Complexity of Creative Telescoping for Bivariate Rational Functions*

Alin Bostan, Shaoshi Chen, Frédéric Chyzak
Algorithms Project-Team, INRIA
Paris-Rocquencourt
78153 Le Chesnay (France)
{alin.bostan,shaoshi.chen,frederic.chyzak}@inria.fr

Ziming Li
Key Laboratory of Mathematics Mechanization,
Academy of Mathematics and System Sciences
100190 Beijing (China)
zmli@mmrc.iss.ac.cn

ABSTRACT

The long-term goal initiated in this work is to obtain fast algorithms and implementations for definite integration in Almkvist and Zeilberger's framework of (differential) creative telescoping. Our complexity-driven approach is to obtain tight degree bounds on the various expressions involved in the method. To make the problem more tractable, we restrict to *bivariate rational* functions. By considering this constrained class of inputs, we are able to blend the general method of creative telescoping with the well-known Hermite reduction. We then use our new method to compute diagonals of rational power series arising from combinatorics.

Categories and Subject Descriptors:
I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulations — *Algebraic Algorithms*

General Terms: Algorithms, Theory.

Keywords: Hermite reduction, creative telescoping.

1. INTRODUCTION

The long-term goal of the research initiated in the present work is to obtain fast algorithms and implementations for the definite integration of general special functions, in a complexity-driven perspective.

As most special-function integrals cannot be expressed in closed form, their evaluation cannot be based on table look-ups only, and even when closed forms are available, they may prove to be intractable in further manipulations. In both cases, the difficulty can be mitigated by representing functions by annihilating differential operators. This motivated Zeilberger to introduce a method now known as *creative telescoping* [20], which applies to a large class of special functions: the D-finite functions [14] defined by sets of linear differential equations of any order, with polynomial coefficients. Zeilberger's method applies in general to multiple integrals and sums.

*We warmly thank the referees for their very helpful comments. — AB and FC were supported in part by the Microsoft Research – Inria Joint Centre, and SC and ZL by a grant of the National Natural Science Foundation of China (No. 60821002).

ISSAC 2010, 25–28 July 2010, Munich, Germany.

A sketch of Zeilberger's method is as follows. Given a D-finite function f of the variables x and y, the definite integral $F(x) = \int_\alpha^\beta f(x,y)\,dy$ is D-finite, and a linear differential equation satisfied by F can be constructed [20]. To explain this, let k be a field of characteristic zero, D_x and D_y be the usual derivations on the rational-function field $k(x,y)$, both restricting to zero on k, and let $k(x,y)\langle D_x, D_y\rangle$ be the ring of linear differential operators over $k(x,y)$. The heart of the method is to solve the *differential telescoping equation* (1) below for $L \in k[x]\langle D_x\rangle \setminus \{0\}$ and $g = R(f)$ for some $R \in k(x,y)\langle D_x, D_y\rangle$. The operator L is called a *telescoper* for f, and g a *certificate* of L for f. Under the assumption

$$\lim_{y\to\alpha} g(x,y) = \lim_{y\to\beta} g(x,y) \quad \text{for } x \text{ in some domain,}$$

$L(x, D_x)$ is then proved to be an annihilator of F.

The main emphasis in works since the 1990's has been on finding telescopers of order minimal over all telescopers for f, which are called *minimal telescopers*. (Two minimal telescopers differ by a multiplicative factor in $k(x)$.) In view of the computational difficulty of solving (1), there has been special attention to subclasses of inputs. Of particular importance is the case of hyperexponential functions, defined by first-order differential equations, studied by Almkvist and Zeilberger in [1]. Their method is a direct differential analogue of Zeilberger's algorithm for the recurrence case [21].

On the other hand, very little is known about the complexity of creative telescoping: the only related result seems to be an analysis in [9] of an algorithm for hyperexponential indefinite integration. In order to get complexity estimates, we simplify the problem by restricting to a smaller class of inputs, namely that of bivariate rational functions. Although restricted, this class already has many applications, for instance in combinatorics, where many nontrivial problems are encoded as diagonals of rational formal power series, themselves expressible as integrals. Our goal thus reads as follows.

Problem *Given $f = P/Q \in k(x,y)\setminus\{0\}$, find a pair (L, g) with $L = \sum_{i=0}^{\rho} \eta_i(x) D_x^i$ in $k[x]\langle D_x\rangle \setminus \{0\}$ and g in $k(x,y)$ such that*

$$L(x, D_x)(f) = D_y(g). \qquad (1)$$

By considering this more constrained class of inputs, we are indeed able to blend the general method of creative telescoping with the well-known Hermite reduction [10].

Essentially two algorithms for minimal telescopers can be found in the literature: The classical way [1] is to apply a

	Method	$\deg_{D_x}(L)$	$\deg_x(L)$	$\deg_x(g)$	$\deg_y(g)$	Complexity	
Minimal	Hermite reduction (new)	$\leq d_y$	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_y^2)$	$\tilde{\mathcal{O}}(d_x d_y^{\omega+3})$	Las Vegas
Telescoper	Almkvist and Zeilberger	$\leq d_y$	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_y^2)$	$\tilde{\mathcal{O}}(d_x d_y^{2\omega+2})$	Las Vegas
Nonminimal	Lipshitz elimination	$\leq 6(d_x+1)(d_y+1)$	$\mathcal{O}(d_x d_y)$	$\mathcal{O}(d_x^2 d_y)$	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_x^{3\omega} d_y^{3\omega})$	deterministic
Telescoper	Cubic size	$\leq 6d_y$	$\mathcal{O}(d_x d_y)$	$\mathcal{O}(d_x d_y)$	$\mathcal{O}(d_y^2)$	$\mathcal{O}(d_x^{\omega} d_y^{3\omega})$	deterministic

Figure 1: Complexity of creative telescoping methods (under Hyp. (H')), together with bounds on output

differential analogue of Gosper's indefinite summation algorithm, which reduces the problem to solving an auxiliary linear differential equation for polynomial solutions. An algorithm developed later in [7] (see also [12]) performs Hermite reduction on f to get an additive decomposition of the form $f = D_y(a) + \sum_{i=1}^{m} u_i/v_i$, where the u_i and v_i are in $k(x)[y]$ and the v_i are squarefree. Then, the algorithm in [1] is applied to each u_i/v_i to get a telescoper L_i minimal for it. The least common left multiple of the L_i's is then proved to be a minimal telescoper for f. This algorithm performs well only for specific inputs (both in practice and from the complexity viewpoint), but it inspired our Lemma 22 via [12].

As a first contribution in this article, we present a new, provably faster algorithm for computing minimal telescopers for bivariate rational functions. Instead of a single use of Hermite reduction as in [12], we apply Hermite reduction to the $D_x^i(f)$'s, iteratively for $i = 0, 1, \ldots$, which yields

$$D_x^i(f) = D_y(g_i) + \frac{w_i}{w} \tag{2}$$

for some factor w of the squarefree part of the denominator of f. If $\eta_0, \ldots, \eta_\rho \in k(x)$ are not all zero and such that $\sum_{i=0}^{\rho} \eta_i w_i = 0$, then the operator $\sum_{i=0}^{\rho} \eta_i D_x^i$ is a telescoper for f, and more specifically, the first nontrivial linear relation obtained in this way yields a minimal telescoper for f.

As a second contribution, we give the first proof of a polynomial complexity for creative telescoping on a specific class of inputs, namely on bivariate rational functions. For *minimal* telescopers, only a polynomial bound on d_x (but none on d_y) was given for special inputs in [7]; more specifically, we derive complexity estimates for all mentioned methods (see Fig. 1), showing that our approach is faster. Furthermore, we analyse the bidegrees of *non minimal* telescopers generated by other approaches: Lipshitz' work [13] can be rephrased into an existence theorem for telescopers with polynomial size; the approach followed in the recent work on algebraic functions [3] leads to telescopers of smaller degree sizes. These are new instances of the philosophy, promoted in [3], that relaxing minimality can produce smaller outputs.

A third contribution is a fast Maple implementation [22], incorporating a careful implementation of the original Hermite reduction algorithm, making use of the special form of w_i/w in (2) and of usual modular techniques (probabilistic rank estimate) to determine when to invoke the solver for linear algebraic equations. Experimental results indicate that our implementation outperforms Maple's core routine.

Note that for the fastest method we propose, denoted by `H1` in Tables 1–3, we chose to output the certificate as a mere sum of (small) rational functions, without any form of normalisation. This choice seems to be uncommon for creative-telescoping algorithms, but a motivation is how the certificate is used in practice: Very often, like for applications to diagonals in § 5, the certificate is actually not needed. In other applications, the next step of the method of creative telescoping is to integrate (1) between α and β, leading to $L(F)(x) = g(x, \alpha) - g(x, \beta)$. Therefore, only evaluations of the certificate are really needed, and normalisation can be postponed to after specialising at α and β.

The end of this section, § 1.1, provides classical complexity results, notation, and hypotheses that will be used throughout. We then study Hermite reduction over $k(x)$ in § 2, proving output degree bounds and a low-complexity algorithm. This is then applied in § 3 to derive our new algorithm for creative telescoping, and to compare its complexity with that of Almkvist and Zeilberger's approach. For nonminimal telescopers, we show the existence of some of lower arithmetic size in § 4: cubic for nonminimal order instead of quartic for minimal order. See the summary in Figure 1, where the low complexity of algorithms for minimal telescopers relies on Storjohann and Villard's algorithms [19], thus inducing a *certified* probabilistic feature. We apply our results to the calculation of diagonals in § 5, and describe our implementation and comment on execution timings in § 5.

1.1 Background on complexity — Notation

We recall basic notation and complexity facts for later use. Let k be again a field of characteristic zero. Unless otherwise specified, all complexity estimates are given in terms of arithmetical operations in k, which we denote by "ops". Let $k[x]_{\leq d}^{m\times n}$ be the set of $m \times n$ matrices with coefficients in $k[x]$ of degree at most d. Let $\omega \in [2, 3]$ be a feasible exponent of matrix multiplication, so that two matrices from $k^{n\times n}$ can be multiplied using $\mathcal{O}(n^\omega)$ ops. Facts 1 and 2 below show the complexity of multipoint evaluation, rational interpolation, and algebraic operations on polynomial matrices using fast arithmetic, where the notation $\tilde{\mathcal{O}}(\cdot)$ indicates cost estimates with hidden logarithmic factors [6, Def. 25.8].

Fact 1 *For $p \in k[x]$ of degree less than n, pairwise distinct $u_0, \ldots, u_{n-1}$ in k, and $v_0, \ldots, v_{n-1} \in k$, we have:*

(i) Evaluating p at the u_i's takes $\tilde{\mathcal{O}}(n)$ ops.

(ii) For $m \in \{1, \ldots, n\}$, constructing $f = s/t \in k(x)$ with $\deg_x(s) < m$ and $\deg_x(t) \leq n - m$ such that $t(u_i) \neq 0$ and $f(u_i) = v_i$ for $0 \leq i \leq n-1$ takes $\tilde{\mathcal{O}}(n)$ ops.

Fact 2 *For M in $k[x]_{\leq d}^{m\times n}$, $d > 0$, we have:*

(i) If $M = \begin{pmatrix} M_1 & M_2 \end{pmatrix}$ is an invertible $n \times n$ matrix with $M_i \in k[x]_{\leq d_i}^{n\times n_i}$, where $i = 1, 2$ and $n_1 + n_2 = n$, then the degree of $\det(M)$ is at most $n_1 d_1 + n_2 d_2$.

(ii) If $M = \begin{pmatrix} M_1 & M_2 \end{pmatrix}$ is not of full rank and with $M_i \in k[x]_{\leq d_i}^{m\times n_i}$, where $i = 1, 2$ and $n_1 + n_2 = n$, then there exists a nonzero $u \in k[x]^n$ with coefficients of degree at most $n_1 d_1 + n_2 d_2$ such that $Mu = 0$.

(iii) The rank r and a basis of the null space of M can be computed using $\tilde{\mathcal{O}}(nmr^{\omega-2}d)$ ops.

(For proofs, see [6, Cor. 10.8, 5.18, 11.6] and [19, Th. 7.3].)

We call *squarefree factorisation* of $Q \in k[x,y]\backslash k[x]$ w.r.t. y the unique product $qQ_1Q_2^2\cdots Q_m^m$ equal to Q for $q \in k[x]$ and $Q_i \in k[x,y]$ satisfying $\deg_y(Q_m) > 0$ and such that the Q_i's are primitive, squarefree, and pairwise coprime. The *square-free part* Q^* *of* Q w.r.t. y is the product $Q_1Q_2\cdots Q_m$. Let Q^- denote the polynomial Q/Q^*, and $\mathrm{lc}_y(Q)$ the leading coefficient of Q w.r.t. y. The following two formulas about Q, Q^*, and Q^- can be proved by mere calculations.

Fact 3 *Let $\hat{Q}_i$ denote Q^*/Q_i. Then we have*

(i) $Q^*D_y(Q^-)/Q^- = \sum_{i=1}^m (i-1)\hat{Q}_iD_y(Q_i) \in k[x,y]$;

(ii) $D_y(Q)/Q^- = \sum_{i=1}^m i\hat{Q}_iD_y(Q_i) \in k[x,y]$.

Let $f = P/Q$ be a nonzero element in $k(x,y)$, where P, Q are two coprime polynomials in $k[x,y]$. The degree of f in x is defined to be $\max\{\deg_x(P), \deg_x(Q)\}$, and denoted by $\deg_x(f)$. The degree of f in y is defined similarly. The *bidegree* of f is the pair $(\deg_x(f), \deg_y(f))$, which is denoted by $\mathrm{bideg}(f)$. The bidegree of f is said to be *bounded (above) by* (α,β), written $\mathrm{bideg}(f) \leq (\alpha,\beta)$, when $\deg_x(f) \leq \alpha$ and $\deg_y(f) \leq \beta$.

We say that $f = P/Q$ is *proper* if the degree of P in y is less than that of Q. For creative telescoping, we may always assume w.l.o.g. that $f = P/Q$ is proper. If not, rewrite $f = D_y(p) + \bar{f}$ with $p \in k(x)[y]$ and $\bar{f}$ proper. A telescoper L for $\bar{f}$ with certificate $\bar{g}$ is a telescoper for f with certificate $L(p) + \bar{g}$.

Hypothesis (H) *From now on, P and Q are assumed to be nonzero polynomials in $k[x,y]$ such that $\deg_y(P) < \deg_y(Q)$, $\gcd(P,Q) = 1$, and Q is primitive w.r.t. y.*

Notation *From now on, we write (d_x, d_y), (d_x^*, d_y^*), and (d_x^-, d_y^-) for the bidegrees of Q, Q^*, and Q^-, respectively.*

The following hypothesis makes our estimates concise.

Hypothesis (H') *Occasionally, we shall require the extended hypothesis: Hypothesis (H) and $\deg_x(P) \leq d_x$.*

2. HERMITE REDUCTION

Let K be a field of characteristic zero, either k or $k(x)$ in what follows. Let $K(y)$ be the field of rational functions in y over K, and D_y be the usual derivation on it. For a rational function $f \in K(y)$, *Hermite reduction* [10] computes rational functions g and $r = a/b$ in $K(y)$ satisfying

$$f = D_y(g) + r, \quad \deg_y(a) < \deg_y(b), \quad b \text{ is squarefree.} \quad (3)$$

Horowitz and Ostrogradsky's method [15, 11] computes the same decomposition as in (3) by solving a linear system. For the details of those methods, see [4, Chapter 2].

Lemma 4 *If f is proper, a pair (g,r) satisfying (3) for proper g, r is unique.*

Proof. This is a consequence of [11, Theorem 2.10] after writing r as a sum $\sum_{i=1}^m \alpha_i/(x-b_i)$ and integrating. ☐

Lemma 5 *Let f be a nonzero rational function in $K(y)$ of degree at most n in y, then Hermite reduction on f can be performed using $\tilde{\mathcal{O}}(n)$ operations in K.*

Proof. See [6, Theorem 22.7]. ☐

In contrast, the method of Horowitz and Ostrogradsky takes $\mathcal{O}(n^\omega)$ operations in K [6, § 22.2]. Thus, Hermite's method is quasi-optimal and asymptotically faster than the former.

From now on, we fix $K = k(x)$ and analyse the complexity of Hermite reduction over $k(x)$ in terms of operations in k. To this end, we use an evaluation-interpolation approach.

2.1 Output size estimates

We derive an upper bound on the bidegrees of g and r satisfying (3) by studying the linear system in [11].

Analysing Hermite reduction (under (H)) shows the existence of $A, a \in k(x)[y]$ with $\deg_y(A) < d_y^-$, $\deg_y(a) < d_y^*$ and

$$\frac{P}{Q} = D_y\left(\frac{A}{Q^-}\right) + \frac{a}{Q^*}. \quad (4)$$

In order to bound the bidegrees of A and a, we reformulate (4) into the equivalent form

$$P = Q^*D_y(A) - \left(\frac{Q^*D_y(Q^-)}{Q^-}\right)A + Q^-a, \quad (5)$$

where $Q^*D_y(Q^-)/Q^-$ is a polynomial in $k[x,y]$ of bidegree at most (d_x^*, d_y^*-1) by Fact 3. Viewing A and a as polynomials in $k(x)[y]$ with undetermined coefficients, we form the following linear system, equivalent to (5),

$$\begin{pmatrix}\mathcal{H}_1 & \mathcal{H}_2\end{pmatrix}\begin{pmatrix}\hat{A}\\ \hat{a}\end{pmatrix} = \hat{P}, \quad (6)$$

where $\mathcal{H}_1 \in k[x]_{\leq d_x^*}^{d_y \times d_y^-}$, $\mathcal{H}_2 \in k[x]_{\leq d_x^-}^{d_y \times d_y^*}$, and $\hat{A}$, $\hat{a}$, and $\hat{P}$ are the coefficient vectors of A, a, and P with sizes d_y^-, d_y^*, and d_y, respectively. Under the constraint of properness of A/Q^- and a/Q^*, (A,a) is unique by Lemma 4. Then (6) has a unique solution, which leads to the following lemma.

Lemma 6 *The matrix $\begin{pmatrix}\mathcal{H}_1 & \mathcal{H}_2\end{pmatrix}$ is invertible over $k(x)$.*

As the matrix $\begin{pmatrix}\mathcal{H}_1 & \mathcal{H}_2\end{pmatrix}$ is uniquely defined by Q, we call it the matrix *associated* with Q, denoted by $\mathcal{H}(Q)$. Let δ be its determinant, so that $\deg_x(\delta) \leq \mu := d_x^*d_y^- + d_x^-d_y^*$ by Fact 2*(i)*. For later use, we also define δ' as the determinant of $\mathcal{H}(Q^{*2})$, so that $\deg_x(\delta') \leq \mu' := 2d_x^*d_y^*$ by Fact 2*(i)* and since $(Q^{*2})^- = Q^*$.

Lemma 7 *There exist $B, b \in k[x,y]$ with $\deg_y(B) < d_y^-$ and $\deg_y(b) < d_y^*$, and such that:*

(i) $\frac{P}{Q} = D_y\left(\frac{B}{\delta Q^-}\right) + \frac{b}{\delta Q^*}$;

(ii) $\deg_x(B) \leq \mu - d_x^* + \deg_x(P)$ *and* $\deg_x(b) \leq \mu - d_x^- + \deg_x(P)$.

Proof. Applying Cramer's rule to (6) leads to *(i)*. Assertion *(ii)* next follows by determinant expansions. ☐

In what follows, we shall encounter proper rational functions with denominator Q satisfying $Q = Q^{*2}$. The following lemma is an easy corollary of Lemma 7 for such functions.

Corollary 8 *Assuming $Q = Q^{*2}$ in addition to Hypothesis (H), there exist $B, b \in k[x,y]$ with $\deg_y(B)$ and $\deg_y(b)$ less than d_y^*, and such that*

(i) $\frac{P}{Q^{*2}} = D_y\left(\frac{B}{\delta' Q^*}\right) + \frac{b}{\delta' Q^*}$;

(ii) $\deg_x(B)$ *and* $\deg_x(b)$ *are bounded by* $\mu' - d_x^* + \deg_x(P)$.

2.2 Algorithm by evaluation and interpolation

We observe that an asymptotically optimal complexity can be achieved by evaluation and interpolation at each step of Hermite reduction over $k(x)$. This inspires us to adapt Gerhard's modular method [8, 9] to $k(x, y)$. Recall that, by Hyp. (H), $Q \in k[x, y]$ is nonzero and primitive over $k[x]$.

Definition *An element $x_0 \in k$ is* lucky *if $\operatorname{lc}_y(Q)(x_0) \neq 0$ and $\deg_y(\gcd(Q(x_0, y), D_y(Q(x_0, y)))) = d_y^-$.*

Lemma 9 *There are at most $d_x(2d_y^* - 1)$ unlucky points.*

Proof. Let $\sigma \in k[x]$ be the d_y^-th subresultant w.r.t. y of Q and $D_y(Q)$. By [9, Corollary 5.5], all unlucky points are in the set $U = \{ x_0 \in k \mid \sigma(x_0) = 0 \}$. By [9, Corollary 3.2*(ii)*], $\deg_x(\sigma) \le d_x(2d_y^* - 1)$. $\square$

Lemma 10 *Let B, b, and δ be the same as in Lemma 7, and let $x_0 \in k$ be lucky. Then $\delta(x_0) \neq 0$ and $(B(x_0, y), b(x_0, y))$ is the unique pair such that*

$$\frac{P(x_0, y)}{Q(x_0, y)} = D_y\left(\frac{B(x_0, y)}{\delta(x_0)Q^-(x_0, y)}\right) + \frac{b(x_0, y)}{\delta(x_0)Q^*(x_0, y)}. \quad (7)$$

Proof. By the luckiness of x_0, $\deg_y(Q(x_0, y)) = d_y$ and $Q(x_0, y)^- = Q^-(x_0, y)$, so $Q(x_0, y)^* = Q^*(x_0, y)$. This implies $\mathcal{H}(Q)(x_0, y) = \mathcal{H}(Q(x_0, y))$, which, by Lemma 6, is invertible over $k(x)$. Hence $\delta(x_0) \neq 0$, and the evaluation at $x = x_0$ of the equality in Lemma 7*(i)* is well-defined. Thus, $(B(x_0, y), b(x_0, y))$ is a solution of (7). Uniqueness follows from Lemma 4. $\square$

Theorem 11 *Algorithm* HermiteEvalInterp *in Figure 2 is correct and takes $\tilde{\mathcal{O}}(d_x d_y^2 + \deg_x(P) d_y)$ ops.*

Proof. Set ν to $d_x(2d_y^* - 1)$. Lemma 9 implies that the $\lambda + 1$ lucky points found in Step 3 are all less than $\lambda + \nu + 1$. By Lemmas 4 and 7*(i)*, $A = B/\delta$ and $a = b/\delta$. By Lemma 10, $A_0 = B(x_0, y)/\delta(x_0)$ and $a_0 = b(x_0, y)/\delta(x_0)$. By Lemma 7*(ii)* and since $\deg_x(\delta) \le \mu$, it suffices to rationally interpolate A and a from values at $\lambda + 1$ lucky points. This shows the correctness. The dominant computation in Step 1 is the gcd, which takes $\tilde{\mathcal{O}}(d_x d_y)$ ops by [6, Cor. 11.9]. For each integer $i \le \lambda + \nu$, testing luckiness amounts to evaluations at x_0 and computing $\gcd(Q(x_0, y), D_y(Q(x_0, y)))$, which takes $\tilde{\mathcal{O}}(d_y)$ ops by Fact 1*(i)* and [6, Cor. 11.6]. Then, generating S in Step 3 costs $\tilde{\mathcal{O}}((\lambda + \nu + 1)d_y)$ ops. By Fact 1*(i)*, evaluations in Step 4 take $\tilde{\mathcal{O}}((\lambda + 1)d_y)$ ops. For each $x_0 \in S$, the cost of the Hermite reduction in Step 4 is $\tilde{\mathcal{O}}(d_y)$ ops by Lemma 5. Thus, the total cost of Step 4 is $\tilde{\mathcal{O}}((\lambda + 1)d_y)$ ops. By Fact 1*(ii)*, Step 5 takes $\tilde{\mathcal{O}}((\lambda + 1)d_y)$ ops. Since $\lambda \le 2d_x d_y + \deg_x(P)$ and $\nu \le 2d_x d_y$, the total cost is as announced. $\square$

As the generic output size of Hermite reduction is proportional to λd_y, which is $\mathcal{O}((d_x d_y + \deg_x(P))d_y)$, Algorithm HermiteEvalInterp has quasi-optimal complexity.

3. MINIMAL TELESCOPERS

We analyse two algorithms for constructing minimal telescopers for bivariate rational functions and their certificates.

Algorithm HermiteEvalInterp(P, Q)

Input: $P, Q \in k[x, y]$ satisfying Hypothesis (H).
Output: $(A, a) \in k(x)[y]^2$ solving (4).

1. Compute $Q^- := \gcd(Q, D_y(Q))$ and $Q^* := Q/Q^-$;
2. Set $\lambda := 2(d_x^* d_y^- + d_y^* d_x^-) + \deg_x(P) - \min\{d_x^-, d_x^*\}$;
3. Set S to the set of $\lambda+1$ smallest nonnegative integers that are lucky for Q;
4. For each $x_0 \in S$, compute $(A_0, a_0) \in k[y]^2$ such that
$$\frac{P(x_0, y)}{Q(x_0, y)} = D_y\left(\frac{A_0}{Q^-(x_0, y)}\right) + \frac{a_0}{Q^*(x_0, y)}$$
using Hermite reduction over k;
5. Compute $(A, a) \in k(x)[y]$ by rational interpolation and return this pair.

Figure 2: Hermite reduction over $k(x)$ via evaluation and interpolation.

3.1 Hermite reduction approach

We design a new algorithm, presented in Figure 3, to compute minimal telescopers for rational functions by basing on Hermite reduction. For $f = P/Q \in k(x, y)$ and $i \in \mathbb{N}$, Hermite reduction decomposes $D_x^i(f)$ into

$$D_x^i(f) = D_y(g_i) + r_i, \quad (8)$$

where $g_i, r_i \in k(x, y)$ are proper. Since the squarefree part of the denominator of $D_x^i(f)$ divides Q^*, so does the denominator of r_i. The following lemma shows that (8) recombines into telescopers and certificates; next, Lemma 13 implies that the first pair obtained in this way by Algorithm HermiteTelescoping in Figure 3 yields a minimal telescoper.

Lemma 12 *The rational functions $r_0, \ldots, r_{d_y^*}$ are linearly dependent over $k(x)$.*

Proof. The constraints on r_i imply $\deg_y(r_i Q^*) < d_y^*$ for all $i \in \mathbb{N}$, from which follows the existence of a nontrivial linear dependence among the r_i's over $k(x)$. $\square$

Lemma 13 *An integer ρ is minimal such that $\sum_{i=0}^{\rho} \eta_i r_i = 0$ for $\eta_0, \ldots, \eta_\rho \in k(x)$ not all zero if and only if $\sum_{i=0}^{\rho} \eta_i D_x^i$ is a minimal telescoper for f with certificate $\sum_{i=0}^{\rho} \eta_i g_i$.*

Proof. Multiplying (8) by η_i before summing yields

$$L(f) = D_y\left(\sum_{i=0}^{\rho} \eta_i g_i\right) + \sum_{i=0}^{\rho} \eta_i r_i \quad \text{for} \quad L := \sum_{i=0}^{\rho} \eta_i D_x^i,$$

where the first two sums are proper. Thus, by Lemma 4, L is a telescoper of order ρ for f with certificate $\sum_{i=0}^{\rho} \eta_i g_i$ if and only if $\sum_{i=0}^{\rho} \eta_i r_i = 0$ with $\eta_\rho \neq 0$. The lemma follows. $\square$

3.1.1 *Order bounds for minimal telescopers*

Lemmas 12 and 13 combine into an upper bound on the order of minimal telescopers for f.

Corollary 14 *Minimal telescopers have order at most d_y^*.*

The bound $6d_y$ is shown in [3] for rational functions of the form $yD_y(Q)/Q$ with $Q \in k[x, y]$. Apagodu and Zeilberger [2] obtain a similar bound for a class of nonrational hyperexponential functions, but their proof does not seem to apply to rational functions, as it heavily relies on the presence of a nontrivial exponential part.

We also derive a lower bound on the order of the minimal telescoper, to be used as an optimisation at the end of § 3.1.3: choosing a lucky $x_0 \in k$, next applying Hermite reduction in $k(y)$ to $D_x^i(f)(x_0, y)$, yields

$$D_x^i(f)(x_0, y) = D_y(g_{0,i}) + r_{0,i}, \tag{9}$$

where $g_{0,i}, r_{0,i} \in k(y)$ are proper and the denominator of $r_{0,i}$ divides $Q^*(x_0, y)$. Let ρ_0 be the smallest integer such that $r_{0,0}, \dots, r_{0,\rho_0}$ are linearly dependent over k.

Lemma 15 *A minimal telescoper has order at least ρ_0.*

Proof. We first claim that $r_{0,i} = r_i(x_0, y)$, for r_i as in (8). Note that the squarefree part w.r.t. y of the denominator of $D_x^i(f)$ divides Q^* for all $i \in \mathbb{N}$. By [9, Cor. 5.5], x_0 is lucky for the denominator of $D_x^i(f)$ for all $i \in \mathbb{N}$. Then, the claim on $r_{0,i}$ follows from Lemma 10 applied to $D_x^i(f)$. Let ρ be the minimal order of a telescoper, then $r_0, \dots, r_\rho$ are linearly dependent over $k(x)$ by Lemma 13. Thus $r_{0,0}, \dots, r_{0,\rho}$ are linearly dependent over k, which implies $\rho_0 \le \rho$. □

3.1.2 Degree bounds for minimal telescopers

To derive degree bounds for g_i and r_i in (8), let δ, δ', μ, and μ' be defined as before Lemma 7, and set $\mu'' = \mu + \mu' - 1$.

Lemma 16 *Let W be in $k[x, y]$ with $\deg_y(W) < d_y^*$. Then, for all $i \in \mathbb{N}$, there exist $B, b \in k[x, y]$ with both $\operatorname{bideg}(B)$ and $\operatorname{bideg}(b)$ bounded by $(\deg_x(W) + \mu'', d_y^* - 1)$, such that*

$$D_x\left(\frac{W}{\delta^{i+1}\delta'^i Q^*}\right) = D_y\left(\frac{B}{\delta^{i+2}\delta'^{i+1} Q^*}\right) + \frac{b}{\delta^{i+2}\delta'^{i+1} Q^*}.$$

Proof. A straightforward calculation leads to

$$D_x\left(\frac{W}{\delta^{i+1}\delta'^i Q^*}\right) = \frac{\tilde W}{\delta^{i+2}\delta'^{i+1} Q^*} - \frac{1}{\delta^{i+1}\delta'^i} \frac{W D_x(Q^*)}{Q^{*2}},$$

where $\operatorname{bideg}(\tilde W) \le (\deg_x(W) + \mu'', d_y^* - 1)$. By Corollary 8, there exist $\tilde B, \tilde b \in k[x, y]$ such that

$$\frac{1}{\delta^{i+1}\delta'^i} \frac{W D_x(Q^*)}{Q^{*2}} = \frac{1}{\delta^{i+2}\delta'^{i+1}}\left(D_y\left(\frac{\delta \tilde B}{Q^*}\right) + \frac{\delta \tilde b}{Q^*}\right),$$

with $\operatorname{bideg}(\tilde B)$ and $\operatorname{bideg}(\tilde b)$ bounded by $(\deg_x(W) + \mu' - 1, d_y^* - 1)$. Setting $(B, b) = (-\delta \tilde B, \tilde W - \delta \tilde b)$ ends the proof. □

Lemma 17 *For $i \in \mathbb{N}$, there exist $B_i, b_i \in k[x, y]$ such that*

$$D_x^i(f) = D_y\left(\frac{B_i}{\delta^{i+1}\delta'^i Q^{*i} Q^-}\right) + \frac{b_i}{\delta^{i+1}\delta'^i Q^*}. \tag{10}$$

Moreover, $\operatorname{bideg}(B_i) \le (\deg_x(P) + \mu + i\mu'' + (i-1)d_x^*, i d_y^* + d_y^- - 1)$ and $\operatorname{bideg}(b_i) \le (\deg_x(P) + \mu + i\mu'' - d_x^-, d_y^* - 1)$.

Proof. We proceed by induction on i. For $i = 0$, the claim follows from Lemma 7. Assume that $i > 0$ and that the claim holds for the values less than i. For brevity, we set $\gamma = \deg_x(P) + \mu$, $F_{i-1} = B_{i-1}/(\delta^i \delta'^{i-1} Q^{*i-1} Q^-)$, and $G_{i-1} = b_{i-1}/(\delta^i \delta'^{i-1} Q^*)$. The induction hypothesis implies

$$D_x^i(f) = D_y D_x(F_{i-1}) + D_x(G_{i-1}),$$

with bidegree bounds on B_{i-1} and b_{i-1}. Fact 3*(i)* implies that $\tilde Q := Q^* D_x(Q^-)/Q^-$ is in $k[x, y]$, with $\operatorname{bideg}(\tilde Q) \le (d_x^* - 1, d_y^*)$. Hence $D_x(1/Q^-) = -\tilde Q/Q$. This observation and an easy calculation imply that

$$D_x(F_{i-1}) = \frac{\tilde B_{i-1}}{\delta^{i+1}\delta'^i Q^{*i} Q^-},$$

where $\tilde B_{i-1} \in k[x, y]$ and $\deg_x(\tilde B_{i-1}) \le \deg_x(B_{i-1}) + \mu'' + d_x^*$. Furthermore, by Lemma 16 there are $\bar B_i, \bar b_i \in k[x, y]$ with bidegrees at most $(\deg_x(b_{i-1}) + \mu'', d_y^* - 1)$, such that

$$D_x(G_{i-1}) = D_y\left(\frac{\bar B_i}{\delta^{i+1}\delta'^i Q^*}\right) + \frac{\bar b_i}{\delta^{i+1}\delta'^i Q^*}.$$

Setting $B_i = \tilde B_{i-1} + \bar B_i Q^{*i-1} Q^-$ and $b_i = \bar b_i$, we arrive at (10). It remains to verify the degree bounds. The induction hypothesis implies that both $\deg_x(\bar B_i)$ and $\deg_x(b_i)$ are bounded by $\gamma + i\mu'' - d_x^-$. It follows that $\deg_x(\bar B_i Q^{*i-1} Q^-)$ is bounded by $\gamma + i\mu'' + (i-1)d_x^*$. Similarly, $\deg_x(\tilde B_{i-1})$ is bounded by $\gamma + i\mu'' + (i-1)d_x^*$, and so is $\deg_x(B_i)$. The bounds on degrees in y are obvious. □

We next derive degree bounds for the minimal telescopers obtained at an intermediate stage of HermiteTelescoping; refined bounds on the output will be given by Theorem 25.

Lemma 18 *Under (H'), Step 2(c) of Algorithm HermiteTelescoping computes a minimal telescoper $L \in k[x]\langle D_x\rangle$ with order ρ and a certificate $g \in k(x, y)$ for P/Q with $\deg_x(L) \in \mathcal{O}(d_x d_y \rho^2)$ and $\operatorname{bideg}(g) \in \mathcal{O}(d_x d_y \rho^2) \times \mathcal{O}(d_y \rho)$.*

Proof. By Lemma 13, we exhibit a minimal telescoper by considering the first nontrivial linear dependence among the a_i's in (10). Let M be the coefficient matrix of the system in (η_i) obtained from $\sum_{i=0}^{\rho} \eta_i a_i = 0$. By Lemma 17, M is of size at most $(\rho + 1) \times d_y^*$ and with coefficients of degree at most $\sigma := d_x + \mu + \rho\mu'' - d_x^-$ in x. Hence, there exists a solution $(\eta_0, \dots, \eta_\rho) \in k[x]^{\rho+1}$ of degree at most $\sigma\rho$ in x by Fact 2*(ii)*. Since $\mu, \mu'' \in \mathcal{O}(d_x d_y)$ and $d_y^* \le d_y$, the degree estimates of L and g are as announced. □

3.1.3 Complexity estimates

We proceed to analyse the complexity of the algorithm in Figure 3 and of an optimisation.

Algorithm HermiteTelescoping(f)

Input: $f = P/Q \in k(x, y)$ satisfying Hypothesis (H).
Output: A minimal telescoper $L \in k[x]\langle D_x\rangle$ with certificate $g \in k(x, y)$.

1. Apply HermiteEvalInterp to f to get (g_0, a_0) such that $f = D_y(g_0) + a_0/Q^*$. If $a_0 = 0$, return $(1, g_0)$.
2. For i from 1 to $\deg_y(Q^*)$ do
   (a) Apply HermiteEvalInterp to $-a_{i-1} D_x(Q^*)/Q^{*2}$ to express it as $D_y(\tilde g_i) + \tilde a_i/Q^*$.
   (b) Set $g_i = D_x(g_{i-1}) + \tilde g_i$ and $a_i = D_x(a_{i-1}) + \tilde a_i$.
   (c) Solve $\sum_{j=0}^{i} \eta_j a_j = 0$ for $\eta_j \in k(x)$ using [19]. If there exists a nontrivial solution, then set $(L, g) := (\sum_{j=0}^{i} \eta_j D_x^j, \sum_{j=0}^{i} \eta_j g_j)$, and break.
3. Compute the content c of L and return $(c^{-1}L, c^{-1}g)$.

Figure 3: Creative telescoping by Hermite reduction

Theorem 19 *Under Hyp. (H'), Algorithm* HermiteTelescoping *in Figure 3 is correct and takes $\tilde{\mathcal{O}}(\rho^{\omega+1} d_x d_y^2)$ ops, where ρ is the order of the minimal telescoper.*

Proof. The formulas in Step 2(a) create the loop invariant $D_x^i(f) = D_y(g_i) + a_i/Q^*$. Correctness then follows from Lemmas 12 and 20. Step 1 takes $\tilde{\mathcal{O}}(d_x d_y^2)$ ops by Theorem 11 under (H'). By Lemma 17, $\deg_x(-a_{i-1} D_x(Q^*)) \in \mathcal{O}(i d_x d_y)$. So the cost for performing Hermite reduction on $-a_{i-1} D_x(Q^*)/Q^{*2}$ in Step 2(a) is $\tilde{\mathcal{O}}(i d_x d_y^2)$ ops by Theorem 11. The bidegrees of g_i and a_i in Step 2(b) are in $\mathcal{O}(i d_x d_y) \times \mathcal{O}(i d_y)$ by Lemma 17. Since adding and differentiating have linear complexity, Step 2(b) takes $\tilde{\mathcal{O}}(i^2 d_x d_y^2)$ ops. For each i, the coefficient matrix of $\sum_{j=0}^{i} \eta_j a_j = 0$ in Step 2(c) is of size at most $(i+1) \times d_y^*$ and with coefficients of degree at most $\deg_x(a_i) \in \mathcal{O}(i d_x d_y)$. Moreover, the rank of this matrix is either i or $i+1$. Then, Step 2(c) takes $\tilde{\mathcal{O}}(i^\omega d_x d_y^2)$ ops by Fact 2*(iii)*. Computing the content and divisions in Step 3 has complexity $\tilde{\mathcal{O}}(d_x d_y \rho^3)$. If the algorithm returns when $i = \rho$, then the total cost is in

$$\sum_{i=0}^{\rho} \tilde{\mathcal{O}}(i^2 d_x d_y^2) + \sum_{i=1}^{\rho} \tilde{\mathcal{O}}(i^\omega d_x d_y^2) \subset \tilde{\mathcal{O}}(\rho^{\omega+1} d_x d_y^2) \text{ ops}, \quad (11)$$

which is as announced. □

An optimisation, based on Lemma 15, consists in guessing the order ρ so as to perform Step 2(c) a few times only: As a preprocessing step, choose $x_0 \in k$ lucky for Q, then detect linear dependence of $\{r_{0,0}, \dots, r_{0,j}\}$ in (9). The minimal j for dependence is a lower bound ρ_0 on ρ. So Step 2(c) is then performed only when $i \geq \rho_0$. In practice, the lower bound ρ_0 computed in this way almost always coincides with the actual order ρ. So normalising the g_i's becomes the dominant step, as observed in experiments. We analyse this optimisation by first estimating the cost for computing ρ_0.

Lemma 20 *Under Hypothesis (H'), computing a lower order bound ρ_0 for minimal telescopers takes $\tilde{\mathcal{O}}(d_x d_y \rho_0^3)$ ops.*

Proof. Since differentiating has linear complexity, the derivative $D_x^i(f)$ takes $\tilde{\mathcal{O}}(i^2 d_x d_y)$ ops. By Fact 1*(i)*, the evaluation $D_x^i(f)(x_0, y)$ takes as much. The cost of Hermite reduction on $D_x^i(f)(x_0, y)$ is $\tilde{\mathcal{O}}(i d_y)$ ops by Lemma 5. By Fact 2*(iii)* with $d = 1$, computing the rank of the coefficient matrix of $\sum_{j=0}^{i} \eta_j r_{0,j}$, with $r_{0,j}$ as in (9), takes $\tilde{\mathcal{O}}(d_y i^{\omega-1})$ ops. Thus, the total cost for computing a lower bound on ρ_0 is $\sum_{i=0}^{\rho_0} \tilde{\mathcal{O}}(i^2 d_x d_y) \in \tilde{\mathcal{O}}(d_x d_y \rho_0^3)$ ops. □

Corollary 21 *For runs such that $\rho_0 = \rho - \mathcal{O}(1)$, the previous optimisation of* HermiteTelescoping *takes $\tilde{\mathcal{O}}(\rho^3 d_x d_y^2)$ ops.*

Proof. In view of Lemma 20, the estimate (11) becomes $\tilde{\mathcal{O}}(d_x d_y \rho_0^3) + \sum_{i=0}^{\rho} \tilde{\mathcal{O}}(i^2 d_x d_y^2) + \sum_{i=\rho_0}^{\rho} \tilde{\mathcal{O}}(i^\omega d_x d_y^2)$, which is $\tilde{\mathcal{O}}(\rho^3 d_x d_y^2) + \tilde{\mathcal{O}}((\rho - \rho_0)\rho^\omega d_x d_y^2)$ ops, whence the result. □

3.2 Almkvist and Zeilberger's approach

We analyse the complexity of Almkvist and Zeilberger's algorithm [1] when restricted to bivariate rational functions. In order to get a telescoper whose order ρ is minimal, the resulting algorithm, denoted RatAZ, solves (1) for increasing, prescribed values of ρ until it gets a solution $(\eta_0, \dots, \eta_\rho, g) \in k(x)^{\rho+1} \times k(x, y)$ with the η_i's not all zero. For the analysis, we start by studying the parameterisation of the differential Gosper algorithm of [1] under the same restriction to $k(x, y)$.

Definition ([9]) *Let K be a field and $a, b \in K[y]$ be nonzero polynomials. A triple $(p, q, r) \in K[y]^3$ is said to be a* differential Gosper form *of the rational function a/b if*

$$\frac{a}{b} = \frac{D_y(p)}{p} + \frac{q}{r} \text{ and } \gcd(r, q - \tau D_y(r)) = 1 \text{ for all } \tau \in \mathbb{N}.$$

For hyperexponential f, a key step in [1] is to *compute* a differential Gosper form of the logarithmic derivative of $F = \sum_{i=0}^{\rho} \eta_i D_x^i(f)$, where the η_i's are undetermined from $k(x)$. In the analogue RatAZ, this form is *predicted* by Lemma 22 below, which is a technical generalisation of a result by Le [12] on F when f has a squarefree denominator.

Write $Q = t(y)T(x, y)$, splitting content and primitive part w.r.t. x. By an easy induction, $D_x^i(f) = N_i/(QT^{*i})$ for $N_i \in k[x, y]$. For this section, set $F = \sum_{i=0}^{\rho} \eta_i D_x^i(f)$, $N = \sum_{i=0}^{\rho} \eta_i N_i T^{*\rho-i}$, and $H = -D_y(Q)/Q^- - \rho t^* D_y(T^*)$.

Lemma 22 *If F is nonzero, the triple (N, H, Q^*) is a differential Gosper form of $D_y(F)/F$.*

Proof. First, observe $F = N/(QT^{*\rho})$ and $Q^* = t^* T^*$. Next, $D_y(F)/F = D_y(N)/N - D_y(Q)/Q - \rho D_y(T^*)/T^*$ is $D_y(N)/N + H/Q^*$. There remains to prove $\gcd(Q^*, H - \tau D_y(Q^*)) = 1$, for any $\tau \in \mathbb{N}$. Recall that the squarefree part Q^* of Q is the product $Q_1 Q_2 \cdots Q_m$ and that $\hat{Q}_i$ denotes Q^*/Q_i. By Fact 3*(ii)*,

$$Z := H - \tau D_y(Q^*) = -\rho t^* D_y(T^*) - \sum_{i=1}^{m} (i + \tau) \hat{Q}_i D_y(Q_i).$$

If Q_j divides t^*, Z reduces to $-(j+\tau)\hat{Q}_j D_y(Q_j)$ modulo Q_j. If not, it reduces to $-(j+\tau)\hat{Q}_j D_y(Q_j) - \rho t^*(D_y(Q_j)T^*/Q_j)$, which rewrites to $-(j+\tau+\rho)\hat{Q}_j D_y(Q_j)$ modulo Q_j. In both cases, Z is coprime with Q^*, as $j > 0$, $\tau \geq 0$, and $\rho \geq 0$. □

By another induction, we observe $\operatorname{bideg}(N_i) \leq (\deg_x(P) + i\deg_x(T^*) - i, d_y + i\deg_y(T^*) - 1)$, so that $\operatorname{bideg}(N) \leq (\deg_x(P) + \rho\deg_x(T^*) - \rho, d_y + \rho\deg_y(T^*) - 1)$.

The next step in RatAZ is, for fixed ρ, to reduce (1) by the change of unknown $g = z/(Q^- T^{*\rho})$, so as to determine all $(\eta_i) \in k(x)^{\rho+1}$ for which the differential equation in z

$$\sum_{i=0}^{\rho} \eta_i N_i T^{*\rho-i} = Q^* D_y(z) + (D_y(Q^*) + H)\, z \quad (12)$$

has a polynomial solution in $k(x)[y]$. For later use, we recall the following consequence of [9, Corollary 9.6].

Lemma 23 *Let $a, b \in K[y]$ be such that $\beta = -\operatorname{lc}_y(b)/\operatorname{lc}_y(a)$ is a nonnegative integer and $\deg_y(b) = \deg_y(a) - 1$. Let $c \in K[y]$ be such that $\beta \geq \deg_y(c) - \deg_y(a) + 1$. If u is a polynomial solution of $aD_y(z) + bz = c$, then $\deg_y(u) \leq \beta$.*

The following lemma generalises [12, Lemma 2] to present a degree bound for z.

Lemma 24 *If $u \in k(x)[y]$ is a solution of (12) for $(\eta_i) \in k(x)^{\rho+1}$, then $\deg_y(u)$ is bounded by $\beta = d_y^- + \rho\deg_y(T^*)$.*

Proof. Let $a = Q^*$ and $b = D_y(Q^*) + H$. By the definition of H, $b = -Q^* D_y(Q^-)/Q^- - \rho t^* D_y(T^*)$. Fact 3*(i)* implies that $\operatorname{lc}_y(b) = -(d_y^- + \rho\deg_y(T^*))\operatorname{lc}_y(a)$. Therefore, $\beta = -\operatorname{lc}_y(b)/\operatorname{lc}_y(a) = d_y^- + \rho\deg_y(T^*)$. As $\deg_y(N) < d_y + \rho\deg_y(T^*)$ and $d_y = d_y^* + d_y^-$, $\beta \geq \deg_y(N) - d_y^* + 1$. The lemma holds by Lemma 23. □

Algorithm $\mathsf{RatAZ}(f)$

INPUT: $f = P/Q \in k(x,y)$ satisfying Hypothesis (H).
OUTPUT: A minimal telescoper $L \in k[x]\langle D_x\rangle$ with certificate $g \in k(x,y)$.

1. Compute $Q^- = \gcd(Q, D_y(Q))$, $Q^* = Q/Q^-$, and T, T^* primitive parts of Q, Q^* w.r.t. x, respectively;
2. Set $(\tilde{N}, N, \beta, H)$ to $(P, P, d_y^-, -Q^* D_y(Q)/Q)$;
3. For $\ell = 0, 1, \ldots$ do
   (a) Set z to $\sum_{j=0}^{\beta} z_j y^j$, extract the linear system $\mathcal{M}\begin{pmatrix}\eta_i & z_j\end{pmatrix}^T = 0$ from (12) (for $\rho = \ell$) and compute a basis S of the null space of $\mathcal{M}$ by [19].
   (b) If S contains a solution $(\eta_0, \ldots, \eta_\ell, s)$ such that $\eta_0, \ldots, \eta_\ell$ are not all nonzero, then set $(L, g) := \left(\sum_{i=0}^{\ell} \eta_i D_x^i, s/(Q^- T^{*\ell})\right)$, and go to Step 4;
   (c) Update $\tilde{N} := D_x(\tilde{N})T^* - \tilde{N}(T^* D_x(T)/T + iD_x(T^*))$, $N := NT^* + \eta_{\ell+1}\tilde{N}$, $\beta := \beta + \deg_y(T^*)$, and $H := H - t^* D_y(T^*)$.
4. Compute the content c of L and return $(c^{-1}L, c^{-1}g)$.

Figure 4: Improved Almkvist–Zeilberger algorithm

We end the present section using the approach of Almkvist and Zeilberger to provide tight degree bounds on the outputs from Algorithms $\mathsf{HermiteTelescoping}$ and RatAZ.

Theorem 25 *Under Hypothesis (H'), there exists a minimal telescoper $L \in k[x]\langle D_x\rangle$ with certificate $g \in k(x,y)$ with $\deg_x(L) \in \mathcal{O}(d_x d_y d_y^*)$ and $\operatorname{bideg}(g) \in \mathcal{O}(d_x d_y d_y^*) \times \mathcal{O}(d_y d_y^*)$.*

PROOF. By Corollary 14, there exists a smallest $\rho \in \mathbb{N}$ at most d_y^*, for which (1) has a solution with the η_i's not all zero. For this ρ, we estimate the size of the polynomial matrix $\mathcal{M}$ derived from (12) by undetermined coefficients. By the remark on N after Lemma 22, we have $\operatorname{bideg}(N) \le (n_x, n_y)$ where $n_x := d_x + \rho \deg_x(T^*) - \rho \in \mathcal{O}(\rho d_x)$ and $n_y := d_y + \rho \deg_y(T^*) - 1 \in \mathcal{O}(\rho d_y)$. The matrix $\mathcal{M}$ contains two blocks $\mathcal{M}_1 \in k[x]_{\le n_x}^{(n_y+1)\times(\rho+1)}$ and $\mathcal{M}_2 \in k[x]_{\le d_x}^{(n_y+1)\times(\beta+1)}$, where $\beta \in \mathcal{O}(\rho d_y)$ is the same as in Lemma 24. By the minimality of ρ, the dimension of the null space of $\mathcal{M}$ is 1. So there exists $u \in k[x]^{n_y+1}$ with coefficients of degree at most $n_x(\rho+1) + d_x(\beta+1) \in \mathcal{O}(d_x d_y d_y^*)$ in x such that $\mathcal{M}\begin{pmatrix}\eta & z\end{pmatrix}^T = 0$, which implies degree bounds in x for L and g. The degree bound in y for g is obvious. $\square$

We now analyse the complexity of the algorithm in Fig. 4.

Theorem 26 *Under Hypothesis (H'), Algorithm RatAZ in Figure 4 is correct and takes $\tilde{\mathcal{O}}(d_x d_y^{\omega} \rho^{\omega+2})$ ops, where ρ is the order of the minimal telescoper.*

PROOF. By the existence of a telescoper, Corollary 14, and Lemma 24, the algorithm always terminates and returns a minimal telescoper L, of order ρ at most d_y^*. Gcd computations dominate the cost of Steps 1 and 2, which take $\tilde{\mathcal{O}}(d_x d_y^2)$ ops. For each $\ell \in \mathbb{N}$, the dominating cost in Step 3 is computing the null space of $\mathcal{M}$. Let $n_y = d_y + \ell \deg_y(T^*) - 1 \in \mathcal{O}(\ell d_y)$ and $n_x = d_x + \ell \deg_x(T^*) \in \mathcal{O}(\ell d_x)$. By the same argument as in the proof of Theorem 25, the matrix $\mathcal{M}$ is of size at most $(n_y+1) \times (\ell + \beta + 2)$ and with coefficients of degree at most n_x. Let r be the rank of $\mathcal{M}$, which is either $\ell + \beta + 2$ or $\ell + \beta + 1$ by construction. Thus, a basis of the null space of $\mathcal{M}$ can be computed within $\tilde{\mathcal{O}}(n_x(n_y+1)(\ell+\beta+2)r^{\omega-2})$ ops by Fact 2*(iii)*. Since $\beta \in \mathcal{O}(\ell d_y)$, $\tilde{\mathcal{O}}(n_x(n_y+1)(\ell+\beta+2)r^{\omega-2})$ is included in $\tilde{\mathcal{O}}(d_x d_y^{\omega} \ell^{\omega+1})$. Since Step 3 terminates at $\ell = \rho$, the total cost of the algorithm is $\sum_{\ell=0}^{\rho} d_x d_y^{\omega} \ell^{\omega+1}$ ops. This is within the announced complexity, $\tilde{\mathcal{O}}(d_x d_y^{\omega} \rho^{\omega+2})$ ops. $\square$

Corollary 27 *Algorithms $\mathsf{HermiteTelescoping}$ and RatAZ in Fig. 3 and 4 both output the primitive minimal telescoper L together with its certificate g, which satisfy $\deg_{D_x}(L) \le d_y^*$, $\deg_x(L), \deg_x(g) \in \mathcal{O}(d_x d_y d_y^*)$, and $\deg_y(g) \in \mathcal{O}(d_y d_y^*)$.*

PROOF. Both algorithms output the primitive minimal telescoper, as they compute a minimal telescoper at an intermediate step, and owing to their last step of content removal. Bounds follow from Corollary 14 and Theorem 25. $\square$

4. NONMINIMAL TELESCOPERS

Here, we discard Hypothesis (H) and trade the minimality of telescopers for smaller total output sizes. To this end, we adapt and slightly extend the arguments in [13] and [3, § 3].

Given $f = P/Q \in k(x,y)$ of bidegree (d_x, d_y), our goal is to find a (possibly nonminimal) telescoper for f. It is sufficient to find a nonzero differential operator $A(x, D_x, D_y)$ that annihilates f. Indeed, any $A \in k[x]\langle D_x, D_y\rangle \setminus \{0\}$ such that $A(f) = 0$ can be written $A = D_y^r(L + D_y R)$, where L is nonzero in $k[x]\langle D_x\rangle$ and $R \in k[x]\langle D_x, D_y\rangle$. If $r = 0$, then clearly L is a telescoper for f; otherwise, $A(f) = 0$ yields $L(f) = D_y(-R(f) - \sum_{i=0}^{r-1} \frac{a_i}{i+1} y^{i+1})$ for some $a_i \in k(x)$, which implies that L is again a telescoper for f. Moreover, in both cases, $\deg_x(L) \le \deg_x(A)$ and $\deg_{D_x}(L) \le \deg_{D_x}(A)$. Furthermore, for any $(i,j,\ell) \in \mathbb{N}^3$, a direct calculation yields

$$x^i D_x^j D_y^\ell(f) = \frac{H_{i,j,\ell}}{Q^{j+\ell+1}}, \tag{13}$$

where $H_{i,j,\ell} \in k[x,y]$ and $\deg_x(H_{i,j,\ell}) \le (j+\ell+1)d_x + i - j$ and $\deg_y(H_{i,j,\ell}) \le (j+\ell+1)d_y - \ell$. From these inequalities, we derive the size and complexity estimates in Figure 1 (bottom half), using two different filtrations of $k[x]\langle D_x, D_y\rangle$.

Lipshitz's filtration ([13]). Let F_ν be the k-vector space of dimension $\mathsf{f}_\nu := \binom{\nu+3}{3}$ spanned by $\{\, x^i D_x^j D_y^\ell \mid i+j+\ell \le \nu \,\}$. By (13), $F_\nu(f)$ is contained in the vector space of dimension $\mathsf{g}_\nu := ((\nu+1)d_x + \nu + 1)((\nu+1)d_y + 1)$ spanned by $\left\{\, \frac{x^i y^j}{Q^{\nu+1}} \mid i \le (\nu+1)d_x + \nu,\ j \le (\nu+1)d_y \,\right\}$. Choosing $\nu = 6(d_x+1)(d_y+1)$ yields $\mathsf{f}_\nu > \mathsf{g}_\nu$; therefore, there exists A in $k\langle x, D_x, D_y\rangle \setminus \{0\}$ with total degree at most $6(d_x+1)(d_y+1)$ in x, D_x, and D_y that annihilates f. Moreover, A is found by linear algebra in dimension $\mathcal{O}((d_x d_y)^3)$.

A better filtration ([3]). Instead of taking total degree, set $F_{\kappa,\nu}$ to the k-vector space of dimension $\mathsf{f}_{\kappa,\nu} := (\kappa+1)\binom{\nu+2}{2}$ generated by $\{\, x^i D_x^j D_y^\ell \mid i \le \kappa,\ j+\ell \le \nu \,\}$. By (13), $F_{\kappa,\nu}(f)$ is contained in the vector space of dimension $\mathsf{g}_{\kappa,\nu} := ((\nu+1)d_x + \kappa + 1)((\nu+1)d_y + 1)$ spanned by $\left\{\, \frac{x^i y^j}{Q^{\nu+1}} \mid i \le (\nu+1)d_x + \kappa,\ j \le (\nu+1)d_y \,\right\}$. Choosing $\kappa = 3d_x d_y$ and $\nu = 6d_y$ results in $\mathsf{f}_{\kappa,\nu} > \mathsf{g}_{\kappa,\nu}$. This implies the existence of A in $k\langle x, D_x, D_y\rangle \setminus \{0\}$ with total degree at most $6d_y$ in D_x and D_y and degree at most $3d_x d_y$ in x that annihilates f. Again, A is found by linear algebra over k, but in smaller dimension $\mathcal{O}(d_x d_y^3)$.

No.	AZ	Abr	RAZ	H1	H2	HO	EI	MG
29	44	72	32	28	36	20	608	528
43	52	76	36	20	24	32	652	584
46	4268	1436	784	492	1288	752	343413	18945
49	474269	34694	20977	10336	36254	22417	∞	652968

Table 1: Creative telescoping on random instances
Timings in ms for algorithms in Table 3 (stopped after 30 min).

5. IMPLEMENTATION AND TIMINGS

We implemented in Maple 13 all the algorithms described; as we used Maple's generic solver `SolveTools:-Linear`, all of our implementations are deterministic.

The evaluation-interpolation algorithm HermiteEvalInterp for Hermite reduction (Fig. 2) does not perform well, mainly because Maple's rational interpolation routines are far too slow. We thus implemented Algorithm HermiteReduce (original version) in [4, § 2.2] (carefully avoiding redundant extended gcd calculations), and noted that it performs better.

We then implemented a variant of Algorithm HermiteTelescoping in Figure 3, using HermiteReduce in place of HermiteEvalInterp, and including the optimisation at the end of § 3.1.3, refined by additional modular calculations.

For a rational function, Algorithm HermiteTelescoping returns the minimal telescoper L and the certificate g. The algorithm separates the computation for L from that for g. Indeed, g is formed by the coefficients of L, g_0, the $\tilde{g}_i$ and their derivatives given in Figure 3. This feature enables us to either return the certificate g as a sum of unnormalised rational functions, or a normalised rational function.

A selection of timings by this implementation and others are given in Table 1; our code, the full table, as well as the random inputs are given in [22]. For our experiments, we exhaustively considered all 49 bidegree patterns in factorisations of denominators $Q_1 \cdots Q_m^m$ ($m \leq 5$) that add up to bidegree (5,5), and generated corresponding random denominators, imposing the integers of the expanded forms to have around 26 digits. Numerators were generated as random bidegree-(5,5) polynomials with coefficients of 26 digits.

Application to diagonals. The diagonal of a formal power series $f = \sum_{i,j \geq 0} f_{i,j} x^i y^j$ in $k[[x,y]]$ is defined to be the power series $\Delta(f) := \sum_{i=0}^{\infty} f_{i,i} x^i$. For a D-finite power series f, it is known to be D-finite [13], and it is even algebraic for a bivariate rational function $f \in k(x,y) \cap k[[x,y]]$ [18, § 6.3]. A linear differential operator $L \in k(x)\langle D_x \rangle$ that annihilates $\Delta(f)$ can then be computed via rational-function telescoping, owing to the following classical lemma from [13].

Lemma 28 *Any telescoper for $f(y, \frac{x}{y})/y$ annihilates $\Delta(f)$.*

By this lemma, it suffices to compute a telescoper without its certificate to get an annihilator. Algorithm HermiteTelescoping is suitable for this task, since it separates computation of telescopers and certificates. Alternatively, for $f = P/Q$, we can compute an annihilator of $\Delta(f)$ either as the differential resolvent of the resultant $\mathrm{Res}_y(Q, P - \tau D_y Q)$, or simply *guess* it from the first terms of the series expansion of $\Delta(f)$.

We compare the various algorithms on an example borrowed from [5] (timings of execution are given in Table 2):

$$f = \frac{1}{1 - x - y - xy(1 - x^d)}, \quad \text{where } d \in \mathbb{N}. \tag{14}$$

All computer calculations have been performed on a Quad-Core Intel Xeon X5482 processor at 3.20GHz, with 3GB of RAM, using up to 6.5GB of memory allocated by Maple.

d	AZ	Abr	RAZ	H1	H2	HO	RR	GHP
4	176	136	100	116	208	108	220	956
8	3032	4244	4380	1976	5344	4396	10336	154409
10	11740	12816	7108	7448	24565	7076	46882	1118313
4	184	168	120	120	220	116	224	1340
8	3540	3704	2540	2092	6976	2516	10348	271480
10	16817	17013	9200	8068	32218	9092	46750	∞

Table 2: Computation of the diagonals of (14)
Timings in ms by creative telescoping of $f(y, x/y)/y$ (upper half) or $f(y/x, x)/x$ (second half). Algorithms listed in Table 3.

AZ	`DETools[Zeilberger]`
Abr	AZ with Abramov's denominator bound by option `gosper_free`
RAZ	Algorithm RatAZ of Fig. 4, with lower-bound prediction
H1	our Hermite-based approach, without certificate normalisation
H2	H1, but with normalised certificate
HO	RAZ, solving (1) by Horowitz–Ostrogradsky
EI	H1 with evaluation and interpolation for calculations over $k(x)$
MG	`Mgfun`'s creative telescoping for general D-finite functions
RR	telescoper computation by resultant and differential resolvent
GHP	telescoper guessing by diagonal expansion and Hermite–Padé

Table 3: List of the algorithms for the experiments

6. REFERENCES

[1] G. Almkvist and D. Zeilberger. The method of differentiating under the integral sign. *J. Symb. Comput.*, 10:571–591, 1990.

[2] M. Apagodu and D. Zeilberger. Multi-variable Zeilberger and Almkvist-Zeilberger algorithms and the sharpening of Wilf-Zeilberger theory. *Adv. in Appl. Math.*, 37(2):139–152, 2006.

[3] A. Bostan, F. Chyzak, B. Salvy, G. Lecerf, and É. Schost. Differential equations for algebraic functions. In *ISSAC'07*, pages 25–32. ACM, New York, 2007.

[4] M. Bronstein. *Symbolic Integration I: Transcendental functions*, volume 1 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, second edition, 2005.

[5] A. Flaxman, A. W. Harrow, and G. B. Sorkin. Strings with maximally many distinct subsequences and substrings. *Electron. J. Combin.*, 11(1):R8, 10 pp., 2004.

[6] J. von zur Gathen and J. Gerhard. *Modern Computer Algebra*. Cambridge University Press, Cambridge, second edition, 2003.

[7] K. O. Geddes and H. Q. Le. An algorithm to compute the minimal telescopers for rational functions (differential-integral case). In *Mathematical Software*, pages 453–463. WSP, 2002.

[8] J. Gerhard. Fast modular algorithms for squarefree factorization and Hermite integration. *Appl. Algebra Engrg. Comm. Comput.*, 11(3):203–226, 2001.

[9] J. Gerhard. *Modular Algorithms in Symbolic Summation and Symbolic Integration (LNCS)*. SpringerVerlag, 2004.

[10] C. Hermite. Sur l'intégration des fractions rationnelles. *Ann. Sci. École Norm. Sup. (2)*, 1:215–218, 1872.

[11] E. Horowitz. Algorithms for partial fraction decomposition and rational function integration. In *SYMSAC'71*, pages 441–457, New York, USA, 1971. ACM.

[12] H. Q. Le. On the differential-integral analogue of Zeilberger's algorithm to rational functions. In *Proc. of the 2000 Asian Symposium on Computer Mathematics*, pages 204–213, 2000.

[13] L. Lipshitz. The diagonal of a D-finite power series is D-finite. *J. Algebra*, 113(2):373–378, 1988.

[14] L. Lipshitz. D-finite power series. *J. Algebra*, 122(2):353–373, 1989.

[15] M. Ostrogradsky. De l'intégration des fractions rationnelles. *Bull. de la classe physico-mathématique de l'Acad. Impériale des Sciences de Saint-Pétersbourg*, 4:145–167, 286–300, 1845.

[16] R. H. Risch. The problem of integration in finite terms. *Trans. Amer. Math. Soc.*, 139:167–189, 1969.

[17] R. H. Risch. The solution of the problem of integration in finite terms. *Bull. Amer. Math. Soc.*, 76:605–608, 1970.

[18] R. P. Stanley. *Enumerative Combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. CUP, 1999.

[19] A. Storjohann and G. Villard. Computing the rank and a small nullspace basis of a polynomial matrix. In *ISSAC'05*, pages 309–316. ACM, New York, 2005.

[20] D. Zeilberger. A holonomic systems approach to special functions identities. *J. Comput. Appl. Math.*, 32:321–368, 1990.

[21] D. Zeilberger. The method of creative telescoping. *J. Symbolic Comput.*, 11(3):195–204, 1991.

[22] http://algo.inria.fr/chen/BivRatCT/, 2010.