{\infty\}$ the valuation of the prime divisor corresponding to x_0, and call it the valuation at $x = x_0$. A local uniformizer at $x_0 \in \mathbb{K}$ is $x - x_0$, and at ∞ it is $\frac{1}{x}$. The corresponding valuation rings are the localization $\mathbb{K}[x]_{(x-x_0)}$ and $\mathbb{K}[\frac{1}{x}]_{(\frac{1}{x})}$ respectively.

Let $u(x) = \prod_{i=1}^{r}(x - x_i)^{n_i} \in \mathbb{K}(x)$ be a rational function, where $n_1, \ldots, n_r \in \mathbb{Z}$. The valuation of u at $x = x_0 \in \mathbb{K}$ is equal to n_i if $x_0 = x_i$ for some $i = 1, \ldots, r$ or is equal to 0 otherwise. The valuation of u at infinity is equal to $-\sum\limits_{i=1}^{r} n_i$, which is also equal to the difference of the degree of the denominator and the degree of the numerator. Therefore the principle divisor of u in $\mathbb{K}(x)$ is $[u] := \sum\limits_{i=1}^{r} n_i q_{x_i} - \left(\sum\limits_{i=1}^{r} n_i\right) q_\infty$, where q_{x_i} denote the prime divisor corresponding with x_i in $\mathbb{K}(x)$. In this particular case we have seen that the degree of a principle divisor is always zero, and we also know that every divisor of degree zero is principle. The first property also holds for a general algebraic function field, while the latter is a particularity of purely transcendental fields of degree one over $\mathbb{K}$.

We also denote by $\mathrm{ord}_{x_0}(u) := -\nu_{x_0}(u)$ and call it the *order of u at x_0* $\in \mathbb{K}\cup\{\infty\}$. The order satisfies the following properties for all $u, v \in \mathbb{K}(x)$:

(i) $\mathrm{ord}_{x_0}(uv) = \mathrm{ord}_{x_0}(u) + \mathrm{ord}_{x_0}(v)$

(ii) $\min\{\mathrm{ord}_{x_0}(u), \mathrm{ord}_{x_0}(v)\} \le \mathrm{ord}_{x_0}(u + v) \le \max\{\mathrm{ord}_{x_0}(u), \mathrm{ord}_{x_0}(v)\}$

(iii) If $\mathrm{ord}_{x_0}(u) \neq 0$, then the order of the derivative is

$$\mathrm{ord}_{x_0}(u') = \begin{cases} \mathrm{ord}_{x_0}(u) + 1\,, & \text{if } x_0 \in \mathbb{K}\,,\\ \mathrm{ord}_{x_0}(u) - 1\,, & \text{if } x_0 = \infty\,. \end{cases}$$

We might extend the domain and the values of a rational function to the affine line $\mathbb{K}\cup\{\infty\}$. Let $u(x) \in \mathbb{K}(x)$ be a rational function. Then the value $u(x_0)$ is defined for all x_0 in $\mathbb{K}$ but roots of the denominator. We may extend the domain of u to the whole affine line $\mathbb{K}\cup\{\infty\}$ as follows.

- If x_0 is a root of the denominator, we define $u(x_0) := \infty$.

- If $x_0 = \infty$ and $u(x) = \frac{a_n x^n + \ldots + a_0}{b_m x^m + \ldots + b_0}$, with $a_n b_m \neq 0$, then

$$u(x_0) := \begin{cases} 0\,, & \text{if } n < m\,, \\ \infty\,, & \text{if } n > m\,, \\ \frac{a_n}{b_m}\,, & \text{if } n = m\,. \end{cases}$$

If $r := \mathrm{ord}_{x_0}(y(x)) > 0$, we say that $y(x)$ has a *pole of order r at $x = x_0$*. In case $r < 0$, x_0 is called a *zero of order r of $y(x)$*. Poles of a rational function are roots of the denominator, and probably at infinity. The degree of a rational function (which is the maximum of the degrees of the numerator and the denominator) is equal to the number of poles in $\mathbb{K} \cup \{\infty\}$ counting multiplicities. We recall partial fraction representation of $y(x)$.

Proposition 2.1.
Every rational function $y(x) \in \mathbb{K}(x)$ can be represented in the form

$$y(x) = \sum_{i=1}^{n} \sum_{j=1}^{r_i} \frac{c_{ij}}{(x - x_i)^j} + \sum_{k=0}^{N} c_i x^i\,. \tag{2}$$

In this formula we use the following notation.

n	*number of poles of $y(x)$ in $\mathbb{K}$,*
$x_1, \ldots, x_n$	*poles of $y(x)$ in $\mathbb{K}$,*
$r_1, \ldots, r_n$	*orders of $y(x)$ at $x_1, \ldots, x_n$ respectively,*
N	*order of $y(x)$ at infinity, and*
c_{ij}, c_k	*coefficients in $\mathbb{K}$*

3 Combinatorial aspect of first-order AODEs without movable poles

In this section we locally study rational solutions of first-order AODEs. We describe how partial fraction decomposition and a bound on the order of a rational solution can be used to define a class of first-order AODEs which can be proven to be without movable poles. For this class the ideas also yield an algorithm for finding the solution by coefficient comparison.

3.1 A bound for the order of the p-adic part

In this subsection, we determine for each $x_0 \in \mathbb{K} \cup \{\infty\}$ an upper bound for the order at $x = x_0$ of a rational solution of the differential equation (1). An algorithm for finding such a bound is then presented.

Theorem 3.1.
Let $F(x, y, z) = \sum_{i,j} f_{ij}(x)y^i z^j \in \mathbb{K}[x][y, z] \setminus \mathbb{K}[x][y]$. Assume further that F is not homogeneous as a polynomial in y and z with coefficients in $\mathbb{K}[x]$. Then for each $x_0 \in \mathbb{K} \cup \{\infty\}$, there is a non-negative integer $\rho = \rho(x_0, F)$, depending only on x_0 and F, such that the order of every rational solution of the differential equation $F(x, y, y') = 0$ at $x = x_0$ does not exceed ρ.

Proof. We are going to determine the order bound $\rho(x_0, F)$ in an algorithmic way. Let us fix an x_0 in $\mathbb{K}$. A bound $\rho(\infty, F)$ for the order at infinity is constructed similarly. Assume that $y(x)$ is a rational solution of the differential equation (1), $F(x, y, y') = 0$, and that $y(x)$ has a pole of order $r > 0$ at $x = x_0$. Then $y'(x)$ has a pole of order $r + 1$ at $x = x_0$. For each $(i, j) \in \mathbb{N}^2$, we denote $\alpha_{ij} = \text{ord}_{x_0}(f_{ij})$, i. e. the order at $x = x_0$ for f_{ij}. Note, that α_{ij} is non-positive. Let

$$\begin{aligned}
E &:= \{(i, j) \in \mathbb{N}^2 \mid f_{ij} \neq 0\} \\
n &:= \max \{i + j \mid (i, j) \in E\} \\
A &:= \{(i, j) \in E \mid i + j = n\} \\
d &:= \max \{j + \alpha_{ij} \mid (i, j) \in A\} \\
D &:= \{(i, j) \in A \mid j + \alpha_{ij} = d\}
\end{aligned}$$

Note, that d is different for the case $x_0 = \infty$. Since F is not homogeneous, $E \setminus A$ is a non-empty set. It is clear that D is also non-empty, and is contained in A. For each $(i, j) \in D$, we rewrite $f_{ij}(x)$ as

$$f_{ij}(x) = a_{ij}(x - x_0)^{-\alpha_{ij}} + h_{ij}(x)\,,$$

where $a_{ij} \in \mathbb{K}$, $h_{ij}(x) \in \mathbb{K}[x]$ such that $\text{ord}_{x_0}(h_{ij}(x)) < \alpha_{ij}$. Since $y = y(x)$ is a solution the differential equation, we have that $F(x, y(x), y'(x)) = 0$. We gather terms of $F(x, y(x), y'(x))$ into different groups as follow:

$$\sum_{(i,j)\in D} f_{ij}(x)y(x)^i y'(x)^j + \sum_{(i,j)\in A\setminus D} f_{ij}(x)y(x)^i y'(x)^j + \sum_{(i,j)\in E\setminus A} f_{ij}(x)y(x)^i y'(x)^j = 0\,.$$

Therefore,

$$\begin{aligned}
\sum_{(i,j)\in D} a_{ij}(x - x_0)^{-\alpha_{ij}} y(x)^i y'(x)^j + \sum_{(i,j)\in D} h_{ij}(x)y(x)^i y'(x)^j + \sum_{(i,j)\in A\setminus D} f_{ij}(x)y(x)^i y'(x)^j \\
= -\sum_{(i,j)\in E\setminus A} f_{ij}(x)y(x)^i y'(x)^j\,. \quad (3)
\end{aligned}$$

The orders at $x = x_0$ of terms in the first sum are equal to $nr + d$, which is always larger than the order of terms in the second and the third sum. It does not mean that the order of the left hand side is equal to $nr + d$. Two cases arise.

Case 1: The order at $x = x_0$ of the first sum in (3) is equal to $nr + d$:
Then the order of the left hand side of (3) is exactly $nr + d$. By comparing with the order of terms on the right hand side, we obtain

$$nr + d \leq \max\left\{(i+j)r + j + \alpha_{ij} \mid (i,j) \in E \setminus A\right\} .$$

Therefore,

$$r \leq \max\left\{\frac{j + \alpha_{ij} - d}{n - i - j} \mid (i,j) \in E \setminus A\right\} .$$

Case 2: The order at $x = x_0$ of the first sum in (3) is smaller than $nr + d$:
Let

$$g(x) := (x - x_0)^{nr+d} \sum_{(i,j)\in D} a_{ij}(x - x_0)^{-\alpha_{ij}} y(x)^i y'(x)^j .$$

Then $g(x_0) = 0$. This property of g leads to an upper bound for r. In order to do that, let $z(x) := (x - x_0)^r y(x)$. Then $z(x_0)$ is neither 0 nor ∞, and

$$y'(x) = \frac{z'(x)(x - x_0) - rz(x)}{(x - x_0)^{r+1}} .$$

Rewriting $g(x)$ in terms of $z(x)$ and then simplifying the result yields

$$g(x) = \sum_{(i,j)\in D} a_{ij} z(x)^i (z'(x)(x - x_0) - rz(x))^j .$$

By substituting $x = x_0$ and dividing by $z(x_0)^n$, we see that r must be a positive integer root of the algebraic equation

$$\sum_{(i,j)\in D} a_{ij}(-1)^j t^j = 0 . \tag{4}$$

Hence, in any case, r must be smaller than either

$$\max\left\{\frac{j + \alpha_{ij} - d}{n - i - j} \mid (i,j) \in E \setminus A\right\} ,$$

or the largest positive integer root of the algebraic equation (4). □

Theorem 3.1 itself is not interesting. What is useful for us is the proof. There, the bound for the order at $x = x_0$ of rational solutions of the differential equation is constructed in an algorithmic way. We summarize this result in Algorithm 1.

Algorithm 1 An order bound for the p-adic part

Input: $x_0 \in \mathbb{K} \cup \{\infty\}$; $F = \sum f_{ij}(x) y^i z^j \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$ non-homogeneous in y and z.

Output: An upper bound $\rho(x_0, F)$ for the order at $x = x_0$ of all rational solutions of the ODE, $F(x, y, y') = 0$.

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E = {(i,j) ∈ ℕ² | f_ij ≠ 0}
n = max {i + j | (i,j) ∈ E}
A = {(i,j) ∈ E | i + j = n}
α_ij = ord_{x_0}(f_ij(x)), for (i,j) ∈ E
if x_0 ∈ 𝕂 then
    d = max {j + α_ij | (i,j) ∈ A}
    D = {(i,j) ∈ A | j + a_ij = d}
    a_ij = f_ij(x) / (x − x_0)^{−α_ij} |_{x=x_0}  for (i,j) ∈ D
    R = the set of positive integer roots of the algebraic equation Σ_{(i,j)∈D} a_ij (−t)^j = 0
    ρ̄ = ⌊max { (j + α_ij − d) / (n − i − j) | (i,j) ∈ E \ A }⌋
else if x_0 = ∞ then
    d = max {α_ij − j | (i,j) ∈ A}
    D = {(i,j) ∈ A | α_ij − j = d}
    a_ij = the leading coefficient of f_ij(x), for (i,j) ∈ D
    R = the set of positive integer roots of the algebraic equation Σ_{(i,j)∈D} a_ij t^j = 0
    ρ̄ = ⌊max { (α_ij − j − d) / (n − i − j) | (i,j) ∈ E \ A }⌋
end if
return ρ = max (R ∪ {ρ̄, 0}).
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3.2 First-order AODEs without movable poles

As we have seen in the previous subsection, once a pole is given, one can compute an upper bound for the order of rational solutions of a given first-order AODE. Unfortunately it is not always easy to find possible candidates for the positions of poles. For a linear ODE, poles of a (rational) solution can be easily determined from the coefficients of the differential equation itself. In general, it is no longer true when we pass to the class of non-linear first-order AODEs. In fact, poles of a rational solution of a first-order AODE may occur at an arbitrary point. For example, for every $c \in \mathbb{K}$, the function $y(x) = \frac{1}{x-c}$ is a rational solution of the Riccati equation $y' + y^2 = 0$.

The task of this section is to collect first-order AODEs for which no unexpected poles occur in their rational solutions. The core of the idea is that we equip the support of F with a certain partial order. The existence of the greatest element decides whether the differential equation is in the class of AODEs we are interested in.

Definition 3.2.
On $\mathbb{N}^2$ we define a relation $\gg$ as follows. For $(i_1, j_1), (i_2, j_2) \in \mathbb{N}^2$, we say $(i_1, j_1) \gg$

(i_2, j_2) *iff either* $i_1 + j_1 = i_2 + j_2$ *and* $j_1 > j_2$*, or* $(i_1 + j_1) - (i_2 + j_2) > \max\{0, j_2 - j_1\}$.

It is easy to check that $\gg$ is a strict partial ordering on $\mathbb{N}^2$, i. e. the following properties hold for all $u, v, w \in \mathbb{N}^2$:

(i) irreflexivity: $u \not\gg u$

(ii) transitivity: if $u \gg v$ and $v \gg w$ then $u \gg w$

(iii) asymmetry: if $u \gg v$ then $v \not\gg u$.

For $u, v \in \mathbb{N}^2$, we say u and v are *comparable* if either $u \gg v$ or $v \gg u$. Otherwise, they are called *incomparable*. Not every pair of elements from $\mathbb{N}^2$ is comparable. In other words, the order $\gg$ is not a total order on $\mathbb{N}^2$. For example, $(2, 0)$ and $(0, 1)$ are incomparable.

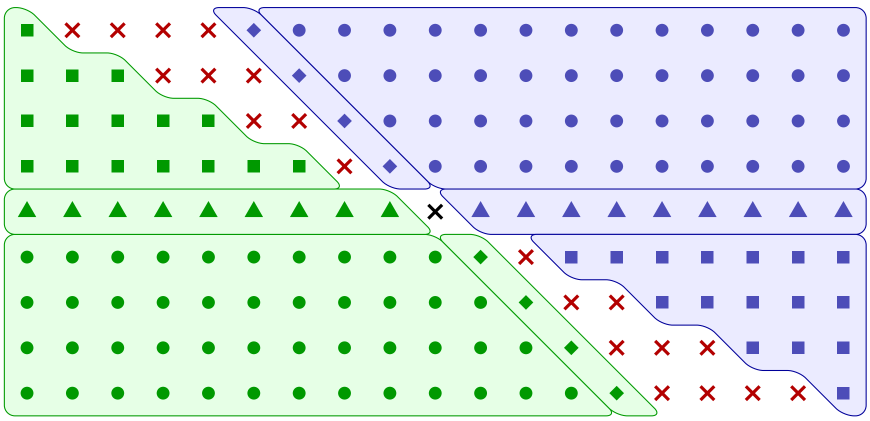

Figure 1: Comparable and incomparable points for a given point

In Figure 1 we consider a given point **×** (in black) and show all the comparable and incomparable points in some surrounding. All smaller points in that eary are highlighted in green (with green background), the greater points are in blue. The symbol at the respective points ilustrates the class of points according to the inequalities in the definiton. The points with symbol **×** are incomparable to the given one. All such incomparable points except the given one are drawn in red. Let (i_1, j_1) be our given point, and (i_2, j_2) a smaller point. Then one of the following cases has to be fulfilled.

- ◆ $i_1 + j_1 = i_2 + j_2$ and $j_1 > j_2$,

or $(i_1 + j_1) - (i_2 + j_2) > \max\{0, j_2 - j_1\}$ and furthermore

- ● $j_2 < j_1$, or
- ▲ $j_2 = j_1$, or
- ■ $j_2 > j_1$.

Note, that this shows, that for a given point, the number of points which are incomparable to this one, is finite.

Let S be a subset of $\mathbb{N}^2$. An element $u \in S$ is called a greatest element of S if $u \gg v$ for every $v \in S \setminus \{u\}$. The set S has at most one greatest element. Since $\gg$ is not a total order, a subset of $\mathbb{N}^2$ may have no greatest element. This motivates the following definition.

Definition 3.3.
Let $F = \sum_{i,j} f_{ij}(x) y^i z^j$ be a polynomial in $\mathbb{K}[x][y, z]$. Let $\mathcal{E}(F) \subseteq N^2$ be the support of F, i. e.

$$\mathcal{E}(F) := \left\{(i, j) \in \mathbb{N}^2 \,|\, f_{ij} \neq 0\right\} .$$

A polynomial F for which the set $\mathcal{E}$ admits a greatest element is called maximally comparable. *We also call the corresponding differential equation maximally comparable.*

Theorem 3.4.
Let $F = \sum_{i,j} f_{ij}(x) y^i z^j \in \mathbb{K}[x][y, z] \setminus \mathbb{K}[x][y]$ be a polynomial, which is non-homogeneous in y and z. We assume that F is maximally comparable and the greatest element of $\mathcal{E} = \mathcal{E}(F)$ is (i_0, j_0). Then poles in $\mathbb{K}$ of a rational solution of the differential equation $F(x, y, y') = 0$ only occur at the zeros of $f_{i_0 j_0}(x)$.

Proof. We prove the theorem by contradiction. Assume that $y(x) \in \mathbb{K}(x)$ is a rational solution of the differential equation $F(x, y, y') = 0$, that $x_0 \in \mathbb{K}$ is a pole of order $r \geq 1$ of $y(x)$, and that $f_{i_0 j_0}(x_0) \neq 0$. Let $A := \{(i, j) \in \mathcal{E} \,|\, i + j = i_0 + j_0\}$. It is clear that $(i_0, j_0) \in A$ and $A \subsetneq \mathcal{E}$. Let us substitute $y(x)$ to the differential equation $F(x, y, y') = 0$ and group terms on the left hand side as follows.

$$f_{i_0 j_0}(x) y(x)^{i_0} y'(x)^{j_0} + \sum_{(i,j) \in A \setminus \{(i_0, j_0)\}} f_{i,j}(x) y(x)^i y'(x)^j = - \sum_{(i,j) \in E \setminus A} f_{i,j}(x) y(x)^i y'(x)^j \,, \quad (5)$$

where the sum on the left hand side is just zero if $A \setminus \{(i_0, j_0)\}$ is the empty set. The order at $x = x_0$ of the first term in (5) is equal to $(i_0 + j_0)r + j_0$, while the orders of terms in the sum on the left hand side are $(i+j)r + j - \nu_{x_0}(f_{ij}) = (i_0 + j_0)r + j - \nu_{x_0}(f_{ij})$. Since (i_0, j_0) is the greatest element in A, the order of the left hand side is always equal to the order of the first term, $(i_0 + j_0)r + j_0$. By comparing with the order of terms on the right hand side, we have

$$(i_0 + j_0)r + j_0 \leq \max \{(i + j)r + j - \nu_{x_0}(f_{ij}) \,|\, (i, j) \in \mathcal{E} \setminus A\} .$$

Therefore,

$$r \leq \max \left\{ \frac{j - j_0 - \nu_{x_0}(f_{ij})}{(i_0 + j_0) - (i + j)} \,|\, (i, j) \in \mathcal{E} \setminus A \right\} . \quad (6)$$

For each $(i, j) \in E \setminus A$, we have $(i_0, j_0) \gg (i, j)$ and $i_0 + j_0 \neq i + j$. Thus $(i_0 + j_0) - (i + j) > \max\{0, j - j_0\} \geq j - j_0 - \nu_{x_0}(f_{ij})$. Combination with (6) yields $r < 1$. This contradicts with the assumption. □

The theorem gives us a necessary condition for a first-order AODE having no "unexpected" poles. Once the condition is fulfilled, candidates for poles will be easily determined. Note, that by Theorem 3.4 maximally comparable AODEs cannot have movable poles. It is not clear whether the inverse direction also holds, but it is not important to us. The previous subsection provides a bound for the order of these pole candidates. Thus we have enough ingredients to determine the form of a rational solution in a certain finite number of indeterminate coefficients. By an ansatz we find all possible rational solutions.

Before we give an algorithm for finding all rational solutions of first-order AODEs without movable poles, we analyze different features of Theorem 3.4 by discussing the following questions.

1. What if F is homogeneous as a polynomial in y and z with coefficients in $\mathbb{K}[x]$?
2. How can we check the existence of the greatest element of $\mathcal{E}$ effectively, and find it in the affirmative case?
3. How likely is an $\mathcal{E}$ which admits a greatest element?

To answer Question 1, let us consider a homogeneous polynomial of degree $n \geq 1$ as a polynomial in y and z,

$$F(x, y, z) := f_{n,0}(x)y^n + f_{n-1,1}(x)y^{n-1}z + \ldots + f_{0,n}(x)z^n \,,$$

where $f_{i,j}(x) \in \mathbb{K}[x]$. We assume F to be irreducible as a polynomial in $\mathbb{K}[x, y, z]$, and consider the differential equation $F(x, y, y') = 0$. The differential equation always has the solution 0. Assume that the differential equation admits a non-zero rational solution $y(x)$, then $\frac{y'(x)}{y(x)}$ is a rational solution of the algebraic equation

$$f_{n,0}(x) + f_{n-1,1}(x)t + \ldots + f_{0,n}(x)t^n = 0 \,.$$

Since F is irreducible, this is only possible if $n = 1$. In case $n = 1$, the differential equation is linear, therefore it can be solved easily by known methods.

For the Question 2, the naive way would be to check the existence of the greatest element of $\mathcal{E}$ by comparing all pairs of its elements. However, there is a much simpler and intuitive way. Figure 2 shows how to proceed. We first take the set of points which have the greatest total degree. Within these we take the element, say $p = (p_1, p_2)$ which has the smallest first component. Now we check for each remaining point (x, y) whether or not it satisfies $y \geq \frac{2p_2+p_1-x}{2}$. If one point does, it is incomparable to the point p and hence, there is no greatest element. Otherwise p is the greatest element.

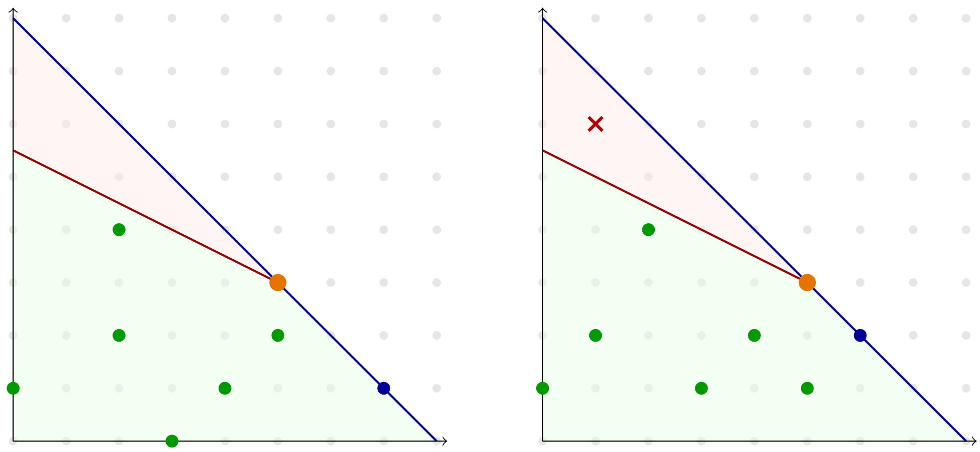

Figure 2: Check the existence of a greatest element

We are going to answer Question 3. We rewrite the differential equation in the form $F_0(x,y) + F_1(x,y)y' + \ldots + F_n(x,y)y'^n = 0$, where n is the degree of F in z and $F_0, \ldots, F_n \in \mathbb{K}[x,y]$. Then $\mathcal{E}$ admits a greatest element if and only if the smaller set $\{(\deg_y F_0, 0), (\deg_y F_1, 1), \ldots, (\deg_y F_n, n)\}$ does. Note, that if $F_i = 0$ for some $i < n$, the point $(\deg_y F_i, i)$ is smaller than $(\deg_y F_n, n)$. Therefore, we can state Question 3 in a different way: Given $n \in \mathbb{N}$, and let $m_0, m_1, \ldots, m_n$ be arbitrary natural numbers, how likely has the set $\{(m_0, 0), (m_1, 1), \ldots, (m_n, n)\}$ a greatest element? In order to answer the last question, we denote

$$A_{ijk} := \Big\{(m_0, m_1, ..., m_n) \in \mathbb{N}^{n+1} \,|\, m_i - m_j = j - i + k\Big\}$$
$$A_{ij} := \bigcup_{k=1}^{j-i} A_{ijk}$$
$$A := \bigcup_{0 \le i < j \le n} A_{ij}$$

It follows immediately from the definition of $\gg$ that for $0 \le i < j \le n$, the pairs (m_i, i) and (m_j, j) are incomparable if and only if $j - i < m_i - m_j \le 2(j-i)$. Therefore, a point $(m_0, m_1, \ldots, m_n) \in \mathbb{N}^{n+1}$ is in A_{ij} if and only if (m_i, i) and (m_j, j) are incomparable. Hence, for every point $(m_0, m_1, \ldots, m_n) \in \mathbb{N}^{n+1} \setminus A$, the set $S := \{(m_0, 0), (m_1, 1), \ldots, (m_n, n)\}$ has no incomparable points, or consequently, S admits a greatest element. We might consider each A_{ijk} as a hyperplane in $\mathbb{N}^{n+1}$. The set A is a union of $\sum_{0 \le i < j \le n} j - i = \frac{1}{6}n(n+1)(n+2)$ such hyperplanes. Therefore, almost all first-order AODEs for a given degree n are maximally comparable.

For instance, let us consider $m, n \in \mathbb{N}$, the pairs $(n, 0)$ and $(m, 1)$ are comparable iff $n - m \neq 2$. In other words, the quasi-linear ODE, $F(x,y,y') = F_0(x,y) + F_1(x,y)y' = 0$, with $F_0, F_1 \in \mathbb{K}[x,y]$, is out of the scope of Theorem 3.4 iff $\deg_y F_0 - \deg_y F_1 = 2$. In the next section, we introduce another approach which covers such a differential equation.

Algorithm 2 results from the above discussion. It finds all rational solutions of first-order maximally comparable AODEs. Example 3.5 illustrates Algorithm 2.

Algorithm 2 Rational solutions of maximally comparable first-order AODEs

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Input: $F = \sum f_{ij}(x)y^i z^j \in \mathbb{K}[x,y,z] \setminus \mathbb{K}[x,y]$ maximally comparable, and (i_0, j_0) the greatest element of the set $\mathcal{E}(F) = \{(i,j) \mid f_{ij} \neq 0\}$

Output: All rational solutions of the ODE, $F(x, y, y') = 0$.

1: $Sol = \emptyset$
2: **if** F is homogeneous of order n as polynomial in y and z **then**
3:     **if** $n > 1$ **then**
4:         $Sol = \{0\}$
5:     **else if** $n = 1$ **then**
6:         Solve the linear ODE, $F(x, y, y') = 0$, and append all rational solutions to Sol.
7:     **end if**
8: **else**
9:     $Pol = \{x_0 \in \mathbb{K} \mid f_{i_0 j_0}(x_0) = 0\}$
10:     Use Algorithm 1 to compute $\rho(x_0, F)$ for all $x_0 \in Pol$
11:     Use Algorithm 1 to compute $\rho(\infty, F)$, a bound for the degree of the polynomial part.
12:     Make an ansatz for

$$y(x) = \sum_{x_0 \in Pol} \sum_{j=1}^{\rho(x_0,F)} \frac{c_{x_0,j}}{(x - x_0)^j} + \sum_{k=0}^{\rho(\infty,F)} c_k x^k \,,$$

    with indeterminate coefficients $c_{x_0,j}, c_k$
13:     Solve the obtained algebraic system in $c_{x_0,j}$ and c_k, and append the results to Sol
14: **end if**
15: **return** Sol

Example 3.5.
We consider the differential equation

$$F(x, y, y') = 3x^8 yy'^2 - 2x^9 y' - x^6 y^3 + 4x^4(x^3 + 2)y + 52x^3 + 152 = 0\,. \qquad (7)$$

Among the terms of F as a polynomial in y and y', the term $3x^8yy'^2$ has the largest power (which is $(1, 2)$) with respect to $\gg$. Therefore, poles of a rational solution of the differential equation (7) occur only at $x = 0$ and probably at infinity. By applying Algorithm 1, the orders of a rational solution at $x = 0$ and infinity are at most 2 and 1 respectively. Making an ansatz with $y(x) = \frac{c_1}{x^2} + \frac{c_1}{x} + c_3 + c_4 x$, and then solving the obtained algebraic system in c_1, c_2, c_3, c_4, we see that the differential equation (7) has only a rational solution $y(x) = \frac{-2}{x^2} - x$.

Remark 3.6.
Another interesting result of the combinatorial aspect is that it provides an algorithm

for determining all polynomial solutions of an arbitrary first-order AODE. It is based on the fact that a non-constant polynomial has only a pole at infinity, and the order of this pole is exactly the degree of the polynomial. Assume that we are looking for all polynomial solutions of the differential equation $F(x, y, y') = 0$. If F is homogeneous as a polynomial in y and y' then either the differential equation has only a zero polynomial solution or it is a linear first-order ODE which can be easily solved by known methods. Otherwise Algorithm 1 provides a bound for the degree of a polynomial solution. All polynomial solutions can be computed by the undeterminate coefficient method.

The question of finding all polynomial solutions of first-order AODEs was already addressed and solved in [15]. In fact this paper considers also higher order AODEs, but for higher order there is no full decision. The methods and results are rather similar to those presented here both restricted to polynomial solutions of first-order AODEs.

4 Algebraic aspect and quasi-linear first-order AODEs

A quasi-linear first-order ODE is a differential equation of the form $y' = f(x, y)$ for some $f \in \mathbb{K}(x, y)$. By multiplying both sides of the differential equation with the denominator of f, we also view it as a first-order AODE of degree 1 in y'. Although Algorithm 2 works on a generic class of first-order AODEs, its scope does not cover the class of quasi-linear first-order ODEs. In particular if the subtraction of the degree of the numerator of f by the degree of the denominator is equal to 2, then Algorithm 2 is invalid. In this section we approach rational solutions of first-order AODEs in the global meaning by using a tool from algebraic function field theory. Together with Algorithm 2, the new approach provides another puzzle piece for an algorithm covering the whole class of quasi-linear first-order ODEs.

The following idea is derived from [2]. In order to find all rational solutions of a quasi-linear first-order ODE, we first study a degree bound for all rational solutions. Once a degree bound is determined, we can make an ansatz for rational solutions with undeterminate coefficients. In [2], Eremenko studies a degree bound for rational solutions of first-order AODEs. He corresponds each first-order AODE with an algebraic function field over $\overline{\mathbb{K}(x)}$. The function field is moreover a differential field with the derivation extended from the usual derivation of $\overline{\mathbb{K}(x)}$. In case the differential function field satisfies the Fuchs condition (without movable critical points), it can be classified up to an isomorphism of differential fields by using the theory of Matsuda [19]. Hence, Eremenko reduces the differential equation to several simpler ones according to the classification, and then estimates a degree bound for rational solutions of such particular cases.

Eremenko [2] theoretically investigates the determination of a degree bound of rational solutions of first-order AODEs. Based on some of his ideas we give a different and more explicit algorithm for actually computing this bound. In the scope of this paper, we study such an algorithm for the class of quasi-linear first-order ODEs. Although our idea is based on Eremenko's results, we study the problem without the theory of

Matsuda on classification of differential function fields. Our algorithm is therefore much simpler.

4.1 Algebraic function fields

We start the discussion by giving an important class of algebraic function fields over the field $\overline{\mathbb{K}(x)}$ of algebraic functions. Let $H \in \mathbb{K}[x, Y, Z]$ be a trivariate polynomial such that H is irreducible as an element in $\overline{\mathbb{K}(x)}[Y, Z]$. The algebraic equation $H(x, Y, Z) = 0$ defines an irreducible algebraic curve in the affine plane $\mathbb{A}^2(\overline{\mathbb{K}(x)})$. The set of all rational functions over the curve is a field and it is isomorphic to the fraction field of the coordinate ring, i. e. $K := \operatorname{Frac}\left(\overline{\mathbb{K}(x)}[Y, Z]/(H)\right)$. The field K is an algebraic function field of degree one over $\overline{\mathbb{K}(x)}$. The set $\mathbb{P}_{K/\overline{\mathbb{K}(x)}}$ of all prime divisors of K over $\overline{\mathbb{K}(x)}$ is in general not easy to determine.

Lemma 4.1. (Eremenko [2])
Let K be an algebraic function field over $\overline{\mathbb{K}(x)}$ of transcendence degree one, and $y, z \in K$ such that $[y]^- \leq [z]^-$. Let $m := \left|\overline{\mathbb{K}(x)}(y, z) : \overline{\mathbb{K}(x)}(z)\right|$ be the degree of the field extension. Then there exists a unique irreducible polynomial $G \in \overline{\mathbb{K}(x)}[Y, Z]$ of the form

$$G(Y, Z) = Y^m - \sum_{\substack{i+j \leq m \\ 0 \leq i \leq m-1 \\ 0 \leq j \leq m}} g_{ij}(x) Y^i Z^j \,,$$

with $g_{ij} \in \overline{\mathbb{K}(x)}$, such that $G(y, z) = 0$ in K.

The result of the following lemma is used in the proof of Lemma 1 in [2] without giving a detailed proof.

Lemma 4.2.
Let K be an algebraic function field over $\overline{\mathbb{K}(x)}$ of transcendence degree one, and $y, z \in K$ such that $[y]^- \leq [z]^-$. Since Lemma 4.1 is applicable we let $G \in \overline{\mathbb{K}(x)}[Y, Z]$ be a polynomial with this property. Denote by $L \subset \overline{\mathbb{K}(x)}$ an algebraic function field over $\mathbb{K}$ containing x and all coefficients g_{ij} of G. Let $P \in \mathbb{P}_{K/\overline{\mathbb{K}(x)}}$ be a prime divisor. If $y(P), z(P)$ belong to L, then

$$\operatorname{supp}_{\mathbb{P}_{L/\mathbb{K}}}\left([y(P)]^-\right) \subseteq \operatorname{supp}_{\mathbb{P}_{L/\mathbb{K}}}\left([z(P)]^-\right) \bigcup \left(\bigcup_{i,j} \operatorname{supp}_{\mathbb{P}_{L/\mathbb{K}}}\left([g_{ij}]^-\right) \right) .$$

Proof. We prove the lemma by contradiction. Assume that the conclusion does not hold. Then there is a prime divisor $q \in \mathbb{P}_{L/\mathbb{K}}$ not in $\operatorname{supp}_{\mathbb{P}_{L/\mathbb{K}}}\left([z(P)]^-\right) \cup \left(\bigcup_{i,j} \operatorname{supp}_{\mathbb{P}_{L/\mathbb{K}}}\left([g_{ij}]^-\right) \right)$

but in $\operatorname{supp}_{\mathbb{P}_{L/\mathbb{K}}}([y(P)]^-)$. This implies that $\nu_q(y(P)) < 0$ and furthermore $\nu_q(z(P)) \geq 0$ and $\nu_q(g_{ij}) \geq 0$ for all i, j. From Lemma 4.1 we have

$$y^m = \sum_{\substack{i+j\leq m \\ 0\leq i\leq m-1 \\ 0\leq j\leq m}} g_{ij}(x) y^i z^j \,.$$

Therefore,

$$\begin{aligned} m \cdot \nu_q(y(P)) &\geq \min_{i,j} \{i \cdot \nu_q(y(P)) + j \cdot \nu_q(z(P)) + \nu_q(g_{ij})\} \\ &\geq (m-1) \cdot \nu_q(y(P)) + \min_{i,j} \{j \cdot \nu_q(z(P)) + \nu_q(g_{ij})\} \,. \end{aligned}$$

It implies that

$$\nu_q(y(P)) \geq \min_{i,j} \{j \cdot \nu_q(z(P)) + \nu_q(g_{ij})\} \,.$$

Because the right hand side is a non-negative integer, the last inequality can not happen.

□

The next theorem reads similar to Lemma 1 in [2]. However, it yields more information on the poles and their order and it explicitly describes the constant. In fact Theorem 4.3 implies Lemma 1 in [2].

Theorem 4.3.
With notations as above. Let $A := \bigcup_{i,j} \operatorname{supp}_{\mathbb{P}_{L/\mathbb{K}}}([g_{ij}]^-) \subseteq \mathbb{P}_{L/\mathbb{K}}$, *and for each* $q \in A$, *let* $\sigma_q = \left\lfloor \max_{i,j} \left\{ \frac{-\nu_q(g_{ij})}{m-i} \right\} \right\rfloor$. *Then,* $\sigma := \sum_{q\in A} \sigma_q \cdot q$ *is an effective divisor in* L, *i. e.* $\sigma \geq 0$. *Let* $P \in \mathbb{P}_{K/\overline{\mathbb{K}(x)}}$ *be a prime divisor. If* $y(P), z(P)$ *belong to* L, *then*

$$[y(P)]^- \leq [z(P)]^- + \sigma$$

as divisors in $\mathbb{P}_{L/\mathbb{K}}$.

Proof. Let q be an arbitrary prime divisor of L such that $q \in \operatorname{supp}_{\mathbb{P}_{L/\mathbb{K}}}([y(P)]^-)$. Then $\nu_q(y(P)) < 0$. We need to prove that:

$$-\nu_q(y(P)) \leq \max_{i,j}\{0, -\nu_q(z(P))\} + \max_{i,j} \left\{ \frac{-\nu_q(g_{ij})}{m-i} \right\}.$$

Due to Lemma 4.1, we have

$$m \cdot \nu_q(y(P)) \geq \min_{i,j} \{i \cdot \nu_q(y(P)) + j \cdot \nu_q(z(P)) + \nu_q(g_{ij})\} \,.$$

Hence,

$$\begin{aligned} \nu_q(y(P)) &\geq \min_{i,j} \left\{ \frac{j}{m-i} \nu_q(z(P)) + \frac{\nu_q(g_{ij})}{m-i} \right\} \\ &\geq \min_{i,j}\{0, \nu_q(z(P))\} + \min_{i,j} \left\{ \frac{\nu_q(g_{ij})}{m-i} \right\} \,. \end{aligned}$$

The last inequality concludes the theorem. □

Theorem 4.3 works for algebraic functions. In fact, we need the result only for rational functions. Application of Theorem 4.3 to rational functions can be seen in Corollary 4.4. We get a result comparable to Lemma 1 in [2] for the case of rational functions but we are able to explicitly determine the constant C.

Corollary 4.4.
Let $y = \frac{y_1}{y_2}, z = \frac{z_1}{z_2} \in \mathbb{K}(x,t)$, where $y_1, y_2, z_1, z_2 \in \mathbb{K}[x,t]$ such that $\gcd(y_1, y_2) = \gcd(z_1, z_2) = 1$. Assume that y_2 divides z_2 and $\deg_t y_1 - \deg_t y_2 \leq \max\{0, \deg_t z_1 - \deg_t z_2\}$. Then there exists a constant $C = C(y,z)$ depending only on y and z such that: for every $t(x) \in \mathbb{K}(x)$, with $z_2(x,t(x)) \neq 0$, we have $\deg y(x,t(x)) \leq \deg z(x,t(x)) + C$.

Proof. We consider y, z as elements of the algebraic function field $K := \overline{\mathbb{K}(x)}(t)$ over $\overline{\mathbb{K}(x)}$. From this point of view, the condition $[y]^- \leq [z]^-$ as divisors in K is equivalent to the assumption that y_2 divides z_2 and $\deg_t y_1 - \deg_t y_2 \leq \max\{0, \deg_t z_1 - \deg_t z_2\}$.

We apply Theorem 4.3 with K as above and $L := \mathbb{K}(x)$. The sets $\mathbb{P}_{K/\overline{\mathbb{K}(x)}}$ and $\mathbb{P}_{L/\mathbb{K}}$ of prime divisors are exactly $\overline{\mathbb{K}(x)} \cup \{\infty\}$ and $\mathbb{K} \cup \{\infty\}$ respectively. For each $t(x) \in \mathbb{K}(x)$, the image of y in L is $y(x,t(x))$. Since the assumptions of Lemma 4.1 are fulfilled, we know that there exists $G \in \overline{\mathbb{K}(x)}[Y,Z]$, determined by coefficients g_{ij}, such that $G(y,z) = 0$. By Remark 4.5 we even know that $G \in \mathbb{K}(x)[Y,Z]$. Note, that A, as defined in Theorem 4.3, is the set of poles of g_{ij} in $\mathbb{K} \cup \{\infty\}$. Theorem 4.3 yields

$$[y(x,t(x))]^- \leq [z(x,t(x))]^- + \sum_{x_0 \in A} \sigma_{x_0} q_{x_0} \,,$$

for all $t(x) \in \mathbb{K}(x)$ such that $z_2(x,t(x)) \neq 0$, where σ_{x_0} is the largest integer which does not exceed $\max\limits_{i,j} \left\{ \frac{-\nu_{x_0}(g_{ij})}{m-i} \right\}$, and q_{x_0} the prime divisor corresponding with x_0 in L. By taking the degree of divisors on both sides, we obtain

$$\deg y(x,t(x)) \leq \deg z(x,t(x)) + \sum_{x_0 \in A} \sigma_{x_0} \,.$$

The constant $C := \sum\limits_{x_0 \in A} \sigma_{x_0}$ depends only on y and z and it is independent of $t(x)$. □

Remark 4.5.
The polynomial G defined in the last proof can be constructed by using Gröbner bases. We first compute a reduced Gröbner basis of the ideal $\langle y_2 Y - y_1, z_2 Z - z_1 \rangle$ in $\mathbb{K}[x,t,Y,Z]$ with respect to lexicographic order such that $t > Y > Z > x$. Let H be an element in the basis with the smallest leading term. Then H must be in $\mathbb{K}[x,Y,Z]$. Finally we consider H as a polynomial in $\mathbb{K}[x][Y,Z]$ and divide H by the leading coefficient (with respect to lexicographic order such that $Y > Z$). The result, which is an irreducible polynomial in $\mathbb{K}(x)[Y,Z]$, is the polynomial G we are looking for.

4.2 Quasi-linear first-order ODEs

As a nice application of Corollary 4.4, we present here an algorithmic way for computing all rational solutions of a quasi-linear first-order ODE

$$y' = f(x, y)\,, \tag{8}$$

for some $f \in \mathbb{K}(x, y)$.

We denote by m and n the degree in y of the denominator and the numerator of f respectively. Let us first consider the differential equation (8) with the additional assumption that $\deg_y f = \max\{n, m\} \geq 3$. We define

$$\hat{f} := \begin{cases} f\,, & \text{if } n \leq m\,, \\ \frac{1}{f}\,, & \text{if } n > m\,, \end{cases}$$

where $\operatorname{denom}(\hat{f})$ denotes the denominator of $\hat{f}$. Considering t, $\hat{f}(x,t)$ and $\operatorname{denom}(\hat{f})(x,t)$ as elements of $\mathbb{K}(x,t)$, we see that the pair of rational functions $t^{\deg_y(f)}, \operatorname{denom}(\hat{f})(x,t)$ satisfies the assumptions of Corollary 4.4. Thus, there is a constant C_1 depending only on f such that

$$\deg\left(t(x)^{\deg_y f}\right) \leq \deg(\operatorname{denom}(\hat{f})(x, t(x))) + C_1$$

and similarly using the functions and $\frac{1}{\operatorname{denom}(\hat{f})(x,t)}, \hat{f}(x,t)$ there exists C_2 such that

$$\deg\left(\frac{1}{\operatorname{denom}(\hat{f})(x, t(x))}\right) \leq \deg(\hat{f}(x, t(x))) + C_2\,.$$

The inequalities hold for all $t(x) \in \mathbb{K}(x)$ for which $f(x, t(x))$ is neither 0 nor ∞. Notice that $\deg \operatorname{denom}(\hat{f})(x, t(x)) = \deg \frac{1}{\operatorname{denom}(\hat{f})(x,t(x))}$ and $\deg \hat{f}(x, t(x)) = \deg f(x, t(x))$. The two inequalities imply

$$\deg_y f \cdot \deg t(x) \leq \deg f(x, t(x)) + C_1 + C_2\,.$$

Now let us assume that $y(x) \in \mathbb{K}(x)$ is a non-constant rational solution of the differential equation (8). Replacing $t(x)$ by $y(x)$ in the last inequality yields

$$\begin{aligned} \deg_y f \cdot \deg y(x) &\leq \deg y'(x) + C_1 + C_2 \\ &\leq 2 \deg y(x) + C_1 + C_2\,. \end{aligned}$$

Since $\deg_y f \geq 3$, we have

$$\deg y(x) \leq \frac{C_1 + C_2}{\deg_y f - 2}\,. \tag{9}$$

The constants C_1, C_2 are determined as in the proof of Corollary 4.4.

The inequality (9) gives an effective way to determine a bound for the degree of a rational solution of the quasi-linear ODE (8) with $\deg_y f \geq 3$. It can be used for making an ansatz, and then computing all rational solutions. Most of such quasi-linear ODEs are also in the scope of Algorithm 2. Note, that in these cases the constants C_1 and C_2 from Corollary 4.4 might be rather high and hence Algorithm 2 is recommended.

Now let us gather the two approaches to propose an algorithm for computing rational solutions of the general quasi-linear ODE (8). Let n and m be the degree of the numerator and the denominator of f, respectively. As we have seen, Algorithm 2 can rationally solve such an ODE in all cases but $n - m = 2$. If $n - m = 2$ and $n \geq 3$, the inequality (9) gives an upper bound for the degree of a rational solution, and then gives a chance to find all of them. The only remaining case is $(n, m) = (2, 0)$. In this case, (8) is a classical Riccati equation. Fortunately algorithms for finding rational solutions of Riccati equations are well established in literature. In the next section, we recall such an algorithm.

We conclude this section by summarizing an algorithm for finding all rational solutions of a given quasi-linear first-order ODE (see Algorithm 3) and giving an example thereof.

Example 4.6.
We consider the differential equation

$$y' = f(x,y) = \frac{x^3y^4 - 5xy - x^3 + 5x^2 - 3}{x^3(y^2 + x)}\,. \tag{10}$$

Let P and Q be the numerator and the denominator of f, respectively. Since $\deg_y P - \deg_y Q = 2$, the differential equation (10) is out of the scope of Algorithm 2. We use the algebraic method described in this section to compute a degree bound for a rational solution. Keeping the notations as in Algorithm 3, we first find the polynomial $G_1 \in \mathbb{K}(x)[Y, Z]$ such that $G_1(y^4, P(x, y)) = 0$. It can be done by Gröbner bases. In fact, we compute a reduced Gröbner basis of the ideal $\langle Y - y^4, Z - P\rangle$ in $\mathbb{K}[y, Y, Z, x]$ with the lexicographic order $y > Y > Z > x$. The polynomial in the basis with the smallest leading term, say G, is the unique one containing only x, Y, Z, and it has the form

$$G = x^{12}Y^4 + \text{ terms of smaller lex order}$$

Therefore, $G_1 = \frac{G}{x^{12}}$. Poles of coefficients of G_1 occur only at $x = 0$. The constant C_1, as defined above, is equal to 1. Similarly, G_2 is a polynomial in $\mathbb{K}(x)[Y, Z]$ such that $G_2\left(\frac{1}{P(x,y)}, \frac{Q(x,y)}{P(x,y)}\right) = 0$ and thus $C_2 = 11$. Then the degree of a rational solution of the differential equation (10) does not exceed $\frac{C_1+C_2}{\deg_y f - 2} = 6$. By using the indeterminate coefficient method, we see that $y(x) = \frac{-1+x}{x}$ is the only rational solution of the differential equation (10).

Algorithm 3 Rational solutions of quasi-linear first-order ODEs

Input: A quasi-linear ODE, $y' = f(x,y)$.
Output: All rational solutions.

$n = \deg_y \operatorname{num}(f)$ and $m = \deg_y \operatorname{denom}(f)$
if $n - m \neq 2$ **then**
  return the result of Algorithm 2
else if $(n,m) = (2,0)$ **then**
  return the result of Algorithm 4
else
  $\hat{f} = \frac{1}{f}$
  $\operatorname{denom}(\hat{f})$ = the denominator of $\hat{f}$
  Construct the polynomials in $\mathbb{K}(x)[Y,Z]$,

$$G_1(Y,Z) = Y^{m_1} - \sum_{i,j} g_{1ij} Y^i Z^j \,, \qquad G_2(Y,Z) = Y^{m_2} - \sum_{i,j} g_{2ij} Y^i Z^j \,,$$

  such that $G_1(t^n, \operatorname{denom}(\hat{f})) = G_2\left(\frac{1}{\operatorname{denom}(\hat{f})}, \hat{f}\right) = 0$, as in Corollary 4.4.
  $A_1 = \{\text{poles of } g_{1ij} \text{ in } \mathbb{K} \mid i,j \in \mathbb{N}\}$
  $A_2 = \{\text{poles of } g_{2ij} \text{ in } \mathbb{K} \mid i,j \in \mathbb{N}\}$
  $C_1 = \sum_{x_0 \in A_1} \left\lfloor \max_{i,j} \left\{ \frac{-\nu_{x_0}(g_{1ij})}{m_1 - i} \right\} \right\rfloor$
  $C_2 = \sum_{x_0 \in A_2} \left\lfloor \max_{i,j} \left\{ \frac{-\nu_{x_0}(g_{2ij})}{m_2 - i} \right\} \right\rfloor$
  $r = \left\lfloor \frac{C_1 + C_2}{n-2} \right\rfloor$
  Make an ansatz

$$y(x) = \frac{a_0 + a_1 x + \ldots + a_r x^r}{b_0 + b_1 x + \ldots + b_r x^r} \,,$$

  with indeterminate coefficients a_i, b_j, and solve the obtained algebraic system.
  return all solutions $y(x)$
end if

4.3 Riccati equations

In this subsection, we restrict our work to the class of Riccati equations. A Riccati equation is a differential equation of the form

$$\omega' = b_0(x) + b_1(x)\omega + b_2(x)\omega^2 \,, \tag{11}$$

where $b_0, b_1, b_2 \in \mathbb{K}(x)$, $b_2 \neq 0$. We normalize (11) by transforming with $y = -b_2(x)\omega - \frac{b_2'(x)}{2b_2(x)} - \frac{b_1(x)}{2}$. The obtained differential equation is

$$y' + y^2 = a(x) \,, \tag{12}$$

where $a = \frac{1}{4}\left(\frac{b_2'}{b_2} + b_1\right)^2 - \frac{1}{2}\left(\frac{b_2'}{b_2} + b_1\right)' - b_0 b_2$. A differential equation of the form (12) is called a rational normal Riccati equation. Since this is always possible we only consider

Riccati equations in normal form and study their rational solutions.

The problem of finding rational solutions of Riccati equations has been intensively studied. An algorithm for finding rational solutions of a Riccati equation can be found for instance in [23, Alg. 2.2,p. 21]. In [14, Case 1] Kovacic also considers rational solutions of Riccati equations, as a step in the computation of Liouvillian solutions of second-order ODEs. Here we summarize the most important aspects of Kovacic's algorithm for Riccati equations.

First we collect necessary conditions for a rational normal Riccati equation having a rational solution. To avoid triviality we always assume that a is not a constant, or equivalently, $a(x)$ has at least one pole in $\mathbb{K} \cup \{\infty\}$.

Proposition 4.7. (Kovacic [14])
If the rational normal Riccati equation (12) *has a rational solution, then*

- *(i) every pole of $a(x)$ on $\mathbb{K}$ must be either a simple pole or a multiple pole of even order,*
- *(ii) the valuation of $a(x)$ at infinity $\nu_\infty(a(x))$ must be even or be greater than or equal to 2.*

Assume that $y(x) \in \mathbb{K}(x)$ is a rational solution of the differential equation (12). A pole of $y(x)$ which is also a pole of $a(x)$ is called a non-movable pole. Otherwise, it is called a movable pole. According to Kovacic's algorithm (see [14]), $y(x)$ must have the form

$$y(x) = \sum_{i=1}^{n} \sum_{j=1}^{r_i} \frac{c_{ij}}{(x - x_i)^j} + \sum_{i=1}^{m} \frac{1}{x - \chi_i} + \sum_{i=0}^{N} d_i x^i \,, \tag{13}$$

where $x_1, \ldots, x_n \in \mathbb{K}$ are poles of $a(x)$, $\chi_1, \ldots, \chi_m \in \mathbb{K}$ are movable poles of $y(x)$, the indices $N, n, r_1, \ldots, r_n$ can be determined from $a(x)$, and each of the vectors $(d_0, d_1, \ldots, d_N)$, $(c_{i1}, \ldots, c_{ir_i})$ over $\mathbb{K}$ is determined up to at most 2 choices. Every such choice determines the potential number m of movable poles.

For details on the possible choices, we separate into several cases depending on the property of $a(x)$ at the pole $x_0 \in \mathbb{K} \cup \{\infty\}$. The first sum in (13) corresponds to the property of $a(x)$ at finite poles $x_1, \ldots, x_n$, while the last sum is defined by the property of $a(x)$ at infinity. In all cases the coefficents can be obtained from Laurent series expansion of a and y. Then we substitute into the ODE and compare coefficients.

Case 1: $x_0 = x_i \in \mathbb{K}$ is a double pole of $a(x)$.
In this case, $y(x)$ has a simple pole at $x = x_i$. The Laurent series expansion of $a(x)$ and $y(x)$ at $x = x_0$ are

$$a(x) = \frac{a_{i2}}{(x - x_i)^2} + \frac{a_{i1}}{(x - x_i)} + \sum_{k=0}^{\infty} a_{i,-k}(x - x_i)^k \,,$$

$$y(x) = \frac{c_{i1}}{(x - x_i)} + \sum_{k=0}^{\infty} c_{i,-k}(x - x_i)^k \,.$$

By substituting these Laurent series to (12) and comparing coefficients of $(x - x_i)^{-2}$ both sides, we see that the c_{i1} has two options

$$c_{i1} = \frac{1 \pm \sqrt{1 + 4a_{i2}}}{2} \,. \tag{14}$$

Notice that c_{i1} is also the residue of $y(x)$ at $x = x_i$.

Case 2: $x_0 = x_i \in \mathbb{K}$ is a multiple pole of degree $2r_i$ of $a(x)$, $(r_i \geq 2)$.
In this case, $y(x)$ has a pole of degree r_i at $x = x_i$. The Laurent series expansion of $a(x)$ and $y(x)$ at $x = x_0$ are

$$a(x) = \frac{a_{i,2r_i}}{(x - x_i)^{2r_i}} + \frac{a_{i,2r_i - 1}}{(x - x_i)^{2r_i - 1}} + \sum_{k=2}^{\infty} \frac{a_{i,2r_i - k}}{(x - x_i)^{2r_i - k}} \,,$$
$$y(x) = \frac{c_{i,r_i}}{(x - x_i)^{r_i}} + \ldots + \frac{c_{i,1}}{(x - x_i)} + \sum_{k=0}^{\infty} c_{i,-k}(x - x_i)^k \,.$$

Again, we substitute these Laurent series to the differential equation (12) and then identify the coefficients of $(x - x_i)^j$ on both sides with $j = 2r_i, 2r_i - 1, \ldots, r_i + 1$. We see that c_{i,r_i} has two possibilities, and once a choices is fixed, $c_{i,r_i - 1}, \ldots, c_{i1}$ are determined uniquely. In particular, the vector $(c_{i,r_i}, \ldots, c_{i1})$ is determined by

$$\begin{cases} c_{i,r_i} = \pm\sqrt{a_{i,2r_i}} \,, \\ c_{i,s} = \dfrac{1}{2c_{i,r_i}} \left(a_{i,r_i + s} - \displaystyle\sum_{j=s+1}^{r_i - 1} c_{i,j} c_{i,r_i + s - j} \right) \,, \qquad s \in \{2, \ldots, r_i - 1\} \,, \\ c_{i,1} = \dfrac{1}{2c_{i,r_i}} \left(a_{i,r_i + 1} - \displaystyle\sum_{j=2}^{r_i - 1} c_{i,j} c_{i,r_i + 1 - j} - r_i c_{i,r_i} \right) \,. \end{cases} \tag{15}$$

The residue of $y(x)$ at $x = x_i$ in this case is c_{i1}.

Case 3: $x_0 = x_i$ is either a simple pole of $a(x)$ or a movable pole.
Then $y(x)$ has a simple pole at $x = x_i$ with the residue 1. The Laurent series expansion of $y(x)$ at x_i is $y(x) = \frac{1}{x - x_i} + \sum_{k=0}^{\infty} c_{i,-k}(x - x_i)^k$.

Case 4: $x_0 = \infty$ satisfying $\nu_\infty(a(x)) = -2N < 0$ for some $N \in \mathbb{N}$.
Then the valuation of $y(x)$ at infinity is $-N$. The Laurent series expansion of $a(x)$ and $y(x)$ at infinity, respectively, are

$$a(x) = a_{2N} x^{2N} + a_{2N-1} x^{2N-1} + \sum_{k=2}^{\infty} a_{2N-k} x^{2N-k} \,,$$
$$y(x) = d_N x^N + d_{N-1} x^{N-1} + \sum_{k=2}^{\infty} d_{N-k} x^{N-k} \,.$$

With the same technique as above, by substituting these Laurent series to the differential equation (12) and comparing corresponding coefficients on both sides, we obtain the following two possibilities for the vector $(d_0, d_1, \ldots, d_N)$:

$$\begin{cases} d_N = \pm\sqrt{a_{2N}}\,, \\ d_s = \dfrac{1}{2d_N}\left(a_{N+s} - \displaystyle\sum_{j=s+1}^{N-1} d_j d_{N+s-j}\right)\,, \quad s \in \{0, \ldots, N-1\}\,. \end{cases} \tag{16}$$

Notice that the residue of $y(x)$ at infinity in this case is

$$-d_{-1} = \frac{-1}{2d_N}\left(a_{N-1} - d_N - \sum_{j=0}^{N-1} d_j d_{N-1-j}\right)\,.$$

Case 5: $x_0 = \infty$ satisfying $\nu_\infty(a(x)) = 0$.
We take $N = 0$. Then the valuation of $y(x)$ at infinity is 0. The Laurent series expansion of $a(x)$ and $y(x)$ at infinity, respectively, are

$$a(x) = a_0 + \frac{a_{-1}}{x} + \sum_{k=2}^{\infty} \frac{a_{-k}}{x^k}\,,$$
$$y(x) = d_0 + \frac{d_{-1}}{x} + \sum_{k=2}^{\infty} \frac{d_{-k}}{x^k}\,.$$

These Laurent series must satisfy the differential equation (12). Therefore $d_0 = \pm\sqrt{a_0}$ and $d_{-1} = \frac{a_{-1}}{2d_0}$. The residue of $y(x)$ at infinity in this case is $-d_{-1} = -\frac{a_{-1}}{2d_0}$

Case 6: $x_0 = \infty$ satisfying $\nu_\infty(a(x)) \geq 2$.
In this case, the valuation of $y(x)$ at infinity is positive. Therefore, we just simply replace the last sum in (13) by zero, i. e. $N = -1$. The Laurent series of $a(x)$ and $y(x)$ at infinity are of the form

$$a(x) = \frac{a_{-\nu_\infty(a(x))}}{x^{\nu_\infty(a(x))}} + \sum_{k=1}^{\infty} \frac{a_{-\nu_\infty(a(x))-k}}{x^{\nu_\infty(a(x))+k}}\,,$$
$$y(x) = \frac{d_{-1}}{x} + \sum_{k=2}^{\infty} \frac{d_{-k}}{x^k}\,,$$

respectively. The possible residue at infinity of $y(x)$ in this case is $-d_{-1} = -\frac{1\pm\sqrt{1+4s_\infty}}{2}$, where $s_\infty = \lim\limits_{x\to 0} \frac{1}{x^2} a\left(\frac{1}{x}\right)$.

Once vectors $(d_{-1}, d_0, \ldots, d_N)$ and $(c_{i1}, \ldots, c_{i,r_i})$ for $i = 1, \ldots, n$ are chosen, the number of non-movable poles can be estimated. By the residue theorem, the sum of all residues of $y(x)$ over $\mathbb{K} \cup \{\infty\}$ is equal to zero. Since the residue of $y(x)$ at movable poles is always equal to 1, the number of movable poles is $m = d_{-1} - \sum_{i=1}^{n} c_{i1}$.

After determining the number m of movable poles, we can make an ansatz. Let $\overline{y}(x) := y(x) - \sum\limits_{i=1}^{m} \frac{1}{x-\chi_i}$, and let $P(x) := (x - \chi_1) \cdot \ldots \cdot (x - \chi_m)$. Then $y(x)$ can be written in the form $\overline{y}(x) + \frac{P'(x)}{P(x)}$. By substituting to the differential equation (12), P must be a polynomial solution of degree m of the following linear second-order ODE:

$$P''(x) + 2\overline{y}(x)P'(x) + (\overline{y}'(x) + \overline{y}(x)^2 - a(x))P(x) = 0\,. \tag{17}$$

Finding all polynomial P of degree m of the differential equation (17) can be done by linear algebra.

Summarize this discussion we can recall Kovacic's algorithm for rationally solving Riccati equations.

Algorithm 4 Rational Solutions of Riccati equations (Kovacic [14])

Input: The Riccati equation $\omega' = b_0(x) + b_1(x)\omega + b_2(x)\omega^2$, with $b_i \in \mathbb{K}(x)$ and $b_2 \neq 0$
Output: The set Sol of all rational solutions.

Set $y = -b_2\omega - \frac{b_2'}{2b_2} - \frac{b_1}{2}$ and transform to the rational normal Riccati equation: $y' + y^2 = a(x)$, where $a = \frac{1}{4}\left(\frac{b_2'}{b_2} + b_1\right)^2 - \frac{1}{2}\left(\frac{b_2'}{b_2} + b_1\right)' - b_0 b_2$
$Sol = \emptyset$, $preSol = \emptyset$
If $a = 0$, then $preSol = preSol \cup \left\{\frac{1}{x-c}\right\}$.
If $a \in \mathbb{K} \setminus \{0\}$, then $preSol := preSol \cup \{\pm\sqrt{a}\}$.
Determine poles of $a(x)$ in $\mathbb{K}$, say $x_1, \ldots, x_n$, and their orders. Compute $\nu_\infty(a(x))$.
Based on the results of the previous step, determine all possible vectors $(d_{-1}, d_0, \ldots, d_N)$, and $(c_{i1}, \ldots, c_{ir_i})$ for $i \in \{1, \ldots, n\}$ as discussed above.
for all possible combinations of these vectors **do**
    Compute $m = d_{-1} - \sum_{i=1}^{n} c_{i1}$.
    Denote $\overline{y}(x) = \sum_{i=1}^{n}\sum_{j=1}^{r_i} \frac{c_{ij}}{(x-x_i)^j} + \sum_{i=0}^{N} d_i x^i$.
    if m is a non-negative integer **then**
        Find polynomial solutions of degree m, say $P(x)$, of the differential equation

$$P'' + 2\overline{y}(x)P' + (\overline{y}'(x) + \overline{y}(x)^2 - a(x))P = 0\,.$$

        For each P, append $\overline{y}(x) + \frac{P'(x)}{P(x)}$ to $preSol$
    end if
end for
For each $y(x) \in preSol$, append $\omega := \frac{-1}{b_2}\left(y + \frac{b_2'}{2b_2} + \frac{b_1}{2}\right)$ to Sol
Return Sol.

5 Geometric aspect and strongly parametrizable first-order AODEs

In this section we study first-order AODEs from a geometric point of view. In particular we work on the class of first-order AODEs which satisfy a certain geometric condition.

They shall be called strongly parametrizable first-order AODEs. Beside, we construct for each such differential equation a quasi-linear first-order ODE which we call the associated differential equation. The key point is that the solution sets of a differential equation and ist associated ODE have a faithful relation.

In the second part of this section we investigate an application of the geometric idea to a specific ODE for computing Zolotarev polynomials.

5.1 Strong parametrizability

We consider the first-order AODE, $F(x, y, y') = 0$, where $F \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$. We are going to approach the differential equation from a geometric point of view. In order to do so, let us consider the algebraic curve $\mathcal{C}$ in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$ defined by $F(x, y, z) = 0$.

Definition 5.1.
$\mathcal{C}$ is called the corresponding curve of the differential equation $F(x, y, y') = 0$.

To connect rational solutions of the differential equation to properties of the corresponding curve, a natural way is using rational parametrization aspects of algebraic curves. We recall here the definition of parametrizations.

Definition 5.2.
Let $\mathcal{C}$ be an irreducible algebraic curve in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$.

- *A rational parametrization, or briefly a parametrization, of the curve $\mathcal{C}$ is a rational map $\mathcal{P} : \mathbb{A}^1(\overline{\mathbb{K}(x)}) \to \mathcal{C} \subseteq \mathbb{A}^2(\overline{\mathbb{K}(x)})$ such that the image of $\mathcal{P}$ is dense in $\mathcal{C}$ w. r. t. Zariski topology.*
- *If furthermore $\mathcal{P}$ is a birational equivalence, it is called a proper parametrization.*
- *If $\mathcal{P}$ has the form $\mathcal{P}(x, t) = (p_1(x, t), p_2(x, t))$ for some $p_1, p_2 \in \mathbb{K}(x, t)$, $\mathcal{P}$ is called a strong parametrization. An algebraic curve having a strong parametrization is called strongly parametrizable.*

Not all algebraic curve have a rational parametrization. It is well-known that an algebraic curve has a rational parametrization if and only if its genus is zero (see [24]). Once an algebraic curve has a rational parametrization, it always admits a proper parametrization. There are several algorithms for computing proper parametrizations of an algebraic curve of genus zero. Readers are referred to see [24] for such an algorithm.

The proper algorithm obtained from Algorithm Optimal-Parametrization in [24, Chap. 5] is, moreover, optimal in the sense that the field of coefficients of the parametrization is as small as possible. In particular, let $\mathcal{C}$ be an algebraic curve in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$ defined by $G(x, y, z) = 0$ for some $G \in \mathbb{K}(x)[y, z]$, and assume that $\mathcal{C}$ is irreducible and of genus zero. The smallest possible coefficient field of a proper parametrization of $\mathcal{C}$ is either $\mathbb{K}(x)$ or a quadratic extension of $\mathbb{K}(x)$. In the first case the curve is strongly parametrizable. In other words, Algorithm Optimal-Parametrization can decide whether a given algebraic curve is strongly parametrizable.

Definition 5.3.
A first-order AODE, $F(x, y, y') = 0$, is called strongly parametrizable if its corresponding curve is strongly parametrizable.

Note, that all quasi-linear first-order ODEs are strongly parametrizable. Furthermore, almost all of the first-order AODEs listed in the collection of Kamke [13] are strongly parametrizable. In fact 89% are strong, irreducible AODEs. The remaining ones consist of two classes. One part contains the reducible AODEs, hence, parametrizability of the factors can be considered. Most of the reducible AODEs have strongly parametrizable factors. The other part consists of AODEs for which the corresponding curve has genus greater than 0. Note, further, that there are only three first-order AODEs in Kamke's collection (1.372, 1.545 and 1.548) which are neither strongly parametrizable, nor maximally comparable. These three equations are autonomous, and hence by Corrolary 5.7 cannot have a rational solution.

The class of first-order AODEs covers around 64 percent of the entire collection of first-order ODEs in Kamke. Some of the remaining ODEs contain arbitrary functions. For certain choices of these functions, the ODEs might be algebraic. For further details on statistical investigations of Kamke's list we refer to [9]. We give here an example from Kamke's collection.

Example 5.4. (Kamke 1.527)
We consider the differential equation

$$F(x, y, y') = -y^5 - xy^4y' + y'^3 = 0\,. \tag{18}$$

Its corresponding curve $F(x, y, z) = 0$ has a strong parametrization

$$\mathcal{P}(t) := \left(\frac{t^3}{x^3(t^2 - x^3)}, \frac{-t^5}{x^4(t^2 - x^3)^2}\right).$$

Therefore, it is a strongly parametrizable first-order AODE.

Next we derive for each strongly parametrizable first-order AODE a quasilinear first-order AODE, for which rational solutions of the one can be obtained from the other. Let us consider the strongly parametrizable first-order AODE (1), $F(x, y, y') = 0$, where $F \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$, such that the corresponding curve, say $\mathcal{C}$, is strongly parametrizable. We might assume that a strong parametrization is given, for instance $\mathcal{P}(t) := (p_1(x, t), p_2(x, t))$ for some $p_1, p_2 \in \mathbb{K}(x, t)$. Let $y(x) \in \mathbb{K}(x)$ be a rational solution of the differential equation. Since $F(x, y(x), y'(x)) = 0$, the pair of rational functions $(y(x), y'(x))$ can be seen as a point on the corresponding curve. Comparing with the image of the rational map $\mathcal{P}$, two cases arise:

Case 1: $(y(x), y'(x)) \notin \operatorname{im} \mathcal{P}$, where $\operatorname{im} \mathcal{P}$ is the image of $\mathcal{P}$. Then $(y(x), y'(x))$ lies on the finite set $\mathcal{C} \setminus \operatorname{im} \mathcal{P}$.

Case 2: $(y(x), y'(x)) = \mathcal{P}(\omega(x))$ for some $\omega(x) \in \overline{\mathbb{K}(x)}$.

Let us go further with **Case 2**. Since $\mathcal{P}$ is birationally equivalent, we know that if such an ω exists, then it must be a rational function. In that case, it satisfies the condition:

$$\begin{cases} p_1(x, \omega(x)) = y(x)\,, \\ p_2(x, \omega(x)) = y'(x)\,. \end{cases}$$

Therefore,

$$\frac{d}{dx} p_1(x, \omega(x)) = p_2(x, \omega(x))\,.$$

By expanding the left hand side, we have

$$\omega'(x) \cdot \frac{\partial p_1}{\partial t}(x, \omega(x)) + \frac{\partial p_1}{\partial x}(x, \omega(x)) = p_2(x, \omega(x))\,.$$

Thus $\omega(x)$ is a rational solution of either the algebraic system

$$\begin{cases} \frac{\partial p_1}{\partial t}(x, \omega) = 0\,, \\ \frac{\partial p_1}{\partial x}(x, \omega) = p_2(x, \omega)\,, \end{cases} \tag{19}$$

or the quasilinear differential equation

$$\omega' = \frac{p_2(x, \omega) - \frac{\partial p_1}{\partial x}(x, \omega)}{\frac{\partial p_1}{\partial t}(x, \omega)}\,. \tag{20}$$

Definition 5.5.
We use notations from above.

- *The system (19) is called the associated algebraic system of the differential equation $F(x, y, y') = 0$, with respect to $\mathcal{P}$.*
- *The quasilinear differential equation (20) is called the associated differential equation of the differential equation $F(x, y, y') = 0$, with respect to $\mathcal{P}$.*

The discussion above yields a nice relation between rational solutions of the differential equation $F(x, y, y') = 0$ and its associated algebraic system and associated differential equation.

Theorem 5.6.
With notations as above, a rational function $y(x) \in \mathbb{K}(x)$ is a solution of the differential equation $F(x, y, y') = 0$ if and only if one of the following holds:

(i) $(y(x), y'(x))$ *lies in the finite set* $\mathcal{C} \setminus \operatorname{im} \mathcal{P}$.

(ii) $y(x) = p_1(x, \omega(x))$ *for some* $\omega(x) \in \mathbb{K}(x)$ *a rational solution of the associated algebraic system.*

(iii) $y(x) = p_1(x, \omega(x))$ *for some* $\omega(x) \in \mathbb{K}(x)$ *a rational solution of the associated differential equation.*

Proof. Clear from above discussion. □

Corollary 5.7.
If an autonomous AODE, $F(y, y') = 0$, has a rational solution then it is strongly parametrizable.

Proof. Let $y(x)$ be a rational solution of the AODE, then $y(x+c)$ is also a solution of the AODE (see [3]) and hence, $(y(x+c), y'(x+c))$ is a strong parametrization. □

As a consequence of Theorem 5.6, the problem of determining all rational solutions of a strongly parametrizable first-order AODE reduced to three smaller ones: determining the complement of $\operatorname{im}\mathcal{P}$ in $\mathcal{C}$, finding rational solutions of a system of two algebraic equations, and finding rational solutions of a quasilinear first-order ODE. The first two problems are well-known, and therefore, quite easy. The last one has been investigated in the previous section. Thus Theorem 5.6 yields the following algorithm for determining all rational solutions of a strongly parametrizable first-order AODE (see Algorithm 5).

Algorithm 5 Rational Solutions of strongly parametrizable first-order AODEs

Input: A strongly parametrizable first-order AODE, $F(x, y, y') = 0$, with a strong parametrization $\mathcal{P} = (p_1, p_2)$ of the associated curve.
Output: All rational solutions.

```
Sol = ∅
Determine the finite set C \ P
for all y(x) ∈ C \ P with y(x) ∈ K(x) and F(x, y(x), y'(x)) = 0 do
    append y(x) to Sol
end for
Find all rational solutions ω = ω(x) of the associated algebraic system
```

$$\begin{cases} \frac{\partial p_1}{\partial x}(x,\omega) = 0\,, \\ \frac{\partial p_1}{\partial t}(x,\omega) = p_2(x,\omega)\,, \end{cases}$$

append $p_1(x, \omega(x))$ to *Sol*

Use Algorithm 3 to rationally solve the associated differential equation

$$\omega' = \frac{p_2(x,\omega) - \frac{\partial p_1}{\partial x}(x,\omega)}{\frac{\partial p_1}{\partial t}(x,\omega)}\,.$$

For each of the rational solution $\omega = \omega(x)$, append $p_1(x, \omega(x))$ to *Sol*

return *Sol*.

Example 5.8.
Let us return to the differential equation (18) *from Example 5.4:*

$$F(x, y, y') = -y^5 - xy^4y' + y'^3 = 0\,.$$

As we have seen, the corresponding curve, say $\mathcal{C}$, has a strong parametrization $\mathcal{P}(t) := \left(\frac{t^3}{x^3(t^2-x^3)}, \frac{-t^5}{x^4(t^2-x^3)^2}\right)$. The rational map $\mathcal{P}^{-1} : \mathcal{C} \to \mathbb{A}^1(\overline{\mathbb{K}(x)})$ given by $\mathcal{P}^{-1}(y,z) = -\frac{x^2y^2}{z}$ is an inverse of $\mathcal{P}$. Therefore, $\mathcal{C} \setminus im(\mathcal{P}) = \emptyset$. The associated algebraic system and the associated differential equation of (18) with respect to $\mathcal{P}$ are

$$\begin{cases} -\frac{\omega^4(\omega^2-5x^3)}{x^4(\omega^2-x^3)^3} = 0\,, \\ -\frac{3\omega^3(\omega^2-2x^3)}{x^4(\omega^2-x^3)^2} = -\frac{\omega^5}{x^4(\omega^2-x^3)^2}\,, \end{cases}$$

and

$$\omega' = \frac{2\omega}{x}\,,$$

respectively. This algebraic system and the differential equation are quite easy to solve. The algebraic system has only the zero solution, while the differential equation has a one-parameter class of rational solution, $\omega(x) = cx^2$. Hence, the set of all rational solutions of the differential equation (18) is the one-parameter class of functions $y(x) = \frac{c^3}{c^2x-1}$.

5.2 An application: Zolotarev Polynomials

In approximation theory Zolotarev polynomials are defined to be those polynomials which deviate least from zero. The precise definition is not important to us here. What is interesting, however, is that Zolotarev polynomials $y(x) = Z_n(x)$ satisfy a first-order AODE (compare [18, 25]),

$$n^2(x-\beta)^2(1-y^2) - (1-x^2)(x-\gamma)(x-\delta)y'^2 = 0 \qquad (21)$$

This AODE is strongly parametrizable over $\mathbb{K}(x)$ by $\mathcal{P}(t) = (p_1, p_2)$ with

$$\mathcal{P} = \left(\frac{4n^2t^2 + (x^2-1)(x-\gamma)(x-\delta)}{4n^2t^2 - (x^2-1)(x-\gamma)(x-\delta)}, \frac{4n^2t(x-\beta)}{4n^2t^2 - (x^2-1)(x-\gamma)(x-\delta)}\right).$$

Using this parametrization in Algorithm 5 we transform the AODE to an ODE of degree 1 in the derivative. Kovacic's algorithm (Algorithm 4) provides a method to solve this ODE. However, we have to deal with the additional parameters β, γ, δ. According to [25] the so called proper Zolotarev polynomials form a one-parameter class of solutions of the above AODE. The algorithm from [9] shows that there is no rational general solution. Hence, there has to be a certain relation between the parameters in order to admit a non-constant polynomial solution. We modify Algorithm 4 slightly in order to attack this additional requirement. In step 11 we need to solve a second order ODE by coefficient comparison. We want to solve the system of equations not only in the coefficients of the undetermined polynomial P but also w. r. t. the parameters β, γ, δ. This means we have an algebraic curve in higher dimensional space. By Gröbner basis computations we can project the space curve to a plane curve which we can try to parametrize.

For $n \in \{2, 3, 4\}$ a rational parametrization of the plane curve can be found and hence, explicit expressions of proper Zolotarev polynomials can be computed. For higher degree this is still subject of further research. According to [25] so far explicit expressions have already been found up to degree 4 and methods have been investigated to determine Zolotarev polynomials algebraically (c. f. for instance [17]). Our method is able to compute explicit expressions also for degrees 5 and 6 and gives an algebraic expression in any case. In this paper we show as a motivating example the case of degree 4, which is interestig enough as an application of our method, but does not require further considerations. For more details as well as for higher degrees and possible issues thereof we refer to [8].

Example 5.9.
We compute the one-parameter class of proper Zolotarev polynomials of degree $n = 4$. Coefficient comparison in Step 11 yields a set of algebraic equations in $\beta, \gamma, \delta, c_0$, where c_0 is the constant coefficient of the polynomial P which solves (17). Note, that in this specific case we have $m = 1$. By Gröbner basis computation we eliminate γ and c_0 and get the equation

$$256\beta^4 - 128\beta^3(3\delta - 1) + 64\beta^2\left(3\delta^2 - 2\delta - 1\right) - 8\beta\left(5\delta^3 - 5\delta^2 - 9\delta + 1\right) + (\delta + 1)^2\left(3\delta^2 - 18\delta + 11\right) = 0\,.$$

This equation defines an algebraic curve which can be properly parametrized by $\mathcal{Q}(t) = \left(\frac{t^4-2t^2-3}{8t}, \frac{t^4+2t^3-1}{2t}\right)$. Hence, we conclude that

$$\beta = \frac{t^4 - 2t^2 - 3}{8t}\,, \quad \gamma = \frac{t^4 - 2t^3 - 1}{2t}\,, \quad \delta = \frac{t^4 + 2t^3 - 1}{2t}\,, \quad c_0 = \frac{1}{2}\left(1 - t - t^2 - t^3\right)\,.$$

Then the solution of the associated ODE is

$$\omega(x) = -\frac{(x+1)\left(-t^3 + t^2 - t + 2x - 1\right)\left(-t^4 - 2t^3 + 2tx + 1\right)}{16t\left(t^3 + t^2 + t - 2x - 1\right)}\,.$$

Hence, we get a proper Zolotarev polynomial

$$Z_4(x,t) = p_1(x, \omega(x)) = \frac{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4}{\left(t^2-1\right)^3\left(t^2+1\right)^2}\,,$$

with

$$a_0 = 2t\left(3t^6 + t^4 + t^2 - 1\right)\,, \qquad a_1 = t^{10} - t^8 - 2t^6 - 10t^4 - 7t^2 + 3\,,$$
$$a_2 = -2t\left(3t^6 + t^4 + t^2 - 5\right)\,, \qquad a_3 = 4\left(3t^4 + 2t^2 - 1\right)\,,$$
$$a_4 = -8t\,.$$

Note, that this is exactly the same expression as described in [25].

6 Conclusion

In constrast to several previous papers here we concentrate on determining all rational solutions of AODEs. For two classes of first-order AODEs we give algorithms for finding all rational solutions. These classes are maximally comparable and strongly parametrizable AODEs. Together they form a big part of the family of first-order AODEs. In Kamke's collection only three of the first-order AODEs are not covered. These three are autonomous and not strongly parametrizable, so they cannot have a rational solution.

References

[1] J. M. Aroca, J. Cano, R. Feng, and X.-S. Gao. Algebraic General Solutions of Algebraic Ordinary Differential Equations. In M. Kauers, editor, *Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation*, ISSAC '05, pages 29–36, New York, 2005. ACM.

[2] A. Eremenko. Rational solutions of first-order differential equations. *Annales Academiae Scientiarum Fennicae. Mathematica*, 23(1):181–190, 1998.

[3] R. Feng and X.-S. Gao. Rational General Solutions of Algebraic Ordinary Differential Equations. In J. Gutierrez, editor, *Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation*, ISSAC '04, pages 155–162, New York, 2004. ACM.

[4] R. Feng and X.-S. Gao. A polynomial time algorithm for finding rational general solutions of first order autonomous ODEs. *Journal of Symbolic Computation*, 41(7):739–762, 2006.

[5] G. Grasegger. Radical Solutions of Algebraic Ordinary Differential Equations. In K. Nabeshima, editor, *Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation*, ISSAC '14, pages 217–223, New York, 2014. ACM.

[6] G. Grasegger, A. Lastra, J. R. Sendra, and F. Winkler. On Symbolic Solutions of Algebraic Partial Differential Equations. In V.P. Gerdt et al., editor, *Computer Algebra in Scientific Computing*, volume 8660 of *Lecture Notes in Computer Science*, pages 111–120. Springer International Publishing, 2014.

[7] G. Grasegger, A. Lastra, J. R. Sendra, and F. Winkler. A solution method for autonomous first-order algebraic partial differential equations. *Journal of Computational and Applied Mathematics*, 300:119–133, 2016.

[8] G. Grasegger and N. T. Vo. An Algebraic-Geometric Method for Computing Zolotarev Polynomials. Technical Report 16-02, RISC, Johannes Kepler University Linz, 2016.

[9] G. Grasegger, N. T. Vo, and F. Winkler. Statistical Investigation of First-Order Algebraic ODEs and their Rational General Solutions. Technical Report 15-19, RISC, Johannes Kepler University Linz, 2015.

[10] G. Grasegger and F. Winkler. Symbolic solutions of first-order algebraic ODEs. In *Computer Algebra and Polynomials*, volume 8942 of *Lecture Notes in Computer Science*, pages 94–104. Springer International Publishing, 2015.

[11] Y. Huang, L. X. C. Ngô, and F. Winkler. Rational General Solutions of Higher Order Algebraic ODEs. *Journal of Systems Science and Complexity*, 26(2):261–280, 2013.

[12] E. Hubert. The General Solution of an Ordinary Differential Equation. In Y. N. Lakshman, editor, *Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation*, ISSAC '96, pages 189–195, New York, 1996. ACM.

[13] E. Kamke. *Differentialgleichungen: Lösungsmethoden und Lösungen.* B. G. Teubner, Stuttgart, 1997.

[14] J. J. Kovacic. An algorithm for solving second order linear homogeneous differential equations. *Journal of Symbolic Computation*, 2:3–43, 1986.

[15] A. A. Krushel'nitskij. Polynomial solutions of algebraic differential equations. *Differential Equations*, 24(12):1393–1398, 1988.

[16] A. Lastra, L. X. C. Ngô, J. R. Sendra, and F. Winkler. Rational General Solutions of Systems of Autonomous Ordinary Differential Equations of Algebro-Geometric Dimension One. *Publicationes Mathematicae Debrecen*, 86(1–2):49–69, 2015.

[17] D. Lazard. Solving Kaltofen's Challenge on Zolotarev's Approximation Problem. In *Proceedings of the 2006 International Symposium on Symbolic and Algebraic Computation*, ISSAC '06, pages 196–203, New York, NY, USA, 2006. ACM.

[18] V. A. Markov. Über Polynome, die in einem gegebenen Intervalle möglichst wenig von Null abweichen. *Mathematische Annalen*, 77:213–258, 1916.

[19] M. Matsuda. *First order algebraic differential equations. A differential algebraic approach*, volume 804 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1980.

[20] L. X. C. Ngô and F. Winkler. Rational general solutions of first order non-autonomous parametrizable ODEs. *Journal of Symbolic Computation*, 45(12):1426–1441, 2010.

[21] L. X. C. Ngô and F. Winkler. Rational general solutions of parametrizable AODEs. *Publicationes Mathematicae Debrecen*, 79(3–4):573–587, 2011.

[22] L. X. C. Ngô and F. Winkler. Rational general solutions of planar rational systems of autonomous ODEs. *Journal of Symbolic Computation*, 46(10):1173–1186, 2011.

[23] F. Schwarz. *Algorithmic Lie theory for solving ordinary differential equations.* Chapman & Hall/CRC, Boca Raton, FL, 2008.

[24] J. R. Sendra, F. Winkler, and S. Pérez-Díaz. *Rational Algebraic Curves, A Computer Algebra Approach*, volume 22 of *Algorithms and Computation in Mathematics.* Springer-Verlag, Berlin Heidelberg, 2008.

[25] A. Shadrin. Twelve proofs of the Markov inequality. In D. K. Dimitrov, G. Nikolov, and R. Uluchev, editors, *Approximation theory: a volume dedicated to Borislav Bojanov*, pages 233–298. Prof. Marin Drinov Academic Publishing House, Sofia, 2004.

[26] N. T. Vo and F. Winkler. Algebraic General Solutions of First Order Algebraic ODEs. In P. V. Gerdt, W. Koepf, M. W. Seiler, and V. E. Vorozhtsov, editors, *Computer Algebra in Scientific Computing*, volume 9301 of *Lecture Notes in Computer Science*, Cham, 2015. Springer International Publishing.