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Integration in Finite Terms of non-Liouvillian Functions

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Introduction

We want to find all linear combinations of given integrands which have an integral that can be expressed in terms of given functions and elementary functions applied to them. We report on recent progress in extending results in [4, 6, 2] to a complete algorithm for more general differential fields, which we call admissible.

Problem: parametric elementary integration
Given a differential field (F, D) and $f_0, \dots, f_m \in F$.
Compute a vector space basis of all $(c_0, \dots, c_m) \in \mathrm{Const}(F)^{m+1}$ such that there exists g from some elementary extension of (F, D) with

$$c_m f_m + \dots + c_0 f_0 = Dg$$

and compute corresponding g's.

Definition: We call a differential field $(F, D) = (C(t_1, \dots, t_n), D)$ admissible, if

1. all t_i are algebraically independent over C,
2. $\mathrm{Const}(F) = C$, and
3. for each t_i and $F_{i-1} := C(t_1, \dots, t_{i-1})$ either
   (a) t_i is a Liouvillian monomial over F_{i-1}, i.e. either $Dt_i \in F_{i-1}$ or $\frac{Dt_i}{t_i} \in F_{i-1}$; or
   (b) there exists $q \in F_{i-1}[t_i]$ with $\deg(q) \geq 2$ such that $Dt_i = q(t_i)$ holds and $Dy = q(y)$ does not have a solution $y \in \overline{F_{i-1}}$.

Algorithm

Structure of one step in the recursive reduction algorithm:
Input integrands from $F_{n-1}(t_n)$ to find elementary integrals over $F_{n-1}(t_n)$

1. Hermite Reduction for reducing the denominators
2. Residue Criterion for computing part of the integral in extensions of $F_{n-1}(t_n)$
3. further reduce integrands by solving auxiliary differential problems in F_{n-1}
4. remaining integrands are from F_{n-1}, reduce elementary integration over $F_{n-1}(t_n)$ to elementary integration over F_{n-1}

Recursive call with integrands from F_{n-1} to find elementary integrals over F_{n-1}

Theorem (Main Result)
Let $(F, D) = (C(t_1, \dots, t_n), D)$ be an admissible differential field involving at most two non-Liouvillian monomials, which then have to be at consecutive positions.
Then we can solve the parametric elementary integration problem over (F, D).

For $(F, D) = (C(t_1, \dots, t_n), D)$ being any admissible differential field we have a good heuristic, for which we could not construct a counterexample so far.

Examples

$$\int \frac{\mathrm{Li}_3(x) - x\mathrm{Li}_2(x)}{(1-x)^2}\,dx = \frac{x}{1-x}\left(\mathrm{Li}_3(x) - \mathrm{Li}_2(x)\right) + \frac{\ln(1-x)^2}{2}$$

$$\int \frac{(a+b)x - a}{x^{a+1}(1-x)^{b+1}} B_x(a,b)\,dx = \frac{B_x(a,b)}{x^a(1-x)^b} + \ln\left(\frac{1-x}{x}\right)$$

$$\int \frac{xE(x)^2}{(1-x^2)(E(x)-K(x))^2}\,dx = \frac{E(x)}{E(x)-K(x)} - \ln(x)$$

$$\int \frac{1}{xJ_n(x)Y_n(x)}\,dx = \frac{\pi}{2}\ln\left(\frac{Y_n(x)}{J_n(x)}\right)$$

Auxiliary Differential Problems

Problem: parametric linear ODEs
Given (F, D) and $a_0, \dots, a_{d-1}, f_0, \dots, f_m \in F$.
Compute a vector space basis of all $(g, c_0, \dots, c_m) \in F \times \mathrm{Const}(F)^{m+1}$ such that

$$D^d g + a_{d-1} D^{d-1} g + \dots + a_0 g = c_0 f_0 + \dots + c_m f_m.$$

Theorem
Let $(F, D) = (C(t_1, \dots, t_n), D)$ be an admissible differential field where only t_n is allowed to be non-Liouvillian.
Then we can solve parametric linear ODEs in (F, D).

Algebraic Representation of Functions

Liouvillian functions:
The class of Liouvillian functions is generated from constants by

- performing rational operations $(+, -, \cdot, /)$,
- taking solutions of polynomials with Liouvillian coefficients (algebraic case),
- (indefinite) integration, and
- applying $\exp$.

Examples: $\log$, $\exp$, trigonometric/hyperbolic functions and their inverses, logarithmic and exponential integrals, polylogarithms, error functions, Fresnel functions, incomplete gamma function, etc.

Functions satisfying a pair of coupled ODEs:

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \qquad (1)$$

Let $\Phi(x) := \begin{pmatrix} y_1(x) & z_1(x) \\ y_2(x) & z_2(x) \end{pmatrix}$ be a fundamental matrix. Apart from $y_1(x)$ we consider

$$t(x) := \frac{y_2(x)}{y_1(x)}, \qquad \tilde{t}(x) := \frac{z_1(x)}{y_1(x)}, \qquad w(x) := \det \Phi(x).$$

These functions satisfy the following system, with (2) being a Riccati equation:

$$\begin{aligned} t'(x) &= -a_{12}(x)t(x)^2 + (a_{22}(x) - a_{11}(x))t(x) + a_{21}(x) && (2) \\ y_1'(x) &= (a_{12}(x)t(x) + a_{11}(x))y_1(x) && (3) \\ w'(x) &= (a_{11}(x) + a_{22}(x))w(x) && (4) \\ \tilde{t}'(x) &= \tfrac{a_{12}(x)}{y_1(x)^2} w(x) && (5) \end{aligned}$$

This system is uncoupled, which makes it fit into the tower framework of our admissible fields.

Examples: orthogonal polynomials, associated Legendre functions, Bessel and Airy functions, complete elliptic integrals, hypergeometric functions, Mathieu functions, Heun functions, etc.

Application to Parameter Integrals

For finding a recurrence equation for the parameter integral $I(n) := \int_a^b f(n,x)\,dx$ we choose $f_i(x) := f(n+i, x)$ as input. Then $c_m f_m + \dots + c_0 f_0 = Dg$ corresponds to

$$c_m(n)I(n+m) + \dots + c_0(n)I(n) = g(n,b) - g(n,a).$$

For finding a differential equation for the parameter integral $I(y) := \int_a^b f(y,x)\,dx$ we choose $f_i(x) := \frac{\partial^i f}{\partial y^i}(y,x)$ as input. Then $c_m f_m + \dots + c_0 f_0 = Dg$ corresponds to

$$c_m(y)I^{(m)}(y) + \dots + c_0(y)I(y) = g(y,b) - g(y,a).$$

Example: For $n \in \mathbb{N}$ and $y > 0$ let us define the following parameter integral

$$A_n(y) := \int_0^1 xP_n(1-2x^2)J_0(yx)\,dx.$$

We can compute by our algorithm, e.g., the relations

$$A_{n+1}(y) = -\tfrac{4(n+1)}{y}A_n'(y) + \tfrac{8n(n+1)-y^2}{y^2}A_n(y), \qquad A_n''(y) + \tfrac{3}{y}A_n'(y) + \tfrac{y^2-4n(n+1)}{y^2}A_n(y) = 0.$$

Specializing $n = 0$ in the definition of $A_n(y)$, our algorithm can explicitly compute the initial value

$$A_0(y) = \tfrac{1}{y}J_1(y).$$

From this we can deduce $A_n(y) = \frac{1}{y}J_{2n+1}(y)$.

Summary

- extended previous decision procedure [6, 3] to include certain non-Liouvillian functions
- can be applied heuristically to certain non-admissible differential fields as well
- solution of parametric linear ODEs based on [5, 1] (joint work with Michael F. Singer at NCSU)
- successful on examples where current computer algebra software fails
- application: certified identities/evaluation of definite integrals by parametric indefinite integration

References

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