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Quelques applications de l'algèbre différentielle et aux différences pour le télescopage créatif

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Some applications of differential-difference algebra to creative telescoping

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Abstract

Since the 1990's, Zeilberger's method of creative telescoping has played an important role in the automatic verification of special-function identities. The long-term goal initiated in this work is to obtain fast algorithms and implementations for definite integration and summation in the framework of this method. Our contributions include new practical algorithms, complexity analyses of algorithms, and theoretical criteria for the termination of existing algorithms.

On the practical side, we present a new algorithm for computing minimal telescopers for bivariate rational functions. This algorithm is based on Hermite reduction. We also improve the classical Almkvist and Zeilberger's algorithm for rational-function inputs. The Hermite-reduction based algorithm and improved Almkvist and Zeilberger's algorithm are analyzed in terms of field operations. Both complexity analysis and experimental results show that our algorithms are superior to other known ones for rational-function inputs.

On the theoretic side, we present a structure theorem for multivariate hyperexponential-hypergeometric functions. This theorem is based on (multivariate) Christopher's theorem for hyperexponential functions, the Ore-Sato theorem for hypergeometric terms, and our generalization of a recent result by Feng, Singer, and Wu on compatible bivariate continuous-discrete rational functions. The structure theorem allows us to decompose a hyperexponential-hypergeometric function as a product of a rational function, several exponential and power functions, and factorial terms. Furthermore, we derive two criteria for the existence of telescopers for bivariate hyperexponential-hypergeometric functions: one is with respect to the continuous variable, and the other with respect to the discrete one. The two criteria solve the termination problem of the continuous-discrete analogue of Zeilberger's algorithms.

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Contents

Chapter 1

Introduction

As most special-function integrals and sums cannot be expressed in closed form, their evaluation cannot be based on table look-ups only. Even when closed forms are available, they may prove to be intractable in further manipulations. In both cases, the difficulty can be mitigated by representing functions by annihilating differential and difference operators. This motivated Zeilberger to introduce a method now known as *creative telescoping* [95], which applies to a large class of special functions. Zeilberger's method has been used extensively in the automatic proofs of special-function identities.

Let us illustrate the basic idea of Zeilberger's method by proving the identity

$$\sum_{n=0}^{+\infty} f(x,n) = \frac{1}{\sqrt{1-4x}}, \quad \text{where } f(x,n) = \binom{2n}{n} x^n \text{ and } x \in [0, 1/4). \tag{1.1}$$

Let D_x and Δ_n denote the derivation with respect to x, and the difference operator with respect to n, respectively. The first step is to find a differential equation satisfied by the sum on the left-hand side of (1.1). To this end, we try to find a nonzero operator $L \in \mathbb{Q}(x)\langle D_x \rangle$ such that

$$L(x, D_x)(f) = \Delta_n(g), \tag{1.2}$$

where g is a function in x and n. Moreover, the method requires that the ratio between g and f is a rational function. For this specific example, we have

$$L = 2 - (1-4x)D_x \quad \text{and} \quad g = \frac{n}{x} \cdot f = nx^{n-1}\binom{2n}{n}.$$

Note that taking sums with respect to n commutes with the application of $L(x, D_x)$ and, additionally, that

$$\sum_{n=0}^{+\infty} \Delta_n(g) = 0.$$

This implies that the sum on the left-hand side of (1.1) satisfies

$$2y(x) - (1-4x)\frac{dy(x)}{dx} = 0.$$

It is easy to verify that the function on the right-hand side of (1.1) also satisfies the above differential equation. Moreover, the identity holds when $x = 0$. By Cauchy's theorem, the identity (1.1) holds for all well-defined values $x \in [0, 1/4)$. The operator L in (1.2) is called a *telescoper* for f with respect to n, and g the *certificate* of L for f.

This thesis focuses on the bivariate case, in which there are four kinds of telescoping problems related to different integration and summation problems.

	$\int / \sum$ problems	Telescoping equations
(P1)	$\int_a^b f(x,y)\,dy$	$L(x, D_x)(f) = D_y(g)$
(P2)	$\int_a^b f(n,y)\,dy$	$L(n, S_n)(f) = D_y(g)$
(P3)	$\sum_{m=a}^b f(m,x)$	$L(x, D_x)(f) = \Delta_m(g)$
(P4)	$\sum_{m=a}^b f(n,m)$	$L(n, S_n)(f) = \Delta_m(g)$

Table 1.1: Four telescoping problems

1.1 Motivation

Since the 1990's, Zeilberger's method has been extensively studied in the literature [95, 96, 15, 92, 63, 6, 7, 65, 28, 16]. The main focus of those studies is on the efficiency and termination of creative-telescoping algorithms.

Related to efficiency, progress and improvements have since been made by Abramov [7], Apagodu [16], Chyzak [35, 33], Le [63, 46, 65], Takayama [85], etc. However, very little is known about the complexity of creative-telescoping algorithms. We believe that complexity analysis helps us understand algorithms, and indicates good ways of improving and implementing them.

This is one of the motivations for Chapter 3. The existing algorithms bind the construction of telescopers with certificates. In certain applications, only telescopers are needed. For bivariate rational-function inputs, our complexity analysis shows that the arithmetic size of certificates is asymptotically larger than that of telescopers. So the complexity could be lower if one can avoid the calculations for certificates in certain applications. This motivates us to find a way in which one could choose to compute or not to compute the certificates according to the user requirements.

By the functionality of Zeilberger-style algorithms, their termination is equivalent to the existence of telescopers. In his pioneering work [95], Zeilberger has proved that telescopers exist for *holonomic* functions. In particular, Wilf and Zeilberger [92] have presented an elementary and constructive proof of the existence of telescopers for *proper hyperexponential-hypergeometric* functions by basing on ideas of Fasenmyer [40] and Verbaeten [89]. By a proper hyperexponential-hypergeometric function, we mean a function that can be written as a product of a polynomial, exponential functions, power functions, and factorial terms. Properness can be detected from the certificates of a hyperexponential-hypergeometric function. Moreover, Wilf and Zeilberger conjectured in [92, page 585] that a hyperexponential-hypergeometric function is holonomic if and only if it is proper. If this conjecture were verified, then one could algorithmically detect the holonomicity of hyperexponential-hypergeometric functions. In the case of several discrete variables, a slightly modified version of the conjecture has been proved independently by Payne in his Ph.D. thesis [73] and by Abramov and Petkovšek [14]. In particular, the case of two variables has also been shown by Hou [54, 55] and by Abramov and Petkovšek [12]. However, holonomicity is only a sufficient condition for the existence of telescopers. In fact, Chyzak, Kauers, and Salvy [34] have listed certain classes of functions that are not holonomic but still have telescopers. Therefore, the challenge is to find theoretical criteria that enable us to algorithmically detect the existence of telescopers.

In view of the theoretical difficulty, special attention has been focused mainly on the subclass of hyperexponential-hypergeometric functions. In the continuous case, the works by Bernstein [17] and Lipshitz [67] show that every hyperexponential function has a telescoper. This implies that Zeilberger's algorithm always succeeds on those inputs. However, the situation in other cases turns out to be more involved. In the discrete case and its q-analogue, the first complete solution to the termination problem is Abramov and Le's criterion [64, 9], which decides whether telescopers exist for a given bivariate rational function in the discrete variables m and n. According to their criterion, the rational function

$$f = \frac{1}{m^2 + n^2}$$

has no telescoper. Soon, the criterion was extended to the general case of bivariate hypergeometric terms by Abramov [5, 6]. Basically, Abramov proves that a hypergeometric term can be written as a sum of a hypergeometric-summable term and a proper one if it has a telescoper [6, Theorem 10]. Similar results have been obtained in the general q-shift case by Chen, Hou and Mu [28]. These results are fundamental for predicting the termination of Zeilberger's algorithm.

A continuous-discrete analogue of Zeilberger's algorithm was presented by Almkvist and Zeilberger [15]. This analogue has been shown to be very useful in the study of orthogonal polynomials [60, Chapters 10–13]. In this setting, not all hyperexponential-hypergeometric functions have telescopers. For example, we will show in this thesis that the rather simple rational function in the continuous variable x and the discrete variable n

$$f = \frac{1}{x + n}$$

has no telescoper with respect to either the continuous variable x or the discrete variable n. Therefore, an existence criterion is also needed in this mixed setting.

1.2 Main results

For a bivariate rational function, we present a new algorithm to compute its minimal telescoper, which is based on Hermite reduction. We also obtain some improvements over the classical algorithm by Almkvist and Zeilberger by extending the idea of Geddes and Le to general rational-function inputs. Moreover, we give the first proof of a polynomial complexity (in terms of field operations) for creative telescoping on this specific class of inputs. Our algorithms are proved to be faster concerning both theoretical complexity and actual performance.

Motivated by a conjecture of Wilf and Zeilberger [92, page 585], we study the possible form of a multivariate hyperexponential-hypergeometric function. For such a function, we prove a structure theorem for its certificates, which generalizes a recent result by Feng, Singer, and

Wu [42, Proposition 5]. Combining our result with the result on multivariate hyperexponential functions in [29, 97] and the Ore-Sato theorem on multivariate hypergeometric terms, we obtain a structure theorem for multivariate hyperexponential-hypergeometric functions, which says that a multivariate hyperexponential-hypergeometric function can be written as a product of a rational function, an exponential function, power functions and factorial terms.

With the help of the result by Feng, Singer, and Wu [42, Proposition 5], we derive two criteria for the existence of telescopers for bivariate hyperexponential-hypergeometric functions. We show that a hyperexponential-hypergeometric function can be written as a sum of a hypergeometric-summable, resp. hyperexponential-integrable, function and a proper one if it has a telescoper with respect to the discrete, resp. continuous, variable. Our criteria are based on standard representations and the two adapted additive decompositions. With them, we can decide in advance the termination of the continuous-discrete analogue of Zeilberger's algorithm for bivariate hyperexponential-hypergeometric functions.

1.3 Outline

In this section, we provide the reader with an outline of this thesis.

Chapter 2. We recall basic notation and facts on differential rings, difference rings and Ore polynomials. Our new algorithm for the construction of minimal telescopers will be based on the Hermite reduction for the integration of the rational function. So we review the classical algorithms for rational-function integration, including Hermite reduction, Ostrogradsky–Horowitz's method, and Rothstein–Trager's algorithm. We also summarize some complexity results for later use.

Chapter 3. We focus on deriving a fast algorithm for computing minimal telescopers for bivariate rational functions. First, we present an optimal algorithm for the Hermite reduction over $k(x)$ by fast evaluation-interpolation strategy. Second, we present a new algorithm for computing the minimal telescoper of a bivariate rational function, based on a bivariate extension of Hermite reduction. Moreover, some improvements over the classical method by Almkvist and Zeilberger [15] are achieved in this chapter. We also analyze the arithmetic complexity of those algorithms. Third, we adapt and slightly extend the arguments by Lipshitz [67] and by

Bostan and others [20] to derive smaller total degree sizes of telescopers. At last, we describe our implementation and show experimental results.

Chapter 4. We first present an algebraic setting for hyperexponential-hypergeometric functions. After that, we review various normal forms of rational functions and introduce a new kind of rational normal forms, which enables us to extend the existing results to multivariate continuous-discrete setting. Our main result is Theorem 4.4.6 on the structure of certificates of a multivariate hyperexponential-hypergeometric function. This result is a generalization of a result by Feng, Singer, and Wu [42, Proposition 5]. At last, we describe a multiplicative form of multivariate hyperexponential-hypergeometric functions.

Chapter 5. We study the existence of telescopers for hyperexponential-hypergeometric functions of one continuous variable and one discrete variable. First, we review the construction of a ring of sequences from [41, 43], which allows us to study the existence problem in an algebraic manner. After that, we introduce standard representations for bivariate hyperexponential-hypergeometric functions and then adapt two additive decompositions [13, 47] to bivariate hyperexponential-hypergeometric functions described by their standard representations. At last, we describe our existence criteria and its algorithmic description with examples.

Chapter 6. We present some conclusions and propose some topics for future work.

1.4 Notation

We shall use the letters $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{Z}$ to denote the set of complex numbers, non-negative integers, rational numbers, real numbers, and integers, respectively. For a field F, we denote its algebraic closure by $\overline{F}$.

Chapter 2

Preliminaries

In this chapter, we recall basic notation and facts on differential and difference rings and Ore polynomials from [56, 21, 70]. Moreover, we review classical techniques for rational-function integration and background on complexity analysis.

2.1 Differential and difference rings

Let R be a commutative ring. A map $\delta : R \to R$ is said to be a *derivation* on R if

$$\delta(a+b) = \delta(a) + \delta(b), \quad \delta(ab) = \delta(a)b + a\delta(b) \quad \text{for all } a, b \in R.$$

The pair (R, δ) is called a *differential ring*. Moreover, if R is a field, R is called a *differential field*. An element $c \in R$ is called a *constant* with respect to δ if $\delta(c) = 0$. All constants with respect to δ form a subring of R, denoted by $C_{\delta,R}$. If R is a field, $C_{\delta,R}$ is also a field. Some basic facts concerning derivations are collected in the next lemma whose proof can be found in any book about differential algebra, e.g. [21]

Lemma 2.1.1. *Let (R, δ) be a differential ring. For $a, b \in R$ and $n \in \mathbb{N}$, we have*

(i) $\delta(1) = 0$;

(ii) $\delta(a^n) = na^{n-1}\delta(a)$;

(iii) If b is invertible, then

$$\delta\left(\frac{a}{b}\right) = \frac{\delta(a)b - a\delta(b)}{b^2}.$$

(iv) Logarithmic derivative identity: if a and b are invertible, then

$$\frac{\delta(a^m b^n)}{a^m b^n} = m\frac{\delta(a)}{a} + n\frac{\delta(b)}{b} \quad \textit{for all } m, n \in \mathbb{Z}.$$

Let $\sigma : R \to R$ be a monomorphism on R. Then the pair (R, σ) is called a *difference ring*. Moreover, if R is a field, R is called a *difference field*. An element $c \in R$ is called a *constant* with respect to σ if $\sigma(c) = c$. All constants with respect to σ form a subring of R, denoted by $C_{\sigma,R}$. If R is a field, $C_{\sigma,R}$ is also a field. The triple (R, δ, σ) is called a *differential-difference ring*. (R, δ, σ) is said to be *orthogonal* [62] if the derivation δ commutes with the monomorphism σ. Unless otherwise specified, all fields in the thesis are of characteristic zero and all differential-difference rings are orthogonal.

2.2 Ore polynomials

Ore polynomials are a common abstraction for linear differential operators, linear difference operators, shifts, and q-shifts [70, 23]. Let (F, δ, σ) be a differential-difference field. The *ring of Ore polynomials* $F[x; \delta, \sigma]$ over F is the univariate polynomial ring with indeterminate x, addition defined as usual, and multiplication being associative, distributive, and satisfying the commutation rule:

$$xf = \sigma(f)x + \delta(f), \quad \text{for any } f \in F. \tag{2.1}$$

By the commutation rule (2.1), the product of two polynomials in $F[x; \delta, \sigma]$ can be decided by association and distribution.

Example 2.2.1. *Let $F(t)$ be the univariate rational-function field in t over F.*

(i) The usual polynomial ring $F(t)[x]$ on which δ is the zero map and σ is the identity map.

(ii) The ring of differential operators $F(t)[x; d/dt, 1]$, where 1 is the identical map.

(iii) The ring of recurrence operators $F(t)[x; 0, \sigma]$, where $\sigma(f(t)) = f(t + 1)$ for any $f \in F(t)$.

(iv) The ring of difference operators $F(t)[x; \Delta, \sigma]$, where $\Delta = \sigma - 1$.

In this thesis, we also denote the ring of differential operators over $F(t)$ by $F(t)\langle D_t\rangle$ and the ring of difference operators over $F(t)$ by $F(t)\langle S_t\rangle$.

If σ is an automorphism, we can perform both left and right Euclidean division on the ring $F[x; \delta, \sigma]$ [70], that is, for any $f, g \in F[x; \delta, \sigma]$, there exist $q, r \in F[x; \delta, \sigma]$ such that $f = qg+r$ or $f = gq + r$, where $\deg(r) < \deg(g)$. This implies that $F[x; \delta, \sigma]$ is a left and right principal ideal domain. A multivariate extension of Ore polynomials is studied in [35, 32, 93].

2.3 Integration of rational functions

Symbolic evaluation of integrals is an active research domain of computer algebra, which revives the study of the integration problems from the algorithmic point of view. The first landmark algorithm, designed by Risch [77, 78] in the late 1960's, can decide whether indefinite integrals of elementary functions are elementary or not. State of the art about integrating transcendental functions has been presented in Bronstein's book [21]. For later use, we review some basic methods for rational-function integration. For more intensive presentations, see the books [21, Chapter 2], [45, Chapter 11], and [90, Chapter 22].

In this section, let F be a field of characteristic zero and $F(x)$ be the rational-function field in x over F. On the field $F(x)$, the derivation δ is defined by setting $\delta(x) = 1$ and $\delta(c) = 0$ for all $c \in F$. In most undergraduate calculus textbooks, we can find that any rational function $f \in \mathbb{C}(x)$ over the field $\mathbb{C}$ of complex numbers has an elementary integral of the form

$$\int f \, dx = g + \sum_{i=1}^{n} \beta_i \log(x - \alpha_i) \tag{2.2}$$

where $g \in \mathbb{C}(x)$ and $\alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, \ldots, n$. Here, we call g the *rational part* of the integral and the sum of logarithms the *logarithmic part* of the integral. In this analytic setting when the ground field is the algebraically closed field $\mathbb{C}$, the problem of rational function integration is well-understood. However, the algorithmic question becomes more involved when the ground field is not algebraically closed. Over an arbitrary field F, one may need to compute objects over an algebraic extension of F. In this case, the method in calculus books has many practical difficulties. For any rational function $f \in F(x)$, it has been shown by Hermite [52] that the rational part of the integral $\int f \, dx$ still lies in $F(x)$ and he presented a method to compute this part without introducing any algebraic extension of F. His method is named *Hermite reduction* in [21, 45, 90]. For the logarithmic part, in general it is required to introduce algebraic extensions of F and then the question is how to compute the minimal algebraic extension of F that is sufficient to express the integral. This question has been independently solved by Trager [87] and Rothstein [80].

2.3.1 Hermite reduction

Hermite reduction is a practical method for computing the rational part, which uses the method of integration by parts in order to reduce the denominator of the integrand to a square-free polynomial. More explicitly, Hermite reduction decomposes $f \in F(x)$ into

$$f = \delta(g) + \frac{a}{b}, \tag{2.3}$$

where $g \in F(x)$ and $a, b \in F[x]$ with $\deg(a) < \deg(b)$ and b squarefree. Such a pair $(g, a/b)$ in (2.3) is called an *additive decomposition* for f (with respect to x). The rational functions g and a/b are called the *rational* and *logarithmic* parts of f, respectively.

We summarize the idea of Hermite reduction following the treatment in [21, Chapter 2.2]. Write the integrand as $f = A/D$ with $A, D \in F[x]$ and $\gcd(A, D) = 1$. Let $D = D_1 D_2^2 \ldots D_n^n$ be the squarefree factorization of D. By computing the partial fraction decomposition for f, we have

$$f = P + \sum_{i=1}^{n} \frac{A_i}{D_i^i} \tag{2.4}$$

where P and the A_i's are in $F[x]$ and $\deg(A_i) < \deg(D_i^i)$ for each i. By the linearity of δ, it is sufficient to consider the same decomposition problem for any fraction of the form

$$\frac{P}{Q^m} \in F[x], \qquad \text{with } m > 1,\ \deg(P) < m \deg(Q) \text{ and } Q \text{ squarefree.}$$

The square-freeness of Q implies the existence of a Bézout relation

$$S\delta(Q) + TQ = 1, \tag{2.5}$$

where $S, T \in F[x]$ can be obtained by the extended Euclidean algorithm. Furthermore, upon multiplication and division with remainder, we get

$$P = SP\delta(Q) + TPQ = (GQ + B)\delta(Q) + TPQ = B\delta(Q) + CQ \tag{2.6}$$

where $C = G\delta(Q) + TP$ with $\deg(C) < (m-1)\deg(Q)$. Now, integration by parts yields

$$\frac{P}{Q^m} = \frac{B\delta(Q) + CQ}{Q^m} = \delta\left(\frac{-(m-1)^{-1}B}{Q^{m-1}}\right) + \frac{C + (m-1)^{-1}\delta(B)}{Q^{m-1}} \tag{2.7}$$

where $\deg(C + (m-1)^{-1}\delta(B)) < (m-1)\deg(Q)$ since $\deg(B) < \deg(Q)$ and $\deg(C) < (m-1)\deg(Q)$. This process is repeated until the denominator is square-free.

In 1975, Mack [69] introduced a variant of Hermite reduction that requires neither partial fraction decomposition nor squarefree factorization, but only extended GCD computation.

The following lemma due to Ostrogradsky [71] shows the uniqueness (up to an additive constant) of additive decompositions. For proofs, one can follow the argument in [45, Theorem 11.4].

Lemma 2.3.1. *Let $f = a/b$ be a nonzero rational function in $F(x)$ such that $a, b \in F[x]$, $\gcd(a, b) = 1$, $\deg(a) < \deg(b)$, and b is squarefree. Then there is no $g \in F(x)$ such that $f = \delta(g)$.*

Corollary 2.3.2. *Let f be a rational function in $F(x)$. Then the pair $(g, a/b)$ satisfying* (2.3) *for f is unique up to adding a constant to g.*

Proof. Assume that $(g_1, a_1/b_1)$ and $(g_2, a_2/b_2)$ are two additive decompositions for f with $a_1/b_1 \neq a_2/b_2$. The difference of logarithmic parts can be written as

$$0 \neq a_1/b_1 - a_2/b_2 = \frac{A}{B}, \quad \text{where } A, B \in F[x],\ \gcd(A, B) = 1, \text{ and } \deg(A) < \deg(B).$$

Since both b_1 and b_2 are squarefree and B divides $\operatorname{lcm}(b_1, b_2)$, B is also squarefree. However,

$$\frac{A}{B} = \delta(g_2 - g_1),$$

which contradicts Lemma 2.3.1. So a_1/b_1 equals a_2/b_2 and then g_1 and g_2 differ by an additive constant. $\square$

For later use, we recall a fact on the logarithmic derivatives of rational functions.

Lemma 2.3.3. *Let $f = a/b \in F(x)$ be such that $a, b \in F[x]$ and $\gcd(a, b) = 1$, then*

$$\frac{\delta f}{f} = \frac{p}{a^* b^*},$$

where a^* and b^* are, respectively, the squarefree parts of a and b, and $p \in F[x]$ with $\deg(p) < \deg(a^* b^*)$ and $\gcd(p, a^* b^*) = 1$.

Proof. Let $a = a_1 a_2^2 \cdots a_m^m$ and $b = b_1 b_2^2 \cdots b_n^n$ be the squarefree factorizations of a and b, respectively. Then $a^* = a_1 a_2 \cdots a_m$ and $b^* = b_1 b_2 \cdots b_n$. By Lemma 2.1.1 (iv), we have

$$\frac{\delta f}{f} = \frac{p}{a^* b^*}, \quad \text{where} \quad p = b^* \sum_{i=1}^{m} \frac{i \delta(a_i) a^*}{a_i} - a^* \sum_{j=1}^{n} \frac{j \delta(b_j) b^*}{b_i} \in F[x].$$

It is easy to see that $\deg(p) < \deg(a^* b^*)$. Since the a_i's and b_j's are squarefree and pairwise coprime, we have

$$\gcd\left(a^*, \sum_{i=1}^{m} \frac{i\delta(a_i)a^*}{a_i}\right) = \gcd\left(b^*, \sum_{j=1}^{n} \frac{j\delta(b_j)b^*}{b_i}\right) = 1,$$

which further implies $\gcd(p, a^* b^*) = 1$. This completes the proof. $\square$

The following lemma will be useful to verify uniqueness properties, in particular, in the proofs of Theorem 4.4.4 and Lemma 4.3.10.

Lemma 2.3.4. *Assume that f is a rational function in $F(x)$, $p_1, \ldots, p_n$ are pairwise coprime polynomials in $F[x]$, and $c_1, \ldots, c_n$ are constants in F. If*

$$\delta(f) = \sum_{i=1}^{n} c_i \frac{\delta(p_i)}{p_i},$$

then $f \in F$ and either $c_i = 0$ or $p_i \in F$ for all i such that $1 \leq i \leq n$.

Proof. Let p_i^* be the squarefree part of p_i for all i such that $1 \leq i \leq n$. By Lemma 2.3.3, we have

$$\sum_{i=1}^{n} c_i \frac{\delta(p_i)}{p_i} = \sum_{i=1}^{n} c_i \frac{q_i}{p_i^*} =: \frac{a}{b},$$

where $a, b \in F[x]$ with $\gcd(a, b) = 1$ and the q_i's are polynomials in $F[x]$ with $\gcd(q_i, p_i^*) = 1$ and $\deg(q_i) < \deg(p_i^*)$. Since the p_i's are pairwise coprime, so are the p_i^*'s. Hence, the denominator b is squarefree and $\deg(a) < \deg(b)$. By Lemma 2.3.1, a/b must be equal to zero and then $f \in F$. By the uniqueness of squarefree partial fraction decomposition, all the fractions $c_i q_i / p_i^*$ are equal to zero. This implies that either $c_i = 0$ or $p_i \in F$ for each i such that $1 \leq i \leq n$. $\square$

Corollary 2.3.5. *Let $c_1, \ldots, c_n \in F$ be linearly independent over $\mathbb{Z}$. If there exist rational functions $f_1, \ldots, f_n \in F(x)$ such that*

$$\sum_{i=1}^{n} c_i \frac{\delta(f_i)}{f_i} = 0,$$

then $f_1, \ldots, f_n$ belong to F.

Proof. Since every element of F is a constant with respect to δ, we may suppose that none of $f_1, \ldots, f_n$ belongs to F, and look for a contradiction. Under this assumption, the sum can be rewritten as

$$\sum_{j=1}^{m} \bar{c}_j \frac{\delta(p_j)}{p_j} = 0$$

where $\bar{c}_j$ is a $\mathbb{Z}$-linear combination of $c_1, \ldots, c_n$, and the p_j's are nontrivial distinct irreducible factors of the denominators of the $\delta(f_i)/f_i$. Since every $\bar{c}_j$ is nonzero, Lemma 2.3.4 implies that every p_j is in F, which is a contradiction. □

2.3.2 Ostrogradsky and Horowitz's method

Ostrogradsky and Horowitz's method [71, 53] computes the additive decomposition of a rational function by solving a linear system. Although this method has asymptotically higher complexity than that of Hermite reduction [90, Section 22.2], it is useful for our complexity analyses in the sequel.

Let $f = P/Q \in F(x)$ be such that $P, Q \in F[x]$ and $\gcd(P, Q) = 1$. After reading out the polynomial part of f, we further assume that $\deg(P) < \deg(Q)$. Let Q^* be the squarefree part of Q and $Q^- = Q/Q^*$. Denote $d_x^* = \deg_x(Q^*)$ and $d_x^- = \deg_x(Q^-)$. According to the functionality of Hermite reduction, one can find two unique polynomials A and a in $F[x]$ with $\deg_x A < \deg(Q^-)$ and $\deg_x a < \deg(Q^*)$ such that

$$\frac{P}{Q} = \delta\left(\frac{A}{Q^-}\right) + \frac{a}{Q^*}. \tag{2.8}$$

Note that A and a satisfy (2.8) if and only if they satisfy the equation

$$P = \delta(A)Q^* - A\widetilde{Q} + aQ^-, \tag{2.9}$$

where $\widetilde{Q} = Q^*\delta(Q^-)/Q^-$ is a polynomial in $F[x]$ of degree $\deg(Q^*) - 1$. Now, equation (2.9) can be solved by the method of undetermined coefficients. Write $P = \sum_{l=0}^{d_x^*+d_x^- -1} P_l x^l$ and set $A = \sum_{i=0}^{d_x^- -1} A_i x^i$, $a = \sum_{j=0}^{d_x^* -1} a_j x^j$ with undetermined coefficients A_i and a_j. Then (2.9) holds if and only if

$$\left(A_{d_x^- -1}, \ldots, A_0, a_{d_x^* -1}, \ldots, a_0\right) M = \left(P_{d_x^*+d_x^- -1}, \ldots, P_0\right), \tag{2.10}$$

where M is a $\deg(Q) \times \deg(Q)$ matrix over F obtained by equating the likewise powers of x

in (2.9). By the uniqueness of A/Q^- and a/Q^*, the system (2.10) has a unique solution, which leads to the following lemma.

Lemma 2.3.6. *The matrix M in* (2.10) *is invertible over F.*

We call the linear system (2.10) the *Ostrogradsky–Horowitz system* and M the *Ostrogradsky–Horowitz matrix* associated with Q. Note that M is uniquely determined by Q. So we denote this matrix by $\mathcal{M}(Q)$.

2.3.3 Residues and Rothstein–Trager resultants

After additive decomposition, the integration problem of rational functions is reduced to computing the integrals of the form

$$\int \frac{a}{b}\,dx, \quad \text{where } a, b \in F[x] \text{ with } \gcd(a,b) = 1,\ \deg(a) < \deg(b) \text{ and } b \text{ squarefree.} \tag{2.11}$$

Over the algebraic closure $\overline{F}$ of F, the rational function a/b above decomposes into

$$\frac{a}{b} = \sum_{i=1}^{n} \frac{\beta_i}{x - \alpha_i}, \quad \text{where } \alpha_i, \beta_i \in \overline{F} \text{ and } b(\alpha_i) = 0 \text{ for } 1 \le i \le n.$$

Consequently, the integral of a/b can be expressed by

$$\int \frac{a}{b}\,dx = \sum_{i=1}^{n} \beta_i \log(x - \alpha_i).$$

By convention, the value β_i is called the *residue* of a/b at the point $x = \alpha_i$. According to the Lagrange formula ([44, page 38] or [45, Exercise 11.8]), the residue of a/b at α_i is

$$\beta_i = \frac{a}{\delta(b)}(\alpha_i) \in F(\alpha_i). \tag{2.12}$$

So we can always express the integrals in (2.11) over the splitting field of the denominator b. However, it is not necessary to compute splitting fields for obtaining the integrals. For instance, the integral below can be expressed without any algebraic extension:

$$\int \frac{2x}{x^2 - 2}\,dx = \log(x + \sqrt{2}) + \log(x - \sqrt{2}) = \log(x^2 - 2).$$

In fact, Rothstein [80] and Trager [87] have shown that the minimal algebraic extension of F for expressing the integrals is the splitting field of the following polynomial

$$R(z) = \text{resultant}_x(b, a - z \cdot \delta(b)) \in F[z]. \tag{2.13}$$

In the literature, $R(z)$ above is called the *Rothstein–Trager resultant* of a/b with respect to x, denoted henceforth by $\mathrm{RT}_x(a/b)$. By the formula (2.12), all the residues of a/b at its poles are the roots of $R(z)$. Moreover, we have the following lemma, which appears implicitly in the literature [45, 21, 90].

Lemma 2.3.7. *Let $a, b \in F[x]$ be such that $\deg(a) < \deg(b)$, $\gcd(a, b) = 1$ and b is squarefree in $F[x]$. Let $R(z)$ be the Rothstein–Trager resultant of a/b with respect to x. Then we have*

(i) all roots of $R(z)$ are nonzero;

(ii) if α_1 and α_2 are two distinct roots of $R(z)$, then p_1 and p_2 are coprime over $\overline{F}$, where

$$p_i = \gcd(b,\, a - \alpha_i \delta(b)) \in F(\alpha_i)[x], \quad \textit{for } i = 1, 2.$$

Proof. The first assertion follows from the fact that $\gcd(a, b) = 1$. For the second one, we suppose p_1 and p_2 are not coprime over $\overline{F}$. Then there exists $\beta \in \overline{F}$ such that $p_1(\beta) = p_2(\beta) = 0$. By the definition of p_1 and p_2, $b(\beta) = 0$ and $(a - \alpha_i\delta(b))(\beta) = 0$ for $i = 1, 2$. Since b is squarefree in $F[x]$, $\delta(b)(\beta) \neq 0$, which implies

$$\alpha_1 = \alpha_2 = \frac{a}{\delta b}(\beta),$$

which is a contradiction with $\alpha_1 \neq \alpha_2$. □

The next theorem is fundamental to design the algorithm for computing the logarithmic part of the integral of a rational function. For proofs, see [21, Theorem 2.4.1] or [90, Theorem 22.8].

Theorem 2.3.8 (Rothstein and Trager's theorem)**.** *Let $a, b \in F[x]$ be such that $\deg(a) < \deg(b)$, $\gcd(a, b) = 1$ and b is squarefree in $F[x]$. Let $R(z)$ be the Rothstein–Trager resultant of a/b with respect to x. Then the integral of rational function a/b can be expressed as*

$$\int \frac{a}{b}\, dx = \sum_{i=1}^{n} c_i \log(g_i),$$

where $c_1, \ldots c_n \in \overline{F}$ are the distinct roots of $R(z)$ and $g_i = \gcd(b, a - c_i\delta(b)) \in F(c_i)[x]$ for all $1 \leq i \leq n$. Moreover, if E is an algebraic extension of F such that we have

$$\int \frac{a}{b}\, dx = \sum_{i=1}^{m} \tilde{c}_i \log(\tilde{g}_i),$$

where $\tilde{c}_1, \ldots, \tilde{c}_m \in E\backslash\{0\}$ and $\tilde{g}_1, \ldots, \tilde{g}_m \in E[x]\backslash E$ are monic, squarefree, and pairwise coprime, then E contains all the roots of $R(z)$.

We conclude this section by the following theorem on the integration of rational functions.

Theorem 2.3.9. *Let f be a nonzero element in $F(x)$. Then there exist $g \in F(x)$, nonzero elements $c_j \in \overline{F}$ and $p_j \in F(c_j)[x] \setminus F(c_j)$ for $j = 1, \ldots, n$ such that*

$$f = \delta(g) + \sum_{j=1}^{n} c_j \frac{\delta(p_j)}{p_j}.$$

Moreover, p_j and p_{j^*} are coprime over $\overline{F}$ whenever $j \neq j^*$.

Proof. By Hermite reduction, there exist $f \in F(x)$, $a, b \in F[x]$ with $\deg a < \deg b$, $\gcd(a, b) = 1$ and b being squarefree such that

$$f = \delta(g) + \frac{a}{b}.$$

If a is zero, then there is nothing to prove. Assume that a is nonzero and let Λ be the set of distinct roots of the Rothstein-Trager resultant $R(z)$ of a/b. By Lemma 2.3.7 (i), all elements in Λ are nonzero. By Theorem 2.3.8,

$$f = \delta(g) + \sum_{\lambda \in \Lambda} \lambda \frac{\delta(p_\lambda)}{p_\lambda},$$

where $p_\lambda = \gcd(b, a - \lambda \delta b) \in F(\lambda)[x] \setminus F(\lambda)$. By Lemma 2.3.7 (ii), p_λ and p_μ are coprime if λ and μ are two distinct elements of Λ. This completes the proof. $\square$

2.4 Background on complexity

In this section, we recall basic notation and complexity results from [90]. These results are useful to analyze algorithms in Chapter 3. The complexity results are expressed using the "big Oh" notation [90, Section 25.7]. For instance, we say that the Karatsuba algorithm for multiplying two polynomials in $\mathbb{Q}[x]$ of degree at most n takes $\mathcal{O}(n^{\log_2 3})$ arithmetic operations in $\mathbb{Q}$. Furthermore, we use the notation $\tilde{\mathcal{O}}(\cdot)$ to indicate cost estimates with hidden logarithmic factors [90, Definition 25.8].

Let k be a field of characteristic zero. Unless otherwise specified, all complexity estimates are given in terms of arithmetic operations in k, which we denote by "ops". Let $k[x]_{\leq d}^{m \times n}$ be the set of $m \times n$ matrices with coefficients in $k[x]$ of degree at most d. Let $\omega \in [2, 3]$ be a feasible exponent of matrix multiplication, so that two matrices from $k^{n \times n}$ can be multiplied using $\mathcal{O}(n^\omega)$ ops.

Facts 2.4.1 and 2.4.2 below show the complexity of multi-point evaluation, rational interpolation, and algebraic operations on polynomial matrices using fast arithmetic. For proofs, see [90, Corollaries 10.8, 5.18, 11.6] and [84, Theorem 7.3].

Fact 2.4.1. *For a polynomial $p \in k[x]$ of degree less than n, pairwise distinct $u_0, \ldots, u_{n-1}$ in k, and $v_0, \ldots, v_{n-1} \in k$, we have:*

(i) Fast evaluation: evaluating p at the u_i's takes $\tilde{\mathcal{O}}(n)$ ops.

(ii) Fast interpolation: for $m \in \{1, \ldots, n\}$, constructing $f = s/t \in k(x)$ with $\deg_x(s) < m$ and $\deg_x(t) \leq n - m$ such that $t(u_i) \neq 0$ and $f(u_i) = v_i$ for $0 \leq i \leq n-1$ takes $\tilde{\mathcal{O}}(n)$ ops.

Fact 2.4.2. *For M in $k[x]_{\leq d}^{m \times n}$ with $d > 0$, we have:*

(i) If $M = \begin{pmatrix} M_1 & M_2 \end{pmatrix}$ is an invertible $n \times n$ matrix with $M_i \in k[x]_{\leq d_i}^{n \times n_i}$, where $i = 1, 2$ and $n_1 + n_2 = n$, then the degree of $\det(M)$ is at most $n_1 d_1 + n_2 d_2$.

(ii) If $M = \begin{pmatrix} M_1 & M_2 \end{pmatrix}$ is not of full rank and with $M_i \in k[x]_{\leq d_i}^{m \times n_i}$, where $i = 1, 2$ and $n_1 + n_2 = n$, then there exists a nonzero $u \in k[x]^n$ with coefficients of degree at most $n_1 d_1 + n_2 d_2$ such that $Mu = 0$.

(iii) The rank r and a basis of the null space of M can be computed using $\tilde{\mathcal{O}}(nmr^{\omega-2}d)$ ops.

The complexity in terms of arithmetic operations in k of Hermite reduction has been analyzed by Yun in [94] and also [90, Theorem 22.7].

Lemma 2.4.3 (Yun, 1977). *Let f be a nonzero rational function in $k(x)$ of degree at most n in x, then Hermite reduction on f can be performed using $\tilde{\mathcal{O}}(n)$ operations in k.*

Chapter 3

Hermite Reduction for Rational-Function Telescoping

3.1 Introduction

Although creative-telescoping algorithms have been now used extensively in modern computer algebra, very little is known about their complexity: some related results seem to be the complexity analyses of an algorithm for hyperexponential indefinite integration in [49], of algorithms for rational and hypergeometric summation in [50, 19], and of an algorithm by Takayama for finding a recurrence for hypergeometric series of a special form in [86]. In order to get complexity estimates, we simplify the problem by restricting to a smaller class of inputs, namely that of bivariate rational functions. Therefore, our goal reads as follows.

Problem 1. *Given a rational function $f \in k(x,y) \setminus \{0\}$, find a nonzero operator L in $k(x)\langle D_x \rangle$ and a rational function g in $k(x,y)$ such that*

$$L(x, D_x)(f) = D_y(g). \tag{3.1}$$

Here, L is called a *telescoper* for f with respect to y and g the *certificate* of L for f. Since D_y commutes with any element in $k(x)\langle D_x \rangle$, we can always choose telescopers with polynomial coefficients in $k[x]$.

By considering this constrained class of inputs, we are indeed able to blend the general method of creative telescoping with the well-known Hermite reduction [52]. Although restricted, this class already has many applications. In 1827, Abel observed that an algebraic function satisfied a linear differential equation [1]. The annihilating differential equations are important in the study of algebraic functions [37, 30, 31]. In [20], it was shown that differential equations for algebraic functions can be computed via rational-function telescoping. In combinatorics, many nontrivial problems are encoded as diagonals of rational formal power series [83, 74]. Differential equations for diagonals of a bivariate rational function can also be constructed via rational-function telescoping.

Essentially two algorithms for computing telescopers of minimal order can be found in the literature: the classical way [15] is to apply a differential analogue of Gosper's indefinite summation algorithm [51], which reduces the problem to solving an auxiliary linear differential equation for polynomial solutions. An algorithm developed later in [46] (see also [63]) performs Hermite reduction on f to get an additive decomposition of the form

$$f = D_y(g) + \sum_{i=1}^{m} \frac{u_i}{v_i},$$

where the u_i and v_i are in $k(x)[y]$ and the v_i are squarefree. Then, the algorithm in [15] is applied to each u_i/v_i to get a minimal telescoper L_i for it. The least common left multiple of the L_i's is proved to be a minimal telescoper for f. This algorithm performs well only for specific inputs (both in practice and from the complexity viewpoint), but it inspired our improvements over Almkvist and Zeilberger's method.

Our contribution. For bivariate rational functions, we present a new and provably faster algorithm for computing minimal telescopers, which is based on Hermite reduction. Over the classical method by Almkvist and Zeilberger, we make some improvements avoiding unnecessary resultant calculations and integer-root finding. We derive complexity estimates of all algorithms described in Figure 3.1, showing that our approach is faster. These complexity results give the first proof of a polynomial complexity (in terms of field operations) for creative telescoping on bivariate rational-function inputs. In the computer algebra system Maple, we have implemented our algorithms for rational-function telescoping, which has been integrated into the existing Maple library Algolib 13.0 (for Maple 13). Our experimental results show that the Hermite-reduction based algorithm outperforms all other known algorithms concerning both the worst-case complexity and the actual timings of implementations.

Algorithms	$\deg_{D_x}(L)$	$\deg_x(L)$	$\deg_x(g)$	$\deg_y(g)$	Complexity
HermiteTelescoping	$\leq d_y$	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_y^2)$	$\tilde{\mathcal{O}}(d_x d_y^{\omega+3})$
RationalAZ	$\leq d_y$	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_y^2)$	$\tilde{\mathcal{O}}(d_x d_y^{2\omega+2})$

Table 3.1: Complexity for rational-function telescoping

The rest of this chapter is organized as follows. In Section 3.2, we recall notation about bivariate polynomials and set some hypotheses used in this chapter. We study Hermite reduction over $k(x)$ in Section 3.3, proving tight output degree bounds and an optimal algorithm via fast evaluation and interpolation. We present a Hermite-reduction based algorithm for minimal telescopers and some improvements over the classical method in Section 3.4. In Section 3.5, we show the existence of telescopers and estimate their order bounds. At last, we describe in Section 3.6 an implementation and show some experimental results.

The work in this chapter is published in [18], which is a joint work with Alin Bostan, Frédéric Chyzak and Ziming Li.

3.2 Notation and hypotheses

In this section, we review some basic notation and introduce some hypotheses for later use.

Let Q be a bivariate polynomial in $k[x,y]\setminus k[x]$. The *squarefree factorisation* of Q with respect to y is the unique product $qQ_1Q_2^2\cdots Q_m^m$ where $q \in k[x]$ and $Q_i \in k[x,y]$ satisfying $\deg_y(Q_m) > 0$ and such that the Q_i's are primitive, squarefree, and pairwise coprime. The *squarefree part* Q^* of Q with respect to y is the product $Q_1Q_2\cdots Q_m$. Let Q^- denote the polynomial Q/Q^*, and $\mathrm{lc}_y(Q)$ the leading coefficient of Q with respect to y. The following two formulas about Q, Q^*, and Q^- can be proved by mere calculations.

Fact 3.2.1. *Let $\hat{Q}_i$ denote Q^*/Q_i. Then we have*

(i) $Q^* D_y(Q^-)/Q^- = \sum_{i=1}^m (i-1)\hat{Q}_i D_y(Q_i) \in k[x,y]$;

(ii) $D_y(Q)/Q^- = \sum_{i=1}^m i\hat{Q}_i D_y(Q_i) \in k[x,y]$.

Let $f = P/Q$ be a nonzero element in $k(x, y)$, where P, Q are two coprime polynomials in $k[x, y]$. The degree of f in x is defined to be $\max\{\deg_x(P), \deg_x(Q)\}$, and denoted by $\deg_x(f)$. The degree of f in y is defined similarly. The *bidegree* of f is the pair $(\deg_x(f), \deg_y(f))$, which is denoted by $\mathrm{b}ideg(f)$. The bidegree of f is said to be *bounded (above) by* (α, β), written $\mathrm{b}ideg(f) \leq (\alpha, \beta)$, when $\deg_x(f) \leq \alpha$ and $\deg_y(f) \leq \beta$.

We say that $f = P/Q$ is *proper* if the degree of P in y is less than that of Q. For creative telescoping, we may always assume w.l.o.g. that $f = P/Q$ is proper. If not, rewrite $f = D_y(p) + \bar{f}$ with $p \in k(x)[y]$ and $\bar{f}$ proper. A telescoper L for $\bar{f}$ with certificate $\bar{g}$ is a telescoper for f with certificate $L(p) + \bar{g}$. So we introduce the following hypothesis.

Hypothesis (H) *From now on, P and Q are assumed to be nonzero polynomials in $k[x, y]$ such that $\deg_y(P) < \deg_y(Q)$, $\gcd(P, Q) = 1$, and Q is primitive with respect to y.*

Notation *From now on, we write (d_x, d_y), (d_x^*, d_y^*), and (d_x^-, d_y^-) for the bidegrees of Q, Q^*, and Q^-, respectively.*

Sometimes, we use the following hypothesis in order to make estimates concise.

Hypothesis (H') Hypothesis (H) and $\deg_x(P) \leq d_x$.

3.3 Hermite reduction for bivariate rational functions

In this section, we will apply Hermite reduction to a bivariate rational function of $k(x, y)$ with respect to y. We present a quasi-optimal algorithm to perform the Hermite reduction by using a fast evaluation-interpolation approach.

3.3.1 Output size estimates

In order to use the evaluation-interpolation strategy, we first derive an upper bound on the degrees of the outputs of the Hermite reduction on a rational function. To this end, we will use the linear system introduced in Ostrogradsky and Horowitz's method.

After reading out the polynomial part, we may assume that $f = P/Q \in k(x, y)$ with P and Q in $k[x, y]$ such that $\gcd(P, Q) = 1$ and $\deg_y(P) < \deg_y(Q)$. Let D_x and D_y denote the usual derivations $\partial/\partial x$ and $\partial/\partial y$ on $k(x, y)$, respectively. Recall that Q^* denotes the squarefree part of Q with respect to y and $Q^- = Q/Q^*$. Set

$$d_x^* = \deg_x(Q^*),\ d_y^* = \deg_y(Q^*),\ d_x^- = \deg_x(Q^-),\ \text{and}\ d_y^- = \deg_y(Q^-).$$

According to Ostrogradsky and Horowitz's method, we make the ansatz

$$\frac{P}{Q} = D_y\left(\frac{A}{Q^-}\right) + \frac{a}{Q^*}, \tag{3.2}$$

where $A, a \in k(x)[y]$ with $\deg_y(A) < d_y^-$ and $\deg_y(a) < d_y^*$. In order to bound the bidegrees of A and a, we reformulate (3.2) into the equivalent form

$$P = Q^* D_y(A) - \left(\frac{Q^* D_y(Q^-)}{Q^-}\right) A + Q^- a, \tag{3.3}$$

where $\widetilde{Q} = Q^* D_y(Q^-)/Q^-$ is a polynomial in $k[x,y]$ of bidegree at most $(d_x^*, d_y^* - 1)$ by Fact 3.2.1. Viewing A and a as polynomials in $k(x)[y]$ with undetermined coefficients, we form the following Ostrogradsky–Horowitz system,

$$M \begin{pmatrix} \hat{A} \\ \hat{a} \end{pmatrix} = \hat{P}, \tag{3.4}$$

where M is the Ostrogradsky–Horowitz matrix associated to Q and $\hat{A}$, $\hat{a}$, and $\hat{P}$ are the coefficient vectors of A, a, and P with sizes d_y^-, d_y^*, and d_y, respectively. From equation (3.3), the matrix M is of the form

$$\begin{pmatrix} M_1 & M_2 \end{pmatrix}, \quad \text{where } M_1 \in k[x]_{\le d_x^*}^{d_y \times d_y^-} \text{ and } M_2 \in k[x]_{\le d_x^-}^{d_y \times d_y^*}.$$

Let Δ be the determinant of M, so that $\deg_x(\Delta) \le \mu := d_x^* d_y^- + d_x^- d_y^*$ by Fact 2.4.2*(i)*. For later use, we also define Δ' as the determinant of $\mathcal{M}(Q^{*2})$, so that $\deg_x(\Delta') \le \mu' := 2 d_x^* d_y^*$ by Fact 2.4.2 (i) and since $(Q^{*2})^- = Q^*$.

Lemma 3.3.1. *Let $f = P/Q \in k(x,y)$ with $P, Q \in k[x,y]$, $\gcd(P,Q) = 1$ and $\deg_y(P) < \deg_y(Q)$. Then there exist $B, b \in k[x,y]$ with $\deg_y(B) < d_y^-$ and $\deg_y(b) < d_y^*$, and such that:*

(i) $f = D_y\left(\frac{B}{\Delta Q^-}\right) + \frac{b}{\Delta Q^*}$;

(ii) $\deg_x(B) \le \mu - d_x^* + \deg_x(P)$ *and* $\deg_x(b) \le \mu - d_x^- + \deg_x(P)$.

Proof. Applying Cramer's rule to (3.4) leads to the first assertion. The second assertion follows by determinant expansions. ∎

Algorithm HermiteEvalInterp

Input: $P, Q \in k[x,y]$ satisfying Hypothesis (H).

Output: $(A, a) \in k(x)[y]^2$ solving (3.2).

1. Compute $Q^- := \gcd(Q, D_y(Q))$ and $Q^* := Q/Q^-$;

2. Set $\lambda := 2(d_x^* d_y^- + d_y^* d_x^-) + \deg_x(P) - \min\{d_x^-, d_x^*\}$;

3. Set S to the set of $\lambda + 1$ smallest nonnegative integers that are lucky for Q;

4. For each $x_0 \in S$, compute $(A_0, a_0) \in k[y]^2$ such that

$$\frac{P(x_0, y)}{Q(x_0, y)} = D_y\left(\frac{A_0}{Q^-(x_0, y)}\right) + \frac{a_0}{Q^*(x_0, y)}$$

   using Hermite reduction over k;

5. Compute $(A, a) \in k(x)[y]$ by rational interpolation and return this pair.

Figure 3.1: Hermite reduction over $k(x)$ via evaluation and interpolation.

In what follows, we shall encounter proper rational functions with denominator Q satisfying $Q = Q^{*2}$. The following lemma is an easy corollary of Lemma 3.3.1 for such functions.

Corollary 3.3.2. *Assume that $Q = Q^{*2}$ additionally in Lemma 3.3.1. Then there exist $B, b \in k[x,y]$ with $\deg_y(B)$ and $\deg_y(b)$ less than d_y^*, and such that*

(i) $\frac{P}{Q^{*2}} = D_y\left(\frac{B}{\Delta' Q^*}\right) + \frac{b}{\delta' Q^*}$;

(ii) $\deg_x(B)$ *and* $\deg_x(b)$ *are bounded by* $\mu' - d_x^* + \deg_x(P)$.

3.3.2 Algorithm by evaluation and interpolation

We observe that an asymptotically optimal complexity can be achieved by evaluation and interpolation at each step of Hermite reduction over $k(x)$. This inspires us to adapt Gerhard's modular method [48, 49] to $k(x, y)$. For simplicity, we further assume that $Q \in k[x,y]$ is nonzero and primitive over $k[x]$.

Definition 3.3.3. *An element $x_0 \in k$ is said to be* lucky *if* $\mathrm{lc}_y(Q)(x_0) \neq 0$ *and the degree in y of* $\gcd(Q(x_0, y), D_y Q(x_0, y))$ *is equal to* $\deg_y(Q^-)$.

Lemma 3.3.4. *There are at most $d_x(2d_y^* - 1)$ unlucky points.*

Proof. Let $\sigma \in k[x]$ be the d_y^-th subresultant with respect to y of Q and $D_y(Q)$. By Corollary 5.5 in [49], all unlucky points are in the set $U = \{ x_0 \in k \mid \sigma(x_0) = 0 \}$. By Corollary 3.2 (*ii*) in [49], $\deg_x(\sigma) \leq d_x(2d_y^* - 1)$. □

Lemma 3.3.5. *Let B, b, and Δ be the same as in Lemma 3.3.1, and let $x_0 \in k$ be lucky. Then $\Delta(x_0) \neq 0$ and $(B(x_0, y), b(x_0, y))$ is the unique pair such that*

$$\frac{P(x_0, y)}{Q(x_0, y)} = D_y\left(\frac{B(x_0, y)}{\Delta(x_0)Q^-(x_0, y)}\right) + \frac{b(x_0, y)}{\Delta(x_0)Q^*(x_0, y)}. \tag{3.5}$$

Proof. By the luckiness of x_0, $\deg_y(Q(x_0, y)) = d_y$ and $Q(x_0, y)^- = Q^-(x_0, y)$. Then we have $Q(x_0, y)^* = Q^*(x_0, y)$. This implies $\mathcal{M}(Q)(x_0, y) = \mathcal{M}(Q(x_0, y))$, which, by Lemma 2.3.6, is invertible over $k(x)$. Hence $\Delta(x_0) \neq 0$, and the evaluation at $x = x_0$ of the equality in Lemma 3.3.1 (*i*) is well-defined. Thus, $(B(x_0, y), b(x_0, y))$ is a solution of (3.5). Uniqueness follows from Corollary 2.3.2. □

Theorem 3.3.6. *Algorithm* HermiteEvalInterp *in Figure 3.1 is correct and takes $\tilde{\mathcal{O}}(d_x d_y^2 + \deg_x(P) d_y)$ operations in k.*

Proof. Set ν to $d_x(2d_y^* - 1)$. Lemma 3.3.4 implies that the $\lambda + 1$ lucky points found in Step 3 are all less than $\lambda + \nu + 1$. By Corollary 2.3.2 and 3.3.1 (*i*), $A = B/\Delta$ and $a = b/\Delta$. By Lemma 3.3.5, $A_0 = B(x_0, y)/\Delta(x_0)$ and $a_0 = b(x_0, y)/\Delta(x_0)$. By Lemma 3.3.1 (*ii*) and since $\deg_x(\Delta) \leq \mu$, it suffices to rationally interpolate A and a from values at $\lambda + 1$ lucky points. This shows the correctness. The dominant computation in Step 1 is the gcd, which takes $\tilde{\mathcal{O}}(d_x d_y)$ ops by Corollary 11.9 in [90]. For each integer $i \leq \lambda + \nu$, testing luckiness amounts to evaluations at x_0 and computing $\gcd(Q(x_0, y), D_y(Q(x_0, y)))$, which takes $\tilde{\mathcal{O}}(d_y)$ ops by Fact 2.4.1 (*i*) and Corollary 11.6 in [90]. Then, generating S in Step 3 costs $\tilde{\mathcal{O}}((\lambda + \nu + 1)d_y)$ ops. By Fact 2.4.1 (*i*), evaluations in Step 4 take $\tilde{\mathcal{O}}((\lambda + 1)d_y)$ ops. For each $x_0 \in S$, the cost of the Hermite reduction in Step 4 is $\tilde{\mathcal{O}}(d_y)$ ops by Lemma 2.4.3. Thus, the total cost of Step 4 is $\tilde{\mathcal{O}}((\lambda + 1)d_y)$ ops. By Fact 2.4.1*(ii)*, Step 5 takes $\tilde{\mathcal{O}}((\lambda + 1)d_y)$ ops. Since $\lambda \leq 2d_x d_y + \deg_x(P)$ and $\nu \leq 2d_x d_y$, the total cost is as announced. □

Remark 3.3.7. *As the generic output size of Hermite reduction is proportional to λd_y, which is equal to $\mathcal{O}((d_x d_y + \deg_x(P))d_y)$, Algorithm* HermiteEvalInterp *has quasi-optimal complexity.*

3.4 Minimal telescopers for bivariate rational functions

Since the 1990's, the main emphasis in existing works [95, 15, 46, 63] has been on finding telescopers of order minimal over all telescopers for f, which are called *minimal telescopers*. For a rational function $f \in k(x, y)$, all its telescopers form a left ideal I_f in the left principal ideal domain $k(x)\langle D_x \rangle$. So a minimal telescoper is a generator of I_f and two minimal telescopers differ by a multiplicative left factor in $k(x)$. Therefore, a rational function has a unique monic minimal telescoper. In this section, we present a new method for computing minimal telescopers for rational functions in $k(x, y)$, which is based on Hermite reduction. Also, we make some improvements over the classical method by Almkvist and Zeilberger. We will show that the arithmetic complexity of Hermite reduction approach is lower than that of the classical one.

3.4.1 Hermite-reduction based method

In some applications, we only need to compute telescopers without certificates. In the next section, we will show that the arithmetic size of certificates is asymptotically larger than that of telescopers. This motivates us to find a way in which one could choose to compute or not to compute the certificates according to the user requirements. To this end, we design a new algorithm, presented in Figure 3.2, to compute minimal telescopers for rational functions by basing on Hermite reduction.

For $f = P/Q \in k(x, y)$ and $i \in \mathbb{N}$, Hermite reduction decomposes $D_x^i(f)$ into

$$D_x^i(f) = D_y(g_i) + r_i, \tag{3.6}$$

where $g_i, r_i \in k(x, y)$ are proper. Since the squarefree part of the denominator of $D_x^i(f)$ divides Q^*, so does the denominator of r_i. The following lemma shows that (3.6) recombines into telescopers and certificates; next, Lemma 3.4.2 implies that the first pair obtained in this way by Algorithm $\mathsf{HermiteTelescoping}$ in Figure 3.2 yields a minimal telescoper.

Lemma 3.4.1. *The rational functions $r_0, \ldots, r_{d_y^*}$ are linearly dependent over $k(x)$.*

Proof. The constraints on r_i imply $\deg_y(r_i Q^*) < d_y^*$ for all $i \in \mathbb{N}$, from which follows the existence of a nontrivial linear dependence among the r_i's over $k(x)$. $\square$

Algorithm HermiteTelescoping

Input: $f = P/Q \in k(x, y)$ satisfying Hypothesis (H).

Output: A minimal telescoper $L \in k[x]\langle D_x \rangle$ with certificate $g \in k(x, y)$.

1. Apply HermiteEvalInterp to f to get (g_0, a_0) such that $f = D_y(g_0) + a_0/Q^*$.
   If $a_0 = 0$, return $(1, g_0)$.

2. For i from 1 to $\deg_y(Q^*)$ do

   (a) Apply HermiteEvalInterp to $-a_{i-1}D_x(Q^*)/Q^{*2}$ to express it as $D_y(\tilde{g}_i) + \tilde{a}_i/Q^*$.

   (b) Set $g_i = D_x(g_{i-1}) + \tilde{g}_i$ and $a_i = D_x(a_{i-1}) + \tilde{a}_i$.

   (c) Solve $\sum_{j=0}^{i} \eta_j a_j = 0$ for $\eta_j \in k(x)$ using [84, Algorithm Nullspace].
   If there exists a nontrivial solution, then set $(L, g) := \left(\sum_{j=0}^{i} \eta_j D_x^j, \sum_{j=0}^{i} \eta_j g_j\right)$, and break.

3. Compute the content c of L and return $(c^{-1}L, c^{-1}g)$.

Figure 3.2: Creative telescoping by Hermite reduction

Lemma 3.4.2. *An integer ρ is minimal such that $\sum_{i=0}^{\rho} \eta_i r_i = 0$ for $\eta_0, \ldots, \eta_\rho \in k(x)$ not all zero if and only if $\sum_{i=0}^{\rho} \eta_i D_x^i$ is a minimal telescoper for f with certificate $\sum_{i=0}^{\rho} \eta_i g_i$.*

Proof. Multiplying (3.6) by η_i before summing yields

$$L(f) = D_y\left(\sum_{i=0}^{\rho} \eta_i g_i\right) + \sum_{i=0}^{\rho} \eta_i r_i \quad \text{for} \quad L := \sum_{i=0}^{\rho} \eta_i D_x^i,$$

where the sum $\sum_{i=0}^{\rho} \eta_i r_i$ is a proper fraction in y with a squarefree denominator with respect to y. Thus, by Corollary 2.3.2, L is a telescoper of order ρ for f with certificate $\sum_{i=0}^{\rho} \eta_i g_i$ if and only if $\sum_{i=0}^{\rho} \eta_i r_i = 0$ with $\eta_\rho \neq 0$. The lemma follows. $\square$

Remark 3.4.3. *In the algorithm HermiteTelescoping described in Figure 3.2, one can choose not to compute the certificate g, because the minimal telescoper and this corresponding certificate are computed separately.*

Order bounds for minimal telescopers

Lemmas 3.4.1 and 3.4.2 combine into a tight upper bound on the order of minimal telescopers for f.

Corollary 3.4.4. *Minimal telescopers have order at most d_y^*.*

We also derive a lower bound on the order of the minimal telescoper, to be used as an optimisation trick in our implementation. First, we choose a lucky element $x_0 \in k$, and then apply Hermite reduction in $k(y)$ to $D_x^i(f)(x_0, y)$, we get

$$D_x^i(f)(x_0, y) = D_y(g_{0,i}) + r_{0,i}, \tag{3.7}$$

where $g_{0,i}, r_{0,i} \in k(y)$ are proper and the denominator of $r_{0,i}$ divides $Q^*(x_0, y)$. Let ρ_0 be the smallest integer such that $r_{0,0}, \ldots, r_{0,\rho_0}$ are linearly dependent over k.

Lemma 3.4.5. *A minimal telescoper has order at least ρ_0.*

Proof. We first claim that $r_{0,i} = r_i(x_0, y)$, for r_i as in (3.6). Note that the squarefree part with respect to y of the denominator of $D_x^i(f)$ divides Q^* for all $i \in \mathbb{N}$. By Corollary 5.5 in [49], x_0 is lucky for the denominator of $D_x^i(f)$ for all $i \in \mathbb{N}$. Then the claim on $r_{0,i}$ follows from Lemma 3.3.5 applied to $D_x^i(f)$. Let ρ be the minimal order of a telescoper, then $r_0, \ldots, r_\rho$ are linearly dependent over $k(x)$ by Lemma 3.4.2. Thus the rational functions $r_{0,0}, \ldots, r_{0,\rho}$ are linearly dependent over k, which implies $\rho_0 \leq \rho$. $\square$

Degree bounds for minimal telescopers

To derive degree bounds for g_i and r_i in (3.6), let Δ, Δ', μ, and μ' be defined as in the paragraph before Lemma 3.3.1, and set $\mu'' = \mu + \mu' - 1$.

Lemma 3.4.6. *Let W be in $k[x, y]$ with $\deg_y(W) < d_y^*$. Then, for all $i \in \mathbb{N}$, there exist $B, b \in k[x, y]$ with both* bi*deg*(B) *and* bi*deg*(b) *bounded by* $(\deg_x(W) + \mu'', d_y^* - 1)$*, such that*

$$D_x\left(\frac{W}{\Delta^{i+1}\Delta'^i Q^*}\right) = D_y\left(\frac{B}{\Delta^{i+2}\Delta'^{i+1}Q^*}\right) + \frac{b}{\Delta^{i+2}\Delta'^{i+1}Q^*}.$$

Proof. A straightforward calculation leads to

$$D_x\left(\frac{W}{\Delta^{i+1}\Delta'^i Q^*}\right) = \frac{\tilde{W}}{\Delta^{i+2}\Delta'^{i+1}Q^*} - \frac{1}{\Delta^{i+1}\Delta'^i}\frac{W D_x(Q^*)}{Q^{*2}},$$

where $\mathrm{b}ideg(\tilde{W}) \leq (\deg_x(W) + \mu'', d_y^* - 1)$. By Corollary 3.3.2, there exist $\tilde{B}, \tilde{b} \in k[x,y]$ such that

$$\frac{1}{\Delta^{i+1}\Delta'^i} \frac{WD_x(Q^*)}{Q^{*2}} = \frac{1}{\Delta^{i+2}\Delta'^{i+1}} \left(D_y\left(\frac{\Delta\tilde{B}}{Q^*}\right) + \frac{\Delta\tilde{b}}{Q^*} \right),$$

where $\mathrm{b}ideg(\tilde{B})$ and $\mathrm{b}ideg(\tilde{b})$ are bounded by $(\deg_x(W) + \mu' - 1, d_y^* - 1)$. The proof is completed by setting $(B, b) = (-\Delta\tilde{B}, \tilde{W} - \Delta\tilde{b})$. $\square$

Lemma 3.4.7. *For $i \in \mathbb{N}$, there exist $B_i, b_i \in k[x,y]$ such that*

$$D_x^i(f) = D_y\left(\frac{B_i}{\Delta^{i+1}\Delta'^i Q^{*i} Q^-}\right) + \frac{b_i}{\Delta^{i+1}\Delta'^i Q^*}. \tag{3.8}$$

Moreover, $\mathrm{b}ideg(B_i) \leq (\deg_x(P) + \mu + i\mu'' + (i-1)d_x^*, id_y^* + d_y^- - 1)$ *and* $\mathrm{b}ideg(b_i) \leq (\deg_x(P) + \mu + i\mu'' - d_x^-, d_y^* - 1)$.

Proof. We proceed by induction on i. For $i = 0$, the claim follows from Lemma 3.3.1. Assume that $i > 0$ and that the claim holds for the values less than i. For brevity, we set $\gamma = \deg_x(P) + \mu$, $F_{i-1} = B_{i-1}/(\Delta^i\Delta'^{i-1}Q^{*i-1}Q^-)$, and $G_{i-1} = b_{i-1}/(\Delta^i\Delta'^{i-1}Q^*)$. The induction hypothesis implies

$$D_x^i(f) = D_yD_x(F_{i-1}) + D_x(G_{i-1}),$$

with bidegree bounds on B_{i-1} and b_{i-1}. Fact 3.2.1*(i)* implies that $\tilde{Q} := Q^*D_x(Q^-)/Q^-$ is in $k[x,y]$, with $\mathrm{b}ideg(\tilde{Q}) \leq (d_x^* - 1, d_y^*)$. Hence $D_x(1/Q^-) = -\tilde{Q}/Q$. This observation and an easy calculation imply that

$$D_x(F_{i-1}) = \frac{\tilde{B}_{i-1}}{\Delta^{i+1}\Delta'^i Q^{*i} Q^-},$$

where $\tilde{B}_{i-1} \in k[x,y]$ and $\deg_x(\tilde{B}_{i-1}) \leq \deg_x(B_{i-1}) + \mu'' + d_x^*$. Furthermore, by Lemma 3.4.6 there are $\bar{B}_i, \bar{b}_i \in k[x,y]$ with bidegrees at most $(\deg_x(b_{i-1}) + \mu'', d_y^* - 1)$, such that

$$D_x(G_{i-1}) = D_y\left(\frac{\bar{B}_i}{\Delta^{i+1}\Delta'^i Q^*}\right) + \frac{\bar{b}_i}{\Delta^{i+1}\Delta'^i Q^*}.$$

Setting $B_i = \tilde{B}_{i-1} + \bar{B}_i Q^{*i-1} Q^-$ and $b_i = \bar{b}_i$, we arrive at (3.8). It remains to verify the degree bounds. The induction hypothesis implies that both $\deg_x(\bar{B}_i)$ and $\deg_x(b_i)$ are bounded by $\gamma + i\mu'' - d_x^-$. It follows that $\deg_x(\bar{B}_i Q^{*i-1} Q^-)$ is bounded by $\gamma + i\mu'' + (i-1)d_x^*$. Similarly, $\deg_x(\tilde{B}_{i-1})$ is bounded by $\gamma + i\mu'' + (i-1)d_x^*$, and so is $\deg_x(B_i)$. The bounds on degrees in y are obvious. $\square$

We next derive degree bounds for the minimal telescopers obtained at an intermediate stage of HermiteTelescoping; refined bounds on the output will be given by Theorem 3.4.16.

Lemma 3.4.8. *Under (H'), Step 2(c) of Algorithm HermiteTelescoping computes a minimal telescoper $L \in k[x]\langle D_x\rangle$ with order ρ and a certificate $g \in k(x,y)$ for P/Q with $\deg_x(L) \in \mathcal{O}(d_x d_y \rho^2)$ and* $\mathrm{bideg}(g) \in \mathcal{O}(d_x d_y \rho^2) \times \mathcal{O}(d_y \rho)$.

Proof. By Lemma 3.4.2, we exhibit a minimal telescoper by considering the first nontrivial linear dependence among the a_i's in (3.8). Let M be the coefficient matrix of the system in (η_i) obtained from $\sum_{i=0}^{\rho} \eta_i a_i = 0$. By Lemma 3.4.7, M is of size at most $(\rho+1) \times d_y^*$ and with coefficients of degree at most $\sigma := d_x + \mu + \rho\mu'' - d_x^-$ in x. Hence, there exists a solution $(\eta_0, \ldots, \eta_\rho) \in k[x]^{\rho+1}$ of degree at most $\sigma\rho$ in x by Fact 2.4.2*(ii)*. Since $\mu, \mu'' \in \mathcal{O}(d_x d_y)$ and $d_y^* \leq d_y$, the degree estimates of L and g are as announced. $\square$

Complexity estimates

We proceed to analyse the complexity of the algorithm in Figure 3.2.

Theorem 3.4.9. *Under Hyp. (H'), Algorithm HermiteTelescoping in Figure 3.2 is correct and takes $\tilde{\mathcal{O}}(\rho^{\omega+1} d_x d_y^2)$ ops, where ρ is the order of the minimal telescoper.*

Proof. The formulas in Step 2(a) create the loop invariant $D_x^i(f) = D_y(g_i) + a_i/Q^*$. Correctness then follows from Lemmas 3.4.1 and 3.4.2. Step 1 takes $\tilde{\mathcal{O}}(d_x d_y^2)$ ops by Theorem 3.3.6 under (H'). By Lemma 3.4.7, $\deg_x(-a_{i-1} D_x(Q^*)) \in \mathcal{O}(i d_x d_y)$. So the cost for performing Hermite reduction on $-a_{i-1} D_x(Q^*)/Q^{*2}$ in Step 2(a) is $\tilde{\mathcal{O}}(i d_x d_y^2)$ ops by Theorem 3.3.6. The bidegrees of g_i and a_i in Step 2(b) are in $\mathcal{O}(i d_x d_y) \times \mathcal{O}(i d_y)$ by Lemma 3.4.7. Since adding and differentiating have linear complexity, Step 2(b) takes $\tilde{\mathcal{O}}(i^2 d_x d_y^2)$ ops. For each i, the coefficient matrix of $\sum_{j=0}^{i} \eta_j a_j = 0$ in Step 2(c) is of size at most $(i+1) \times d_y^*$ and with coefficients of degree at most $\deg_x(a_i) \in \mathcal{O}(i d_x d_y)$. Moreover, the rank of this matrix is either i or $i+1$. Then, Step 2(c) takes $\tilde{\mathcal{O}}(i^\omega d_x d_y^2)$ ops by Fact 2.4.2*(iii)*. Computing the content and divisions in Step 3 has complexity $\tilde{\mathcal{O}}(d_x d_y \rho^3)$. If the algorithm returns when $i = \rho$, then the total cost is in

$$\sum_{i=0}^{\rho} \tilde{\mathcal{O}}(i^2 d_x d_y^2) + \sum_{i=1}^{\rho} \tilde{\mathcal{O}}(i^\omega d_x d_y^2) \subset \tilde{\mathcal{O}}(\rho^{\omega+1} d_x d_y^2) \text{ ops}, \tag{3.9}$$

which is as announced. $\square$

An optimisation, based on Lemma 3.4.5, consists in guessing the order ρ so as to perform Step 2(c) a few times only: As a preprocessing step, choose $x_0 \in k$ lucky for Q, then detect linear dependence of $\{r_{0,0}, \ldots, r_{0,j}\}$ in (3.7). The minimal j with dependence is a lower bound ρ_0 on ρ. So Step 2(c) is then performed only when $i \geq \rho_0$. In practice, the lower bound ρ_0 computed in this way almost always coincides with the actual order ρ. So normalising the g_i's becomes the dominant step, as observed in experiments. We analyse this optimisation by first estimating the cost for computing ρ_0.

Lemma 3.4.10. *Under Hypothesis (H'), computing a lower order bound ρ_0 for minimal telescopers takes $\tilde{\mathcal{O}}(d_x d_y \rho_0^3)$ ops.*

Proof. Since differentiating has linear complexity, the derivative $D_x^i(f)$ takes $\tilde{\mathcal{O}}(i^2 d_x d_y)$ ops. By Fact 2.4.1*(i)*, the evaluation $D_x^i(f)(x_0, y)$ takes as much. The cost of Hermite reduction on $D_x^i(f)(x_0, y)$ is $\tilde{\mathcal{O}}(i d_y)$ ops by Lemma 2.4.3. By Fact 2.4.2*(iii)* with $d = 1$, computing the rank of the coefficient matrix of $\sum_{j=0}^{i} \eta_j r_{0,j}$, with $r_{0,j}$ as in (3.7), takes $\tilde{\mathcal{O}}(d_y i^{\omega-1})$ ops. Thus, the cost for computing a lower bound on ρ_0 is $\sum_{i=0}^{\rho_0} \tilde{\mathcal{O}}(i^2 d_x d_y) \in \tilde{\mathcal{O}}(d_x d_y \rho_0^3)$ ops. □

Corollary 3.4.11. *Assume that $\rho_0 = \rho - \mathcal{O}(1)$. Then the previous optimisation of* HermiteTelescoping *takes $\tilde{\mathcal{O}}(\rho^3 d_x d_y^2)$ ops.*

Proof. In view of Lemma 3.4.10, the estimate (3.9) becomes

$$\tilde{\mathcal{O}}(d_x d_y \rho_0^3) + \sum_{i=0}^{\rho} \tilde{\mathcal{O}}(i^2 d_x d_y^2) + \sum_{i=\rho_0}^{\rho} \tilde{\mathcal{O}}(i^{\omega} d_x d_y^2),$$

which is $\tilde{\mathcal{O}}(\rho^3 d_x d_y^2) + \tilde{\mathcal{O}}((\rho - \rho_0)\rho^{\omega} d_x d_y^2)$ ops, whence the result. □

3.4.2 Improved Almkvist and Zeilberger's method

We analyse the complexity of Almkvist and Zeilberger's algorithm [15] when restricted to bivariate rational functions. In order to get a telescoper whose order ρ is minimal, the resulting algorithm, denoted RationalAZ, solves the telescoping equation

$$\sum_{i=0}^{\rho} \eta_i D_x^i(f) = D_y(g)$$

for increasing, prescribed values of ρ until it gets a solution $(\eta_0, \dots, \eta_\rho, g) \in k(x)^{\rho+1} \times k(x,y)$ with the η_i's not all zero. For the analysis, we start by studying the parametrisation of the differential Gosper algorithm of [15] under the same restriction to $k(x,y)$.

Definition 3.4.12 ([49]). *Let K be a field and $a, b \in K[y]$ be nonzero polynomials. A triple $(p,q,r) \in K[y]^3$ is said to be a* differential Gosper form *of the rational function a/b if*

$$\frac{a}{b} = \frac{D_y(p)}{p} + \frac{q}{r} \quad \textit{and} \quad \gcd(r, q - \tau D_y(r)) = 1 \quad \textit{for all } \tau \in \mathbb{N}.$$

For hyperexponential f, a key step in [15] is to compute a differential Gosper form of the logarithmic derivative of $F = \sum_{i=0}^{\rho} \eta_i D_x^i(f)$, where the η_i's are undetermined from $k(x)$. In the analogue $\mathsf{RationalAZ}$, this form is *predicted* by Lemma 3.4.13 below, which is a technical generalisation of a result by Le [63] on F when f has a squarefree denominator.

For $f = P/Q$, write $Q = t(y)T(x,y)$, splitting content and primitive part with respect to x. By an easy induction, $D_x^i(f) = N_i/(Q{T^*}^i)$ for $N_i \in k[x,y]$. For this section, set

$$F = \sum_{i=0}^{\rho} \eta_i D_x^i(f), \quad N = \sum_{i=0}^{\rho} \eta_i N_i {T^*}^{\rho-i}, \quad \text{and} \quad H = -D_y(Q)/Q^- - \rho t^* D_y(T^*).$$

Lemma 3.4.13. *If F is nonzero, the triple (N, H, Q^*) is a differential Gosper form of $D_y(F)/F$.*

Proof. First, observe $F = N/(Q{T^*}^\rho)$ and $Q^* = t^*T^*$. Next, $D_y(F)/F = D_y(N)/N - D_y(Q)/Q - \rho D_y(T^*)/T^*$ is $D_y(N)/N + H/Q^*$. It remains to prove $\gcd(Q^*, H - \tau D_y(Q^*)) = 1$, for any $\tau \in \mathbb{N}$. Recall that the squarefree part Q^* of Q is the product $Q_1 Q_2 \cdots Q_m$ and that $\hat{Q}_i$ denotes Q^*/Q_i. By Fact 3.2.1*(ii)*,

$$Z := H - \tau D_y(Q^*) = -\rho t^* D_y(T^*) - \sum_{i=1}^{m} (i+\tau)\hat{Q}_i D_y(Q_i).$$

If Q_j divides t^*, Z reduces to $-(j+\tau)\hat{Q}_j D_y(Q_j)$ modulo Q_j. If not, it reduces to $-(j+\tau)\hat{Q}_j D_y(Q_j) - \rho t^*(D_y(Q_j)T^*/Q_j)$, which rewrites to $-(j+\tau+\rho)\hat{Q}_j D_y(Q_j)$ modulo Q_j. In both cases, Z is coprime with Q^*, as $j > 0$, $\tau \geq 0$, and $\rho \geq 0$. $\square$

By induction, we have b*ideg*$(N_i) \leq (\deg_x(P) + i\deg_x(T^*) - i, d_y + i\deg_y(T^*) - 1)$, so that b*ideg*$(N) \leq (\deg_x(P) + \rho\deg_x(T^*) - \rho, d_y + \rho\deg_y(T^*) - 1)$.

Algorithm RationalAZ

Input: $f = P/Q \in k(x, y)$ satisfying Hypothesis (H).

Output: A minimal telescoper $L \in k[x]\langle D_x\rangle$ with certificate $g \in k(x, y)$.

1. Compute $Q^- = \gcd(Q, D_y(Q))$, $Q^* = Q/Q^-$, and T, T^* primitive parts of Q, Q^* with respect to x, respectively;
2. Set $(\tilde{N}, N, \beta, H)$ to $(P, P, d_y^-, -Q^* D_y(Q)/Q)$;
3. For $\ell = 0, 1, \ldots$ do
   (a) Set z to $\sum_{j=0}^{\beta} z_j y^j$, extract the linear system $\mathcal{M}\begin{pmatrix}\eta_i & z_j\end{pmatrix}^T = 0$ from (3.10) (for $\rho = \ell$) and compute a basis S of the null space of $\mathcal{M}$ by [84].
   (b) If S contains a solution $(\eta_0, \ldots, \eta_\ell, s)$ such that $\eta_0, \ldots, \eta_\ell$ are not all nonzero, then set $(L, g) := \left(\sum_{i=0}^{\ell} \eta_i D_x^i, s/(Q^- T^{*\ell})\right)$, and go to Step 4;
   (c) Update $\tilde{N} := D_x(\tilde{N})T^* - \tilde{N}\left(T^* D_x(T)/T + iD_x(T^*)\right)$, $N := NT^* + \eta_{\ell+1}\tilde{N}$, $\beta := \beta + \deg_y(T^*)$, and $H := H - t^* D_y(T^*)$.
4. Compute the content c of L and return $(c^{-1}L, c^{-1}g)$.

Figure 3.3: Improved Almkvist–Zeilberger algorithm

The next step in RationalAZ is, for fixed ρ, to reduce (3.1) by the change of unknown $g = z/(Q^- T^{*\rho})$, so as to determine all $(\eta_i) \in k(x)^{\rho+1}$ for which the differential equation in z

$$\sum_{i=0}^{\rho} \eta_i N_i T^{*\rho-i} = Q^* D_y(z) + (D_y(Q^*) + H)\, z \tag{3.10}$$

has a polynomial solution in $k(x)[y]$. For later use, we recall the following consequence of Corollary 9.6 in [49].

Lemma 3.4.14. *Let $a, b \in K[y]$ be such that $\beta = -\mathrm{lc}_y(b)/\mathrm{lc}_y(a)$ is a nonnegative integer and $\deg_y(b) = \deg_y(a) - 1$. Let $c \in K[y]$ be such that $\beta \geq \deg_y(c) - \deg_y(a) + 1$. If u is a polynomial solution of $aD_y(z) + bz = c$, then $\deg_y(u) \leq \beta$.*

The following lemma generalizes Lemma 2 in [63] to present a degree bound for z.

Lemma 3.4.15. *If $u \in k(x)[y]$ is a solution of* (3.10) *for $(\eta_i) \in k(x)^{\rho+1}$, then $\deg_y(u)$ is bounded by $\beta = d_y^- + \rho \deg_y(T^*)$.*

Proof. Let $a = Q^*$ and $b = D_y(Q^*) + H$. By the definition of H, $b = -Q^* D_y(Q^-)/Q^- - \rho t^* D_y(T^*)$. Fact 3.2.1*(i)* implies that $\mathrm{lc}_y(b) = -(d_y^- + \rho \deg_y(T^*))\mathrm{lc}_y(a)$. Therefore,

$$\beta = -\mathrm{lc}_y(b)/\mathrm{lc}_y(a) = d_y^- + \rho \deg_y(T^*).$$

As $\deg_y(N) < d_y + \rho \deg_y(T^*)$ and $d_y = d_y^* + d_y^-$, $\beta \geq \deg_y(N) - d_y^* + 1$. The lemma holds by Lemma 3.4.14. $\square$

We end the present section using the approach of Almkvist and Zeilberger to provide tight degree bounds on the outputs from Algorithms HermiteTelescoping and RationalAZ.

Theorem 3.4.16. *Under Hypothesis (H'), there exists a minimal telescoper $L \in k[x]\langle D_x \rangle$ with certificate $g \in k(x, y)$ with $\deg_x(L) \in \mathcal{O}(d_x d_y d_y^*)$ and* b*ideg*$(g) \in \mathcal{O}(d_x d_y d_y^*) \times \mathcal{O}(d_y d_y^*)$.

Proof. By Corollary 3.4.4, there exists a smallest $\rho \in \mathbb{N}$ at most d_y^*, for which (3.1) has a solution with the η_i's not all zero. For this ρ, we estimate the size of the polynomial matrix $\mathcal{M}$ derived from (3.10) by undetermined coefficients. By the remark on N after Lemma 3.4.13, we have b*ideg*$(N) \leq (n_x, n_y)$ where $n_x := d_x + \rho \deg_x(T^*) - \rho \in \mathcal{O}(\rho d_x)$ and $n_y := d_y + \rho \deg_y(T^*) - 1 \in \mathcal{O}(\rho d_y)$. The matrix $\mathcal{M}$ contains two blocks $\mathcal{M}_1 \in k[x]_{\leq n_x}^{(n_y+1)\times(\rho+1)}$ and $\mathcal{M}_2 \in k[x]_{\leq d_x}^{(n_y+1)\times(\beta+1)}$, where $\beta \in \mathcal{O}(\rho d_y)$ is the same as in Lemma 3.4.15. By the minimality of ρ, the dimension of the null space of $\mathcal{M}$ is 1. So there exists $u \in k[x]^{n_y+1}$ with coefficients of degree at most $n_x(\rho+1) + d_x(\beta+1) \in \mathcal{O}(d_x d_y d_y^*)$ in x such that $\mathcal{M}\begin{pmatrix}\eta & z\end{pmatrix}^T = 0$, which implies degree bounds in x for L and g. The degree bound in y for g is obvious. $\square$

We now analyse the complexity of the algorithm in Fig. 3.3.

Theorem 3.4.17. *Under Hypothesis (H'), Algorithm RationalAZ in Figure 3.3 works correctly and takes $\tilde{\mathcal{O}}(d_x d_y^{\omega} \rho^{\omega+2})$ ops, where ρ is the order of the minimal telescoper.*

Proof. By the existence of a telescoper, Corollary 3.4.4, and Lemma 3.4.15, the algorithm always terminates and returns a minimal telescoper L, of order ρ at most d_y^*. Gcd computations dominate the cost of Steps 1 and 2, which take $\tilde{\mathcal{O}}(d_x d_y^2)$ ops. For each $\ell \in \mathbb{N}$, the dominating cost in Step 3 is computing the null space of $\mathcal{M}$. Let $n_y = d_y + \ell \deg_y(T^*) - 1 \in \mathcal{O}(\ell d_y)$ and $n_x = d_x + \ell \deg_x(T^*) \in \mathcal{O}(\ell d_x)$. By the same argument as in the proof of Theorem 3.4.16, the matrix $\mathcal{M}$ is of size at most $(n_y+1)\times(\ell+\beta+2)$ and with coefficients of degree at most n_x. Let r be the rank of $\mathcal{M}$, which is either $\ell+\beta+2$ or $\ell+\beta+1$ by construction. Thus, a basis of the null space of $\mathcal{M}$ can be computed within $\tilde{\mathcal{O}}(n_x(n_y+1)(\ell+\beta+2)r^{\omega-2})$ ops by Fact 2.4.2*(iii)*. Since $\beta \in \mathcal{O}(\ell d_y)$, $\tilde{\mathcal{O}}(n_x(n_y+1)(\ell+\beta+2)r^{\omega-2})$ is included in $\tilde{\mathcal{O}}(d_x d_y^{\omega} \ell^{\omega+1})$. Since Step 3 terminates at $\ell = \rho$, the total cost of the algorithm is $\sum_{\ell=0}^{\rho} d_x d_y^{\omega} \ell^{\omega+1}$ ops. This is within the announced complexity, $\tilde{\mathcal{O}}(d_x d_y^{\omega} \rho^{\omega+2})$ ops. □

Corollary 3.4.18. *Algorithms* HermiteTelescoping *and* RationalAZ *in Figures 3.2 and 3.3 both output the primitive minimal telescoper L together with its certificate g, which satisfy* $\deg_{D_x}(L) \leq d_y^*$, $\deg_x(L)$ *and* $\deg_x(g) \in \mathcal{O}(d_x d_y d_y^*)$, *and* $\deg_y(g) \in \mathcal{O}(d_y d_y^*)$.

Proof. Both algorithms output the primitive minimal telescoper, as they compute a minimal telescoper at an intermediate step, and owing to their last step of content removal. Bounds follow from Corollary 3.4.4 and Theorem 3.4.16. □

3.5 Non-minimal telescopers for bivariate rational functions

In this section, we trade the minimality of telescopers for smaller total output sizes. To this end, we adapt and slightly extend the arguments in [67] and [20, Section 3].

Let $f = P/Q$ be a nonzero element in $k(x,y)$ such that $P, Q \in k[x,y]$ and $\gcd(P,Q) = 1$. Denote $d_x = \max\{\deg_x(P), \deg_x(Q)\}$ and $d_y = \max\{\deg_y(P), \deg_y(Q)\}$. Our next goal is to find a nonzero operator $A(x, D_x, D_y)$ in $k[x]\langle D_x, D_y\rangle$ such that $A(f) = 0$. To this end, we consider the k-vector space $\mathbb{W}_N$ generated by the set $\{\, x^i D_x^j D_y^\ell \mid i+j+\ell \leq N \,\}$ over k. By an easy combinatorial counting, the dimension of $\mathbb{W}_N$ is $\binom{N+3}{3}$ over k. Furthermore, for any $(i,j,\ell) \in \mathbb{N}^3$, a direct calculation yields

$$x^i D_x^j D_y^\ell(f) = \frac{P_{i,j,\ell}}{Q^{i+j+\ell+1}}, \tag{3.11}$$

where $P_{i,j,\ell} \in k[x,y]$ with $\deg_x(P_{i,j,\ell}) \leq (i+j+\ell+1)d_x + i$ and $\deg_y(P_{i,j,\ell}) \leq (i+j+\ell+1)d_y$. So the set $\mathbb{W}_N(f)$ is included in the set

$$\mathbb{V}_N = \operatorname{span}_k \left\{ \frac{x^i y^j}{Q^{N+1}} \mid i \leq (N+1)d_x + N,\ j \leq (N+1)d_y \right\},$$

where the dimension of $\mathbb{V}_N$ is $(N+1)(d_x+1)((N+1)d_y+1)$ over k. Define a linear map ϕ : $\mathbb{W}_N \to \mathbb{V}_N$ by $\phi(L) = L(f)$ for any $L \in \mathbb{W}_N$. Choosing $N = 6(d_x+1)(d_y+1)$ yields the inequality

$$\binom{N+3}{3} > (N+1)(d_x+1)((N+1)d_y+1),$$

which implies the kernel of ϕ is nontrivial whenever $N \geq 6(d_x+1)(d_y+1)$. Therefore, there exists a nonzero operator A in $k[x]\langle D_x, D_y\rangle$ with total degree at most $6(d_x+1)(d_y+1)$ in x, D_x, and D_y that annihilates f. Moreover, A can be found by solving a linear system of size $\mathcal{O}((d_x d_y)^3)$ over k. The following lemma shows that one can construct a telescoper for f from any y-free annihilator $A(x, D_x, D_y)$ of f.

Lemma 3.5.1. *Let f be a nonzero rational function in $k(x,y)$ and $A \in k[x]\langle D_x, D_y\rangle$ be a nonzero operator such that $A(f) = 0$. Then there exists a nonzero operator $L(x, D_x)$ in $k[x]\langle D_x\rangle$ such that $L(f) = D_y(g)$ for some $g \in k(x,y)$.*

Proof. Since D_y commutes with x and D_x, we can write $A = D_y^m(L(x,D_x)+D_yM)$, where $m \in \mathbb{N}$, $M \in k[x]\langle D_x, D_y\rangle$, and L is a nonzero operator in $k[x]\langle D_x\rangle$. By a differential extension of a "non-commutative" trick, used by Wegschaider in [91, Theorem 3.2], there exist $w \in k[y]$ and nonzero $r \in k$ such that

$$wD_y^m = D_yQ + r, \tag{3.12}$$

where $Q \in k[y]\langle D_y\rangle$. In particular, $r = (-1)^m m! \neq 0$ if we take $w = y^m$. Using the fact $rD_y = D_y r$ and (3.12), we find

$$\frac{y^m}{(-1)^m m!}A = L + D_yG, \quad \text{where } G \in k[x,y]\langle D_x, D_y\rangle.$$

Since $A(f) = 0$, $L(f) = D_y(-G(f))$. Note that $g = -G(f)$ is still in $k(x,y)$. This completes the proof. □

From Lipshitz's argument, we see that there exists a telescoper of order in $\mathcal{O}(d_x d_y)$. Now, we slightly extend the argument in [20, Section 3] to show that a telescoper of order linear in d_y exists. Instead of taking total degree, set $\mathbb{W}_{N_x,N_\partial}$ to the k-vector space

$$\mathbb{W}_{N_x,N_\partial} = \operatorname{span}_k \left\{ x^i D_x^j D_y^\ell \mid i \leq N_x, j+\ell \leq N_\partial \right\}.$$

where the dimension of $\mathbb{W}_{N_x,N_\partial}$ is $(N_x+1)\binom{N_\partial+2}{2}$ over k. By (3.11), the set $\mathbb{W}_{N_x,N_\partial}(f)$ is included in the set

$$\mathbb{V}_{N_x,N_\partial} = \operatorname{span}_k \left\{ \frac{x^i y^j}{Q^{N+1}} \mid i \le (N_\partial+1)d_x + N_x, j \le (N_\partial+1)d_y \right\}.$$

where the dimension of $\mathbb{V}_{N_x,N_\partial}$ is $((N_\partial+1)d_x+N_x+1)((N_\partial+1)d_y+1)$ over k. Choosing $N_x = 3d_xd_y$ and $N_\partial = 6d_y$ yields the inequality

$$(N_x+1)\binom{N_\partial+2}{2} > ((N_\partial+1)d_x+N_x+1)((N_\partial+1)d_y+1).$$

Therefore, there exists a nonzero operator A in $k[x]\langle D_x, D_y\rangle$ with degree at most $3d_xd_y$ in x and total degree at most $6d_y$ in D_x and D_y that annihilates f. Again, A can be found by solving a linear system, but of smaller size $\mathcal{O}(d_xd_y^3)$ over k. By the construction in Lemma 3.5.1, the order of L is bounded by $6d_y$, which is linear in d_y.

Remark 3.5.2. *The bound $6d_y$ has been shown in [20] for rational functions $yD_y(Q)/Q$ with $Q \in k[x,y]$. Apagodu and Zeilberger [16] obtain a similar bound for a class of nonrational hyperexponential functions, but their proof does not seem to apply to rational functions, as it heavily relies on the presence of a nontrivial exponential part.*

3.6 Implementation and experiments

We describe in this section an implementation of algorithms HermiteTelecoping and RationalAZ, and compare ours with Maple's routine.

3.6.1 Implementation and examples

The evaluation-interpolation algorithm HermiteEvalInterp for Hermite reduction in Figure 3.1 does not perform well, mainly because Maple's rational interpolation routines are far too slow. We thus implemented Algorithm HermiteReduce (original version) in [21, Section 2.2] (carefully avoiding redundant extended gcd calculations), and noted that it performs better. We then implemented a variant of Algorithm HermiteTelescoping in Figure 3.2, using HermiteReduce in place of HermiteEvalInterp, and including the optimisation at the end of § 3.4.1, refined by additional modular calculations. The improved Almkvist-Zeilberger's algorithm RationalAZ is also implemented. Those functions form the module RationalCT.

```
> eval(RationalCT);
```

$$\begin{array}{l}\textbf{module}()\\ \textbf{export}\ \mathit{SquareFreeParFrac},\ \mathit{HermiteReduce},\ \mathit{HermiteTelescoping},\ \mathit{RationalAZ};\\ \textbf{option}\ \mathit{package};\\ \textbf{description}\text{“Computing the minimal telescoper for a rational function”};\\ \textbf{end module}\end{array}$$

The function SquareFreeParFrac is used to compute the partial fraction decomposition of a rational function with respect to the squarefree factorization of its denominator.

```
> f :=1/(x^3+5*x^2+8*x+4);
```

$$f := \frac{1}{x^3 + 5\,x^2 + 8\,x + 4}$$

```
> SquareFreeParFrac(f, x, 'pfd');
```

$$1,\ [x,\ 0,\ [[x+2,\ [1,\ -1],\ [1,\ -1]],\ [x+1,\ [1,\ 1]]]]$$

```
> pfd;
```

$$-\frac{1}{x+2} - \frac{1}{(x+2)^2} + \frac{1}{x+1}$$

The function HermiteReduce returns the additive decomposition of a rational function with respect to the specific variable.

```
> f := 1/(-y+y^2+x)^2;
```

$$f := \frac{1}{(-y + y^2 + x)^2}$$

```
> HermiteReduce(f, y);
```

$$[\frac{-1 + 2\,y}{(4\,x - 1)\,(-y + y^2 + x)},\ \frac{2}{(4\,x - 1)\,(-y + y^2 + x)}]$$

In Maple, the function for computing minimal telescopers is DEtools[Zeilberger], which works for any hyperexponential inputs.

```
> f := 1/(-y+y^2+x);
```

$$f := \frac{1}{-y + y^2 + x}$$

```
> DEtools[Zeilberger](f, x, y, Dx);
```

$$[2 + Dx\,(4\,x - 1),\ -\frac{-1 + 2\,y}{-y + y^2 + x}]$$

Our implementation of the algorithms HermiteTelescoping and RationalAZ is as follows.

```
> HermiteTelescoping(f, x, y, Dx);
```

$$[2 + Dx\,(4\,x - 1),\ -\frac{-1 + 2\,y}{-y + y^2 + x}]$$

```
> HermiteTelescoping(f, x, y, Dx, 'No_Certificate');
```

$$2 + Dx\,(4\,x - 1)$$

```
> RationalAZ(f, x, y, Dx);
```

$$[2 + Dx\,(4\,x - 1),\ -\frac{-1 + 2\,y}{-y + y^2 + x}]$$

3.6.2 Experimental results

Now, we show some timings by our implementation and others.

Random rational functions

We use the function randpoly in Maple to generate the numerator P and the denominator Q of a testing rational function with P and Q having the same degree in x and y. We test the following set of rational functions:

$$f = \frac{P}{Q}, \quad d = \deg_x(P) = \deg_y(P) \in \{1, 2, \ldots, 7\}.$$

For brevity, we denote

- AZ: the function DEtools[Zeilberger] in Maple 13;
- Hermite: Hermite-reduction based method;
- RatAZ: improved Almkvist–Zeilberger algorithm.

d	1	2	3	4	5	6	7
AZ	0.054	0.158	2.731	64.75	619.0	$>$ hr	$>$ hr
RatAZ	0.019	0.059	0.402	4.461	34.13	220.5	792.1
Hermite	0.016	0.057	0.398	2.664	18.80	106.2	422.5

Table 3.2: Timings on random examples (in seconds)

Differential equations for algebraic functions

Let $\alpha(x)$ be a univariate algebraic function over $k(x)$ with minimal polynomial P in $k[x, y]$. The following lemma shows that one can compute differential equations for algebraic functions via rational-function telescoping. For the proof, see [20, Proposition 2].

Lemma 3.6.1. *If L is a telescoper for $yD_y(P)/P$ with respect to y, then $L(\alpha(x)) = 0$.*

In the next table, we show the timings on a set of polynomials in $\mathbb{Z}[x, y]$, which are also generated by the function randpoly. Let d denote the total degree of a polynomial in x and y. The column gfun (Maple 13) shows the timing by the function algeqtodiffeq in Maple package gfun and gfun (Algo) by the one in the updated Algolib 13.0 of INRIA Algorithms project.

d	4	5	6	7	8	9	10
RatAZ	0.30	1.05	4.90	21.6	69.5	237.	846.
Hermite	0.21	0.94	4.53	20.5	84.7	231.	864.
gfun (Maple 13)	0.14	0.75	6.92	79.6	1661	$>$ hr	$>$ hr
gfun (Algo)	0.10	0.46	2.44	12.2	52.7	157.	464.

Table 3.3: Timings on computing differential equations for algebraic functions (in seconds)

Differential equations for diagonals

For a bivariate rational power series

$$f = \sum_{i,j\geq 0} f_{i,j}x^i y^j \in k(x, y) \cap k[[x, y]],$$

define its diagonal by

$$\operatorname{diag}(f) := \sum_{i=0}^{\infty} f_{i,i}x^i.$$

The following lemma shows that one can compute differential equations for diagonals of bivariate rational functions via rational-function telescoping. For the proof, see [67].

Lemma 3.6.2. *If L is a telescoper for $f(y, x/y)/y$ with respect to y, then $L(\operatorname{diag}(f)) = 0$.*

First, we compare the various algorithms on an example borrowed from [74]:

$$f = \frac{1}{1 - x - y - xy(1 - x^d)}, \quad \text{where } d \in \mathbb{N}.$$

d	8	9	10	11	12	13	14	15
AZ	3.53	6.33	13.6	38.5	68.1	145.	263.	368.
RatAZ	5.27	4.63	8.72	16.9	36.1	55.4	99.4	352.
Hermite	2.33	4.52	8.71	18.6	36.1	65.3	121.	169.

Table 3.4: Timing on Pemantle and Wilson's example (in seconds)

The next test is on an example from Stanley's book [83] related to the plane walks. Let $S_d = \{(i, j) \in \mathbb{N}^2 | i + j = d\}$, consider

$$f(x, y, d) = \frac{1}{1 - \sum_{(i,j)\in S_d} x^i y^j}, \quad \text{where } 11 \leq d \leq 20.$$

From the table 3.5, we observed that the order of minimal telescopers is $d/2$ when d is even.

d	11	12	13	14	15	16	17	18	19	20
AZ	48.7	5.72	144.	12.4	400.	23.9	1016.	46.7	$>$ hr.	81.2
RatAZ	43.8	5.61	129.	11.8	269.	27.9	663.4	45.8	2976.	88.4
Hermite	11.7	2.55	31.9	5.71	91.3	12.8	227.8	21.1	617.9	40.3
Order	11	6	13	7	15	8	17	9	19	10

Table 3.5: Timings on the plane walk examples (in seconds)

We conjecture that this pattern is true for all even d and expect a combinatorial explanation.

Chapter 4

Structure of Multivariate Hyperexponential-Hypergeometric Functions

4.1 Introduction

Multivariate hyperexponential-hypergeometric functions are a generalization of usual exponential functions and hypergeometric terms. In their paper [92], Wilf and Zeilberger observed that a large class of identities on special functions are related to the integrals or sums of *proper hyperexponential-hypergeometric* functions. Moreover, they have shown that elegant and computer-constructible proofs for those identities can be obtained efficiently via Zeilberger's method of creative telescoping. Hyperexponential-hypergeometric functions also play an important role in factoring modules over Laurent-Ore algebras [93].

Motivation. It is useful to decompose a hyperexponential-hypergeometric function as a product of "simpler" functions of the same kind. For example, Christopher [29] decomposes a hyperexponential function and uses its multiplicative form to compute Liouvillian first integrals [29]. The Ore-Sato theorem [70, 81], which describes the multiplicative structure of hypergeometric terms, is rediscovered and used by Payne [73], Hou [54], Abramov, and Petkovšek [14] to prove that Wilf and Zeilberger's conjecture holds for hypergeometric terms. Feng, Singer, and Wu present a multiplicative form for bivariate hyperexponential-hypergeometric function to study Liouvillian solutions of linear difference-differential equations [43, 42]. We will decompose a multivariate hyperexponential-hypergeometric function in the hope of studying the general Wilf and Zeilberger's conjecture.

Our contribution. We decompose a multivariate hyperexponential-hypergeometric function into a product of a rational function, a hyperexponential function, a hypergeometric term and some "simpler" functions. Our result is a generalization of a result by Feng, Singer and Wu [42, Proposition 5]. Combining this result with multivariate extension of Christopher's theorem [29, 97] and the Ore-Sato theorem, we present a structure theorem for multivariate hyperexponential-hypergeometric functions.

The rest of this chapter is organized as follows. We describe in Section 4.2 an algebraic setting for hyperexponential-hypergeometric functions. Two rational normal forms are reviewed and a new one is introduced in Section 4.3. We describe a structure of compatible rational functions in Section 4.4. Based on this structure, we obtain a multiplicative form of hyperexponential-hypergeometric functions in Section 4.5.

An earlier version of this chapter can be found in [27], which is a joint work (in progress) with Ziming Li.

4.2 Algebraic setting

We describe an algebraic setting for hyperexponential-hypergeometric functions, which is introduced and used in [62, 22, 93, 66]. We will regard hyperexponential-hypergeometric functions as elements in some differential-difference extensions over the field of multivariate rational functions.

4.2.1 Hyperexponential-hypergeometric functions

Let A be a commutative ring. Recall that a derivation δ on A is an additive map from A to itself satisfying the Leibniz rule

$$\delta(ab) = a\delta(b) + \delta(a)b \quad \text{for all } a, b \in A.$$

Let Δ be a finite set of derivations and automorphisms from A to itself. The pair (A, Δ) is called a *differential-difference ring*, or Δ-*ring* for short. If the maps in Δ commute pairwise, then (A, Δ) is said to be *orthogonal* [62].

Let k be a field of characteristic zero. For brevity, we let $\mathbf{x}$ stand for the continuous variables $(x_1, \ldots, x_m)$ and $\mathbf{y}$ for the discrete variables $(y_1, \ldots, y_n)$. Let $k(\mathbf{x}, \mathbf{y})$ be the field of rational functions in $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ over k. On the field $k(\mathbf{x}, \mathbf{y})$, the derivations δ_i $(1 \leq i \leq m)$ and shift operators σ_j $(1 \leq j \leq n)$ are defined for all $f \in k(\mathbf{x}, \mathbf{y})$ by

$$\delta_i(f) = \frac{\partial f}{\partial x_i} \quad \text{and} \quad \sigma_j(f) = f(\mathbf{x}, y_1, \ldots, y_{j-1}, y_j + 1, y_{j+1}, \ldots, y_n).$$

Put $\Delta = \{\delta_1, \ldots, \delta_m, \sigma_1, \ldots, \sigma_n\}$ and note that the elements of Δ commute pairwise over $k(\mathbf{x}, \mathbf{y})$. So the pair $(k(\mathbf{x}, \mathbf{y}), \Delta)$ is a Δ-field.

A ring $\mathcal{R}$ is called a Δ-extension of the field $k(\mathbf{x}, \mathbf{y})$ if $\mathcal{R}$ contains $k(\mathbf{x}, \mathbf{y})$, all derivations can be extended to $\mathcal{R}$, all the shift operators can be extended to monomorphisms to $\mathcal{R}$, and the extended operators also commute pairwise over $\mathcal{R}$. The set of all extended derivations and shift operators is still denoted by Δ. The ring $\mathcal{R}$ is said to be *simple* if it contains no ideal closed under all maps in Δ except the zero ideal and the whole ring. An element c of $\mathcal{R}$ is called a *constant* with respect to a derivation $\delta \in \Delta$ if $\delta(c) = 0$ and a constant with respect to a shift operator $\sigma \in \Delta$ if $\sigma(c) = c$. An element c of $\mathcal{R}$ is called a *constant* if it is a constant with respect to all operators in Δ. The set of all constants of $\mathcal{R}$, denoted by $C_{\mathcal{R}}$, is a subring of $\mathcal{R}$. The ring $C_{\mathcal{R}}$ is a subfield if $\mathcal{R}$ is a field.

Definition 4.2.1. *Let $\mathcal{R}$ be a Δ-extension of $k(\mathbf{x}, \mathbf{y})$. A nonzero element h of $\mathcal{R}$ is said to be* hyperexponential-hypergeometric *over $k(\mathbf{x}, \mathbf{y})$ if there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in k(\mathbf{x}, \mathbf{y})$ such that $b_1 \cdots b_n \neq 0$ and*

$$\delta_1(h) = a_1 h, \ \ldots, \ \delta_m(h) = a_m h, \quad \textit{and} \quad \sigma_1(h) = b_1 h, \ \ldots, \ \sigma_n(h) = b_n h. \tag{4.1}$$

The rational functions a_i and b_j above are called the certificates *of h with respect to x_i and y_j, respectively.*

According to the definition above, it is easy to see that derivatives, shifts, and products of hyperexponential-hypergeometric functions are also hyperexponential-hypergeometric. If $\mathcal{R}$ is simple, then any hyperexponential-hypergeometric element in $\mathcal{R}$ is invertible [66, Lemma 2.1].

Since all elements in Δ commute pairwise, the certificates a_i's and b_j's in Definition 4.2.1 satisfy three sets of *integrability conditions*:

$$\delta_i(a_j) = \delta_j(a_i) \quad \text{for } 1 \leq i < j \leq m, \tag{4.2}$$

$$\sigma_i(b_j)b_i = \sigma_j(b_i)b_j \quad \text{for } 1 \leq i < j \leq n, \tag{4.3}$$

$$\frac{\delta_i(b_j)}{b_j} = \sigma_j(a_i) - a_i \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n. \tag{4.4}$$

In the discrete case, rational functions $b_1, \ldots, b_n \in k(\mathbf{y})$ in [14] are said to be *compatible* if they satisfy the integrability conditions (4.3). We follow this and call $a_1, \ldots, a_m, b_1, \ldots, b_n$ in $k(\mathbf{x}, \mathbf{y})$ *compatible* if they satisfy the integrability conditions (4.2), (4.3), and (4.4). As opposed to the treatment in [14], we regard the discrete variables $y_1, \ldots, y_n$ as indeterminates, and, thus, the function $h = |y_1 - y_2|$ is excluded in the thesis.

4.2.2 First-order fully integrable systems

A Δ-extension of $k(\mathbf{x}, \mathbf{y})$ may not be an integral domain in general. For this reason, we need to construct a simple Δ-ring, in which any hyperexponential-hypergeometric element is invertible. We specialize the construction in [22, Section 3] for finitely many first-order systems.

Definition 4.2.2. *Let $a_1, \ldots, a_m, b_1, \ldots, b_n$ be rational functions in $k(\mathbf{x}, \mathbf{y})$. A first-order system*

$$\delta_1(z) = a_1 z, \ \ldots, \ \delta_m(z) = a_m z, \quad \sigma_1(z) = b_1 z, \ \ldots, \ \sigma_n(z) = b_n z, \tag{4.5}$$

is said to be fully integrable *over $k(\mathbf{x}, \mathbf{y})$ if $b_1 \cdots b_n \neq 0$ and $a_1, \ldots, a_m, b_1, \ldots, b_n$ are compatible.*

According to Theorem 2 in [22], given a finite number of first-order fully integrable systems, there exists a simple differential-difference extension $\mathcal{R}$ of $k(x, n)$ such that $\mathcal{R}$ contains a nonzero solution of each system. Moreover, the subring of constants in $\mathcal{R}$ is equal to k if k is algebraically closed. Let $\mathcal{H}(\mathbf{a}, \mathbf{b})$ denote the solution space of the first-order fully integrable system (4.5) in such an extension $\mathcal{R}$. By Theorem 2 in [22], $\mathcal{H}(\mathbf{a}, \mathbf{b})$ is one-dimensional over k if k is algebraically closed. In the rest of this chapter, hyperexponential-hypergeometric functions will be regarded as elements in such an extension $\mathcal{R}$. Therefore, it is legitimate to add, multiply and invert hyperexponential-hypergeometric functions. Following Definition 2 in [14], two hyperexponential-hypergeometric functions are said to be *conjugate* if they have the same

Function	Expression	Conjugates given by $\mathcal{H}(\mathbf{a},\mathbf{b})$-Notation
exponential	$\exp(f)$	$\mathcal{H}(\delta_1(f),\ldots,\delta_m(f),1,\ldots,1)$
constant power	β^λ	$\mathcal{H}(\lambda\frac{\delta_1\beta}{\beta},\ldots,\lambda\frac{\delta_m\beta}{\beta},1,\ldots,1)$
symbolic power	β^{y_j}	$\mathcal{H}\left(y_j\frac{\delta_1\beta}{\beta},\ldots,y_j\frac{\delta_m\beta}{\beta},1,\ldots,1,\beta,1,\ldots,1\right)$
factorial	$(\lambda)_{\mathbf{e}\cdot\mathbf{y}}$	$\mathcal{H}\left(0,\ldots,0,\prod_{\ell=0}^{e_1-1}(\mathbf{e}\cdot\mathbf{y}+\lambda+\ell),\ldots,\prod_{\ell=0}^{e_n-1}(\mathbf{e}\cdot\mathbf{y}+\lambda+\ell)\right)$

(In the table above, $f,\beta\in k(\mathbf{x})$, $\lambda\in k$, and $\mathbf{e}\in\mathbb{Z}^n$.)

Table 4.1: Familiar functions and their $\mathcal{H}(\mathbf{a},\mathbf{b})$-representations

certificates. Two conjugate functions can only differ by a nonzero multiplicative element in k if k is algebraically closed.

The following lemma shows some basic properties of the space $\mathcal{H}(\mathbf{a},\mathbf{b})$. They are derived directly from the definition.

Lemma 4.2.3. *(i) For any hyperexponential-hypergeometric function g over $k(\mathbf{x},\mathbf{y})$, we have*

$$g\mathcal{H}(a_1,\ldots,a_m,b_1,\ldots,b_n)=\mathcal{H}\left(a_1+\frac{\delta_1(g)}{g},\ \ldots,\ a_m+\frac{\delta_m(g)}{g},\ b_1\frac{\sigma_1(g)}{g},\ \ldots,\ b_n\frac{\sigma_n(g)}{g}\right).$$

(ii) $\mathcal{H}(\mathbf{a},\mathbf{b})\mathcal{H}(\tilde{\mathbf{a}},\tilde{\mathbf{b}})=\mathcal{H}(\mathbf{a}+\tilde{\mathbf{a}},\mathbf{b}\tilde{\mathbf{b}})$, where $\mathbf{a}+\tilde{\mathbf{a}}$ and $\mathbf{b}\tilde{\mathbf{b}}$ are defined termwise.

(iii) $\delta_i(\mathcal{H}(\mathbf{a},\mathbf{b}))=a_i\mathcal{H}(\mathbf{a},\mathbf{b})$ for $1\le i\le m$ and $\sigma_j(\mathcal{H}(\mathbf{a},\mathbf{b}))=b_j\mathcal{H}(\mathbf{a},\mathbf{b})$ for $1\le j\le n$.

Some examples for hyperexponential-hypergeometric functions are listed in Table 4.1.

4.3 Rational normal forms

In this section, we review two normal forms for univariate rational functions in [13, 10, 47]. Those normal forms have played an important role in the minimal decompositions of hyperexponential and hypergeometric functions [47, 12, 13, 10]. In order to study the compatible multivariate rational functions in the next sections, we introduce a new kind of rational normal forms for the continuous-discrete rational functions in $k(\mathbf{x},\mathbf{y})$.

4.3.1 Differential and shift rational normal forms

Let F be a field of characteristic zero. The field $F(z)$ of univariate rational functions is equipped with both differential and difference structures by

$$\delta(f(z)) = \frac{d(f(z))}{dz} \quad \text{and} \quad \sigma(f(z)) = f(z+1) \quad \text{for all } f \in F(z)\ .$$

A polynomial $P \in F[z]$ is said to be *squarefree* with respect to z over F if $\gcd(P, \delta(P)) = 1$. It is said to be *shift-free* if $\gcd(P, \sigma^i(P)) = 1$ for all $i \in \mathbb{Z} \setminus \{0\}$. In other words, any two roots of a shift-free polynomial have a non-integer distance. In the following, we collect basic facts concerning rational functions whose denominators are squarefree or shift-free, which has appeared implicitly in the literature [2, 71, 52, 53].

Lemma 4.3.1. *Let $f = P/Q$ be in $F(z)$ with $\gcd(P, Q) = 1$ and $\deg(P) < \deg(Q)$. Then*

(i) If Q is squarefree and $f = \delta(g)$ for some $g \in F(z)$, then $f = 0$.

(ii) If Q is shift-free and $f = \sigma(g) - g$ for some $g \in F(z)$, then $f = 0$.

Definition 4.3.2 (Differential-reduced, shift-reduced)**.** *A rational function $f = P/Q \in F(z)$ with $\gcd(P, Q) = 1$ is said to be* differential-reduced *with respect to z over F if*

$$\gcd(Q, P - i\delta(Q)) = 1, \quad \textit{for all } i \in \mathbb{Z}.$$

It is said to be shift-reduced *with respect to z over F if*

$$\gcd(P, \sigma^i(Q)) = 1, \quad \textit{for all } i \in \mathbb{Z}.$$

The following lemmas show basic properties of differential-reduced and shift-reduced rational functions, respectively. Those results are seemingly classical in [51, 15, 47, 13], but we still present their proofs for completeness.

Lemma 4.3.3. *Let $f = P/Q \in F(z)$ be differential-reduced with respect to z, where $P, Q \in F[z]$ and $\gcd(P, Q) = 1$. If $f = \delta(g)/g$ for some $g \in F(z)$, then $g \in F$ and $f = 0$.*

Proof. Suppose that $g \in F(z) \setminus F$. Then

$$f = \frac{\delta(g)}{g} = \sum_{i=1}^{s} \frac{m_i}{z - \alpha_i}, \quad \text{where } m_i \in \mathbb{Z} \text{ and } \alpha_i \in \overline{F} \text{ for all } i \text{ with } 1 \le i \le s.$$

By Theorem 2.3.8, the m_i's are roots of $\mathrm{RT}_z(f)$. So $\gcd(Q, P - m_i\delta(Q)) \neq 1$, which contradicts the assumption that f is differential-reduced. So $g \in F$ and therefore $f = 0$. □

Lemma 4.3.4. *Let $f \in F(z)$ be shift-reduced with respect to z over F. If $f = \sigma(g)/g$ for some $g \in F(z)$, then $g \in F$ and $f = 1$.*

Proof. Let $g = a/b$ with $a, b \in F[z]$ and $\gcd(a, b) = 1$. Suppose that g is not in F. Then either a or b has a positive degree. Assume that the degree of a is positive. Then there exists an element $\alpha \in \overline{F}$ such that α is a root of a, but $\alpha - 1$ is not. Moreover, there exists a nonnegative integer ℓ such that $\alpha + \ell$ is a root of a but $\alpha + \ell + 1$ is not.

Set $\beta = \alpha + \ell$. Since $f = \sigma(g)/g$,

$$f = \frac{\sigma(a)b}{a\sigma(b)}.$$

We have that $\alpha - 1$ is a root of $\sigma(a)$, but not a root of $a\sigma(b)$ by the definition of α and since $\gcd(\sigma(a), \sigma(b)) = 1$. Similarly, β is a root of a, but not a root of $\sigma(a)b$ by the definition of β and since $\gcd(a, b) = 1$. It follows that $\alpha - 1$ is a root of the numerator of f, while β is a root of the denominator of f. Since $\beta - (\alpha - 1)$ is an integer, f is not shift-reduced, a contradiction. Hence, a belongs to F. In the same vein, b belongs to F. □

Definition 4.3.5 (Differential rational normal form)**.** *For $f \in F(z)$, call a pair $(K, S) \in F(z)^2$ a* differential rational normal form *(abbreviated as DRNF) of f if $f = K + \delta(S)/S$ and K is differential-reduced with respect to z over F. If, in addition, the denominators of K and S are coprime, then the pair (K, S) is said to be a* strict DRNF *of f.*

Definition 4.3.6 (Shift rational normal form)**.** *For $f \in F(z)$, call a pair $(K, S) \in F(z) \times F(z)$ a* shift rational normal form *(abbreviated as SRNF) of f if $f = K \cdot \sigma(S)/S$ and K is shift-reduced with respect to z over F. Let $k_1 = \mathrm{num}(K)$, $k_2 = \mathrm{den}(K)$, $s_1 = \mathrm{num}(S)$ and $s_2 = \mathrm{den}(S)$. If, furthermore,*

$$\gcd(k_1, \sigma(s_2)s_1) = \gcd(k_2, \sigma(s_1)s_2) = 1,$$

then the pair (K, S) is said to be a strict SRNF *of f.*

Both strict DRNF's and strict SRNF's exist and can be computed efficiently [47, 13]. Let $h(z)$ be a hyperexponential or hypergeometric function over $F(z)$ with certificate f in $F(z)$. Any differential or shift rational normal form (K, S) of f leads to a multiplicative factorization of h in the form

$$h(z) = Sh', \quad \text{where the certificate of } h' \text{ is } K.$$

From this factorization, one can perform two kinds of minimal decompositions on h, which will be reviewed in the next chapter.

4.3.2 $\mathbb{Y}$-rational normal forms

We introduce a new normal form for rational functions in $k(\mathbf{x}, \mathbf{y})$, which will help us derive the structure of certificates in the continuous-discrete setting.

For brevity, we set $F_i = k(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_m, \mathbf{y})$ for all $i \in \{1, \ldots, m\}$, and abbreviate "irreducible partial fraction decomposition" as IPFD.

For a nonzero element $r \in k(\mathbf{x}, \mathbf{y})$, its IPFD with respect to x_i can be written as

$$r = p_0 + \sum_{i=1}^{\ell} \sum_{j=1}^{n_i} \frac{p_{ij}}{q_i^j}$$

where p_0, p_{ij}, q_i are in $F_i[x_i]$ such that $\deg_{x_i}(p_{ij}) < \deg_{x_i}(q_i)$ and q_i is monic and irreducible.

Let $\mathbb{Y}$ be the additive group generated by $1, y_1, \ldots, y_n$. Let r and its IPFD with respect to x_i be given above. We define r_S to be the sum of all fractions of the form $p_{i,1}/q_i$ with the following two properties

(i) $p_{i,1} = y\delta_i(q_i)$ for some nonzero element $y \in \mathbb{Y}$, and

(ii) $n_i = 1$, that is, q_i^2 does not divide the denominator of r.

Furthermore, set $r_K = r - r_S$. We call r_K the $\mathbb{Y}$-*kernel*, and r_S the $\mathbb{Y}$-*shell* of r with respect to x_i. For completeness, the $\mathbb{Y}$-kernel and $\mathbb{Y}$-shell of zero are both defined to be zero.

Note that a $\mathbb{Y}$-shell with respect to x_i can be written as

$$r_S = \frac{\delta_i(f)}{f} + \sum_{j=1}^{n} y_j \frac{\delta_i(u_j)}{u_j}$$

for some $f, u_j \in k(\mathbf{x}, \mathbf{y})$. Moreover, the denominators of the $\mathbb{Y}$-kernel and $\mathbb{Y}$-shell of a nonzero rational function are coprime.

Lemma 4.3.7. *Let r be a nonzero rational function in $k(\mathbf{x}, \mathbf{y})$ with $\mathbb{Y}$-kernel r_K and $\mathbb{Y}$-shell r_S with respect to x_i. Let $g = y\delta_i(q)/q$, where y is a nonzero element of $\mathbb{Y}$, and q is a monic and irreducible polynomial in $F_i[x_i]$ with positive degree.*

(i) If q is a divisor of the denominator of r_K, then the $\mathbb{Y}$-kernel and $\mathbb{Y}$-shell of $r+g$ are r_K+g and r_S, respectively.

(ii) If q is not a divisor of the denominator of r_K, then the $\mathbb{Y}$-kernel and $\mathbb{Y}$-shell of $r+g$ are r_K and r_S+g, respectively.

Proof. Since the denominator q of g is irreducible, it suffices to look at q-expansions of r instead of its IPFD. The q-adic expansion of r with respect to x_i can be written as:

$$r = \frac{p_\ell}{q^\ell} + \cdots + \frac{p_2}{q^2} + \frac{p_1}{q} + \text{higher terms}, \tag{4.6}$$

where $p_\ell, \ldots, p_2, p_1 \in F_i[x_i]$ with degrees less than the degree of q in x_i. It follows that the q-adic expansion of $r+g$ with respect to x_i is

$$r+g = \frac{p_\ell}{q^\ell} + \cdots + \frac{p_2}{q^2} + \frac{p_1 + y\delta(q)}{q} + \text{higher terms}. \tag{4.7}$$

Assume that q is a divisor of the denominator of r_K. Then one of $p_1, p_2, \ldots, p_\ell$ is nonzero. If one of $p_2, \ldots, p_\ell$ is nonzero, then there is no fraction with denominator q appearing in the IPFD of the $\mathbb{Y}$-shell of $r+g$. The first assertion holds. If $p_2, \ldots, p_\ell$ are all equal to zero, then p_1 is nonzero. By the definition of $\mathbb{Y}$-shells, p_1 is not a product of $\delta(q)$ and a nonzero element in $\mathbb{Y}$, and nor is $p_1 + y\delta_i(q)$. Hence, the fraction $(p_1 + y\delta_i(q))/q$ appears in the IPFD of the $\mathbb{Y}$-kernel of $r+g$. The first assertion holds again.

Assume that q is not a divisor of the denominator of r_K. Then p_1 in (4.6) is of the form $y'\delta_i(q)$ for some $y' \in \mathbb{Y}$ (noting that y' may possibly be zero). Moreover, all $p_2, \ldots, p_\ell$ are equal to zero. Hence, (4.7) becomes

$$r+g = (y'+y)\frac{\delta_i(q)}{q} + \text{higher terms}.$$

It follows that the $\mathbb{Y}$-shell of $r+g$ is equal to r_S+g. □

We now define five additive subgroups of $k(\mathbf{x},\mathbf{y})$ in order to suppress complicated expressions that would appear. Let

$$L_i = \left\{ \frac{\delta_i(f)}{f} \mid f \in k(\mathbf{x},\mathbf{y}),\ f \neq 0 \right\},$$

$$M_i = \left\{ \sum_{\ell=1}^{n} y_\ell \frac{\delta_i(u_\ell)}{u_\ell} \mid u_\ell \in k(\mathbf{x}),\ u_\ell \neq 0 \right\},$$

and

$$M_{i,j} = \left\{ \sum_{\ell=1}^{j-1} y_\ell \frac{\delta_i(v_\ell)}{v_\ell} + \sum_{\ell=j+1}^{n} y_\ell \frac{\delta_i(v_\ell)}{v_\ell} \mid v_\ell \in k(\mathbf{x}, y_j),\, v_\ell \neq 0 \right\}.$$

Moreover, let $N_i = L_i + M_i + k(\mathbf{x})$ and $N_{i,j} = L_i + M_{i,j} + k(\mathbf{x}, y_j)$.

Remark 4.3.8. *Let G be one of the subgroups $L_i, M_i, N_i, M_{i,j}, N_{i,j}$. If r belongs to G, so does every fraction appearing in the IPFD of r with respect to x_i.*

The next lemma is rather technical. It tells us how the $\mathbb{Y}$-shell and $\mathbb{Y}$-kernel of an element in $N_{i,j}$ look like.

Lemma 4.3.9. *Let $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Let a be a nonzero element of $N_{i,j}$ with $\mathbb{Y}$-kernel a_K and $\mathbb{Y}$-shell a_S with respect to x_i. Then there exists $u \in k(\mathbf{x}, y_j)$ such that*

$$a_K \equiv 0 \mod M_{i,j} + k(\mathbf{x}, y_j) \quad \textit{and} \quad a_S \equiv y_j \frac{\delta_i(u)}{u} \mod L_i + M_{i,j}.$$

Proof. Without loss of generality, assume that $i = 1$ and $j = n$. All the IPFD's, $\mathbb{Y}$-kernels and $\mathbb{Y}$-shells in the proof are with respect to x_1. Our goal is to show that

$$a_K \equiv 0 \mod M_{1,n} + k(\mathbf{x}, y_n) \quad \text{and} \quad a_S \equiv y_n \frac{\delta_1(u)}{u} \mod L_1 + M_{1,n} \tag{4.8}$$

for some $u \in k(\mathbf{x}, y_n)$.

Since $a \in N_{1,n}$, there exists $r \in k(\mathbf{x}, y_n)$ such that

$$a \equiv r \mod L_1 + M_{1,n}.$$

Assume that r_K and r_S are the $\mathbb{Y}$-kernel and $\mathbb{Y}$-shell of r, respectively. Then

$$r_K \in k(\mathbf{x}, y_n) \quad \text{and} \quad r_S = \frac{\delta_1(f)}{f} + y_n \frac{\delta_1(u)}{u}.$$

where $f, u \in k(\mathbf{x}, y_n)$. Using congruences, we have

$$r_K \equiv 0 \mod M_{1,n} + k(\mathbf{x}, y_n) \quad \text{and} \quad r_S \equiv y_n \frac{\delta_1(u)}{u} \mod L_1 + M_{1,n} \tag{4.9}$$

To proceed, we denote by p the denominator of r_K, which is viewed as a polynomial in x_1 over $k(x_2, \ldots, x_m, y_n)$. Let $b = a - r$, which is in $L_1 + M_{1,n}$. If $b = 0$, then there is nothing to prove. Assume that $b \neq 0$, and that g is a fraction appearing in the IPFD of b. By the definitions of L_1 and $M_{1,n}$, we have that $g = y\delta_1(q)/q$, where y is a nonzero element in $\mathbb{Y}$ free of y_n, and q is a monic and irreducible polynomial in x_1 over $k(x_2, \ldots, x_m, \mathbf{y})$. We now make a case distinction.

Case 1. If q is a divisor of p, then we set $r'_K = r_K + g$ and $r'_S = r_S$.

Case 2. If q is not a divisor of p, then we set $r'_K = r_K$ and $r'_S = r_S + g$.

Let $r' = r'_K + r'_S$. By Lemma 4.3.7, r'_K and r'_S are the $\mathbb{Y}$-kernel and $\mathbb{Y}$-shell of r', respectively. In Case *1*, q is a factor of p. So q is in $k(\mathbf{x}, y_n)$. Accordingly, g is in $M_{1,n} + k(\mathbf{x}, y_n)$. It follows from (4.9) that

$$r'_K \equiv 0 \mod M_{1,n} + k(\mathbf{x}, y_n) \quad \text{and} \quad r'_S \equiv y_n \frac{\delta_1(u)}{u} \mod L_1 + M_{1,n}.$$

By Remark 4.3.8, g is in $L_1 + M_{1,n}$, which, together with (4.9), implies that the above two congruences also hold in Case *2*.

Since $a - r' = a - r - g$ and $g \in L_1 + M_{1,n}$, we have

$$a \equiv r' \mod L_1 + M_{1,n}.$$

Let $b' = a - r'$. Then the number of fractions in the IPFD of b' is less than that in the IPFD of b. Repeating the same argument on b', we see that (4.8) follows from an easy induction. $\square$

The lemma below is useful to generalize Proposition 5 in [42].

Lemma 4.3.10. *Let $i \in \{1, \ldots, m\}$ and $p, q \in \{1, \ldots, n\}$ with $p \neq q$. Then*

$$N_{i,p} \cap N_{i,q} = N_i.$$

Proof. Without loss of generality, we assume that $i = 1$, $p = 1$, $q = n$ and $n > 1$. Since

$$M_1 \subset M_{1,1} + k(\mathbf{x}, y_1) \quad \text{and} \quad M_1 \subset M_{1,n} + k(\mathbf{x}, y_n),$$

we have N_1 is included in the intersection $N_{1,1}$ and $N_{1,n}$ by definition. It suffices to show the opposite inclusion. Assume that a is a nonzero element in the intersection of $N_{1,1}$ and $N_{1,n}$. Let a_S and a_K be, respectively, the $\mathbb{Y}$-shell and $\mathbb{Y}$-kernel of a with respect to x_1. By Lemma 4.3.9, the $\mathbb{Y}$-shell of a with respect to x_1 is of the form

$$a_S = \frac{\delta_1(f_1)}{f_1} + \sum_{j=1}^{n} y_j \frac{\delta_1(v_j)}{v_j} = \frac{\delta_1(f'_1)}{f'_1} + \sum_{j=1}^{n} y_j \frac{\delta_1(v'_j)}{v'_j},$$

where $f_1, f_1' \in k(\mathbf{x}, \mathbf{y})$, $v_j \in k(\mathbf{x}, y_1)$ and $v_j' \in k(\mathbf{x}, y_n)$ for all j with $1 \le j \le n$. Since the elements $1, y_1, \ldots, y_n$ are $\mathbb{Z}$-linearly independent constants with respect to δ_1, Corollary 2.3.5 implies that

$$\frac{\delta_1(f_1)}{f_1} = \frac{\delta_1(f_1')}{f_1'} \quad \text{and} \quad \frac{\delta_1(v_j)}{v_j} = \frac{\delta_1(v_j')}{v_j'}$$

for all j with $1 \le j \le n$. Since the differential equation

$$\delta_1(z) = \frac{\delta_1(v_j')}{v_j'} z$$

has a solution $v_j \in k(\mathbf{x}, y_1)$ and its coefficients belong to $k(\mathbf{x}, y_n)$, it must have a solution u_j in $k(\mathbf{x})$. Hence,

$$a_S = \frac{\delta_1(f_1)}{f_1} + \sum_{j=1}^{n} y_j \frac{\delta_1(u_j)}{u_j} \in N_1.$$

It remains to show that the $\mathbb{Y}$-kernel of a with respect to x_1 belongs to N_1. By Lemma 4.3.9, the $\mathbb{Y}$-kernel of a is of the form

$$a_K = y_n \frac{\delta_1(w_n)}{w_n} + \sum_{\ell=2}^{n-1} y_\ell \frac{\delta_1(w_\ell)}{w_\ell} + s = y_1 \frac{\delta_1(w_1')}{w_1'} + \sum_{\ell=2}^{n-1} y_\ell \frac{\delta_1(w_\ell')}{w_\ell'} + s',$$

where $w_2, \ldots, w_n, s \in k(\mathbf{x}, y_1)$ and $w_1', \ldots, w_{n-1}', s' \in k(\mathbf{x}, y_n)$. Note that a_K is a polynomial in y_1 over $k(\mathbf{x}, y_2, \ldots, y_n)$ whose degree (in y_1) is less than or equal to 1 by the second equality given above. So $w_2, \ldots, w_n$ in the first equality can be taken as elements in $k(\mathbf{x})$. For, otherwise, the denominator of a_K would involve y_1, a contradiction. It follows that

$$s = cy_1 + d,$$

where $c, d \in k(\mathbf{x})$. In the same vein, we have

$$a_K = y_1 \frac{\delta_1(w_1')}{w_1'} + \sum_{\ell=2}^{n-1} y_\ell \frac{\delta_1(w_\ell')}{w_\ell'} + c' y_n + d',$$

where $w_1', \ldots, w_{n-1}'$, and c', d' are chosen to be in $k(\mathbf{x})$. It follows that $c = \delta_1(w_1')/w_1'$. Consequently,

$$a_K = y_1 \frac{\delta_1(w_1')}{w_1'} + \sum_{\ell=2}^{n} y_\ell \frac{\delta_1(w_\ell)}{w_\ell} + d,$$

which belongs to N_1. We have proved that both a_S and a_K are in N_1, and so is a. $\square$

The lemma below can be viewed as a discrete analogue of Lemma 4.3.10.

Lemma 4.3.11. *Let $p, q \in \{1, \ldots, m\}$ with $p \neq q$, and $j \in \{1, \ldots, n\}$. If a nonzero rational function $b \in k(\mathbf{x}, \mathbf{y})$ can be written as*

$$b = \frac{\sigma_j(f_p)}{f_p}\beta_p\alpha_p = \frac{\sigma_j(f_q)}{f_q}\beta_q\alpha_q \tag{4.10}$$

for some $f_p, f_q \in k(\mathbf{x}, \mathbf{y})$, $\beta_p, \beta_q \in k(\mathbf{x})$, $\alpha_p \in k(x_p, \mathbf{y})$ and $\alpha_q \in k(x_q, \mathbf{y})$, then there exist $f \in k(\mathbf{x}, \mathbf{y})$, $\alpha \in k(\mathbf{y})$, and $\beta \in k(\mathbf{x})$ such that

$$b = \frac{\sigma_j(f)}{f}\beta\alpha.$$

Proof. Without loss of generality, we may assume that both α_p and α_q are shift-reduced with respect to y_j. Suppose that

$$\frac{\alpha_p}{\alpha_q} = \frac{\sigma_j(g)}{g}\frac{1}{u}, \tag{4.11}$$

in which $(g, 1/u)$ is a shift rational normal form of α_p/α_q with respect to y_j. Then g belongs to $k(\mathbf{y})$, because x_p and x_q are two distinct indeterminates, and α_p, α_q are shift-reduced. It follows from (4.10) and (4.11) that

$$\frac{\sigma_j(w)}{w} = \frac{\beta_q}{\beta_p}u,$$

where $w = gf_p/f_q$. Since u is shift-reduced with respect to y_j, and $\beta_p, \beta_q \in k(\mathbf{x})$, the above equation and Lemma 4.3.4 imply that $\sigma_j(w) = w$, and

$$\frac{\beta_q}{\beta_p}u = 1,$$

and, hence, $u \in k(\mathbf{x})$. By (4.11), $\alpha_p \in k(x_p, \mathbf{y})$ is a product of an element in $k(\mathbf{y})$, an element in $k(\mathbf{x})$, and α_q in $k(x_q, \mathbf{y})$. By the uniqueness of the factorization of rational functions, α_p can be written as a product cd for some $c \in k(x_p)$ and $d \in k(\mathbf{y})$. Setting $f = f_p$, $\beta = \beta_p c$, and $\alpha_p = d$ yields the lemma. ◻

4.4 Structure of compatible rational functions

The goal of this section is to show the following theorem, which describes the structure of compatible rational functions. To this end, let us recall a useful notation from Payne's thesis [73].

For any integers $a, b \in \mathbb{Z}$ and a sequence of expressions A_i, define

$$\prod_{i}{}_a^b A_i = \begin{cases} \prod_{i=a}^{b-1} A_i, & \text{if } b > a; \\ 1, & \text{if } b = a; \\ 1/\prod_{i=b}^{a-1} A_i, & \text{if } b < a. \end{cases}$$

Theorem 4.4.1. *Assume that* $a_1, \ldots, a_m, b_1, \ldots, b_n \in k(\mathbf{x}, \mathbf{y})$ *are* $m+n$ *rational functions such that* $b_1 \cdots b_n \neq 0$*, and that all the equalities in* (4.2), (4.3) *and* (4.4) *hold. Then there exist* $f \in k(\mathbf{x}, \mathbf{y})$*, univariate rational functions* $r_v \in k(z)$ *for each* v *in a finite set* $V \subset \mathbb{Z}^n$*,* $c_1, \ldots, c_L \in \bar{k}$*,* $g_0, \beta_1, \ldots, \beta_n \in k(\mathbf{x})$*, and* $g_1, \ldots, g_L \in \bar{k}(\mathbf{x})$ *such that*

$$a_i = \delta_i(g_0) + \frac{\delta_i(f)}{f} + \sum_{\ell=1}^{L} c_\ell \frac{\delta_i(g_\ell)}{g_\ell} + \sum_{j=1}^{n} y_j \frac{\delta_i(\beta_j)}{\beta_j} \quad \text{for all } i \text{ with } 1 \leq i \leq m, \tag{4.12}$$

and

$$b_j = \frac{\sigma_j(f)}{f} \beta_j \prod_{v \in V} \prod_{p}{}_{0}^{v_j} r_v(\mathbf{y} \cdot v + p) \quad \text{for all } j \text{ with } 1 \leq j \leq n. \tag{4.13}$$

where $\mathbf{y} \cdot v$ *denotes the inner product* $y_1 v_1 + \ldots + y_n v_n$*.*

Before proving this theorem, we review two special cases in Sections 4.4.1 and 4.4.2.

4.4.1 The Ore-Sato theorem

The structure of rational solutions of the recurrence equation

$$R_1(m, n+1) R_2(m, n) = R_1(m, n) R_2(m+1, n)$$

has been described by Ore [70]. The multivariate extension of Ore's theorem was obtained by Sato [81] in the 1960s when he developed the theory of prehomogeneous vector spaces. In the process of proving the discrete case of Wilf and Zeilberger's conjecture on holonomic hypergeometric terms, the Ore-Sato theorem was discovered and proved again by Payne in his Ph.D. thesis [73] and independently by Abramov and Petkovšek [14]. In particular, the case of two variables has also been shown by Hou [54, 55] and by Abramov and Petkovšek [12]. Their results reveal a multiplicative structure of the certificates $b_1, \ldots, b_n$ of a hyperexponential-hypergeometric function from its integrability conditions (4.3). The following presentation of the Ore-Sato theorem is taken from Payne's thesis [73, Theorem 2.8.4].

Theorem 4.4.2 (Ore-Sato theorem)**.** *Let* $b_1, \ldots, b_n \in k(\mathbf{y})$ *be nonzero rational functions such that*

$$b_i \sigma_i(b_j) = b_j \sigma_j(b_i), \quad \text{for } 1 \leq i < j \leq n.$$

Then there exist a rational function $f \in k(\mathbf{y})$, a finite set $V \subset \mathbb{Z}^n$, and univariate rational functions r_v in $k(z)$ for each $v \in V$ such that for all j with $1 \le j \le n$,

$$b_j = \frac{\sigma_j(f)}{f} \prod_{v \in V} \prod\nolimits_{p}^{v_j}{}_{0}\, r_v(\mathbf{y} \cdot v + p),$$

where $\mathbf{y} \cdot v$ denotes the inner product $y_1 v_1 + \ldots + y_n v_n$.

4.4.2 Alternative proof of multivariate Christopher's theorem

A multivariate hyperexponential function $h(\mathbf{x})$ is a nonzero solution of the first-order fully integrable system

$$\delta_1(z) = a_1 z, \quad \ldots, \quad \delta_m(z) = a_m z,$$

where $a_1, \ldots, a_m$ are compatible rational functions in $k(\mathbf{x})$, i.e., they satisfy the integrability conditions (4.3). In his refinement of Singer's theorem [82] on Liouvillian first integrals, Christopher [29] has described a possible form for those compatible a_i's in the bivariate case.

Theorem 4.4.3 (Christopher, 1999). *Any bivariate hyperexponential function $h(x_1, x_2)$ over the field $\mathbb{C}(x_1, x_2)$ can be written as*

$$\exp(f) \prod_{\ell=1}^{L} g_\ell^{c_\ell}, \quad \text{where } f, g_\ell \in \mathbb{C}(x_1, x_2) \text{ and } c_\ell \in \mathbb{C}.$$

Consequently, the two certificates a_1 and a_2 of $h(x_1, x_2)$ can be written as

$$a_i = \delta_i(f) + \sum_{\ell=1}^{L} c_\ell \frac{\delta_i(g_\ell)}{g_\ell}, \quad \text{for } i = 1, 2.$$

Zoladek [97] has shown the multivariate extension of Christopher's theorem by using a result in Cerveau and Mattei's book [25]. We offer an alternative proof involving only rational functions.

Theorem 4.4.4 (Multivariate Christopher's theorem). *Let $a_1, \ldots, a_m \in k(\mathbf{x})$ be rational functions such that*

$$\delta_i(a_j) = \delta_j(a_i), \quad \text{for } 1 \le i < j \le m.$$

Then there exist $f \in k(\mathbf{x})$, nonzero elements $c_\ell \in \overline{k}$, and $g_\ell \in k(c_\ell)(\mathbf{x})$ for $1 \le \ell \le n$ such that

$$a_i = \delta_i(f) + \sum_{\ell=1}^{n} c_\ell \frac{\delta_i(g_\ell)}{g_\ell}, \quad \text{for } i = 1, \ldots, m.$$

The proof proceeds by induction on m. To this end, we first present a lemma for the induction step. In the proof of the lemma below, we use the formula:

$$\delta_i\left(\frac{\delta_j(f)}{f}\right) = \delta_j\left(\frac{\delta_i(f)}{f}\right) \quad \text{for all nonzero } f \in k(\mathbf{x}) \text{ and } 1 \le i < j \le m. \tag{4.14}$$

Lemma 4.4.5. *Let K denote $k(x_2, x_3, \ldots, x_m)$ and $a_1, \ldots, a_m \in k(\mathbf{x})$ be rational functions such that*

$$\delta_i(a_j) = \delta_j(a_i) \quad \textit{for all } i, j \textit{ with } 1 \le i < j \le m.$$

Then there exist $f \in k(\mathbf{x})$, $A_\ell \in K$ with $\ell = 2, \ldots, m$, nonzero elements $c_j \in \overline{k}$ and $p_j \in K(c_j)[x_1] \setminus K(c_j)$ with $1 \le j \le n$ for some finite $n \in \mathbb{N}$ such that $\delta_i(A_j) = \delta_j(A_i)$ for all i, j with $2 \le i < j \le m$, and

$$a_1 = \delta_1(f) + \sum_{j=1}^{n} c_j \frac{\delta_1(p_j)}{p_j},$$

$$a_\ell = \delta_\ell(f) + \sum_{j=1}^{n} c_j \frac{\delta_\ell(p_j)}{p_j} + A_\ell \quad \textit{for } \ell = 2, \ldots, m.$$

Moreover, the p_j's are pairwise coprime polynomials over $K(c_1, \ldots, c_n)$.

Proof. Lemma 2.3.9 asserts that the theorem holds for $m = 1$. Applying Lemma 2.3.9 to $a_1 \in K(x_1)$ yields that there exist $f \in k(\mathbf{x})$, nonzero elements $c_j \in \overline{K}$ and $p_j \in K(c_j)[x_1] \setminus K(c_j)$ with $1 \le j \le n$ for some $n \in \mathbb{N}$ such that

$$a_1 = \delta_1(f) + \sum_{j=1}^{n} c_j \frac{\delta_1(p_j)}{p_j}. \tag{4.15}$$

Moreover, the c_j's are the distinct roots of the Rothstein-Trager resultant of the logarithmic part A/D of a_1 with respect to x_1 and $p_j = \gcd(D, A - c_j\delta_1(D))$. So the p_j's are pairwise coprime polynomials over $K(c_1, \ldots, c_n)$ by Lemma 2.3.7.

We show that all the c_j's are in $\overline{k}$. For all ℓ with $2 \le \ell \le m$, the commutative formula (4.14) implies

$$\delta_\ell(a_1) = \delta_1(\delta_\ell(f)) + \delta_1\left(\sum_{j=1}^{n} c_j \frac{\delta_\ell(p_j)}{p_j}\right) + \sum_{j=1}^{n} \delta_\ell(c_j) \frac{\delta_1(p_j)}{p_j}.$$

Now, it follows from the integrability condition $\delta_\ell(a_1) = \delta_1(a_\ell)$ that

$$\delta_1\left(a_\ell - \delta_\ell(f) - \sum_{j=1}^{n} c_j \frac{\delta_\ell(p_j)}{p_j}\right) = \sum_{j=1}^{n} \delta_\ell(c_j) \frac{\delta_1(p_j)}{p_j}.$$

Since $p_j \in K(c_j)[x_1] \setminus K(c_j)$, we have $\delta_1(p_j) \neq 0$. By Lemma 2.3.4, for all ℓ with $2 \leq \ell \leq m$, we have $\delta_\ell(c_j) = 0$ and

$$a_\ell = \delta_\ell(f) + \sum_{j=1}^{n} c_j \frac{\delta_\ell(p_j)}{p_j} + A_\ell, \quad \text{for some } A_\ell \in K(c_1, \ldots, c_n).$$

So the c_j's are in $\overline{k}$ by Lemma 3.3.2 in [21]. Moreover, for all ℓ with $2 \leq \ell \leq m$, the sum

$$\sum_{j=1}^{n} c_j \frac{\delta_\ell(p_j)}{p_j}$$

is in $k(\mathbf{x})$, because the c_j's are distinct roots of some Rothstein-Trager resultant with coefficients in $k(x_2, \ldots, x_m)$ and the sum above is invariant under every permutation of the c_j's. This implies the A_ℓ's are in K. The integrability conditions $\delta_i(a_j) = \delta_j(a_i)$ imply that $\delta_i(A_j) = \delta_j(A_i)$ for all i, j with $2 \leq i < j \leq m$. This completes the proof. □

Now, we present the proof of Theorem 4.4.4.

Proof. We proceed by induction on m. Lemma 2.3.9 asserts the theorem holds for the base case $m = 1$. We assume that $m \geq 2$ and that the theorem holds for $m - 1$. Let K denote $k(x_2, x_3, \ldots, x_m)$. By Lemma 4.4.5, there exist $f \in k(\mathbf{x})$, $A_\ell \in K$ for $\ell = 2, \ldots, m$, nonzero elements $c_j \in \overline{k}$ and $p_j \in K(c_j)[x_1] \setminus K(c_j)$ with $1 \leq j \leq n$ for some finite $n \in \mathbb{N}$ such that $\delta_i(A_j) = \delta_j(A_i)$, for all i, j with $2 \leq i < j \leq m$, and

$$a_1 = \delta_1(f) + \sum_{j=1}^{n} c_j \frac{\delta_1(p_j)}{p_j},$$
$$a_\ell = \delta_\ell(f) + \sum_{j=1}^{n} c_j \frac{\delta_\ell(p_j)}{p_j} + A_\ell, \quad \text{for } \ell = 2, \ldots, m.$$

By the induction hypothesis, for the $m - 1$ rational functions A_ℓ, there exist $\bar{f} \in K$, nonzero elements $\bar{c}_j \in \overline{k}$ and $\bar{p}_j \in K(c_j)$ for $j = 1, \ldots, \bar{n}$ such that

$$A_\ell = \delta_\ell(\bar{f}) + \sum_{j=1}^{\bar{n}} \bar{c}_j \frac{\delta_\ell(\bar{p}_j)}{\bar{p}_j}, \quad \text{for all } \ell \text{ with } 2 \leq \ell \leq m.$$

Since the $\bar{f}$ and $\bar{p}_j$'s are free of x_1, we have

$$a_i = \delta_i(f + \bar{f}) + \sum_{j=1}^{n} c_j \frac{\delta_i(p_j)}{p_j} + \sum_{j=1}^{\bar{n}} \bar{c}_j \frac{\delta_i(\bar{p}_j)}{\bar{p}_j}, \quad \text{for all } i \text{ with } 1 \leq i \leq m.$$

This completes the proof. □

4.4.3 Multivariate extension of Feng-Singer-Wu's lemma

In the bivariate continuous-discrete case, Feng, Singer, and Wu [42, Proposition 5] have shown that there exist $f \in k(x_1, y_1)$, $\alpha \in k(y_1)$, and $\beta, \gamma \in k(x_1)$ such that

$$a_1 = \frac{\delta_1(f)}{f} + y_1 \frac{\delta_1(\beta)}{\beta} + \gamma \quad \text{and} \quad b_1 = \frac{\sigma_1(f)}{f} \beta\alpha.$$

These two equalities are used to develop algorithms for computing Liouvillian solutions of prime-order linear difference-differential equations in [42]. We generalize Feng-Singer-Wu's result to the multivariate case.

Theorem 4.4.6. *Assume that $a_1, \ldots, a_m, b_1, \ldots, b_n$ are in $k(\mathbf{x}, \mathbf{y})$ with $b_1 \cdots b_n \neq 0$. If all the equalities in (4.2), (4.3) and (4.4) hold, then there exist $f \in k(\mathbf{x}, \mathbf{y})$, $\beta_1, \ldots, \beta_n \in k(\mathbf{x})$, $\gamma_1, \ldots, \gamma_m \in k(\mathbf{x})$ and $\alpha_1, \ldots, \alpha_n \in k(\mathbf{y})$ such that*

$$a_i = \frac{\delta_i(f)}{f} + \sum_{j=1}^{n} y_j \frac{\delta_i(\beta_j)}{\beta_j} + \gamma_i \quad \text{for all } i \text{ with } 1 \leq i \leq m, \tag{4.16}$$

and

$$b_j = \frac{\sigma_j(f)}{f} \beta_j \alpha_j \quad \text{for all } j \text{ with } 1 \leq j \leq n. \tag{4.17}$$

Moreover, all the equalities in (4.2), (4.3) and (4.4) remain valid if a_i is replaced by γ_i and b_j is replaced by α_j for all i with $1 \leq i \leq m$ and j with $1 \leq j \leq n$.

Before presenting a proof of the theorem, we look at a few special cases. If $n = 0$, then we may set $f = 1$ and $\gamma_i = a_i$ for all i with $1 \leq i \leq m$. Similarly, if $m = 0$, then we set $f = 1$, $\beta_j = 1$ and $b_j = \alpha_j$ for all j with $1 \leq j \leq n$. Hence, the theorem holds if either $n = 0$ or $m = 0$. If $m = n = 1$, then the theorem holds by Proposition 5 in [42]. Note that k is assumed to be algebraically closed in [42]. But the proposition holds without this assumption when one reads their proof carefully. Hence, it suffices to show the theorem when $m > 1$ and $n \geq 1$ or when $m \geq 1$ and $n > 1$.

The proof proceeds by induction on m and n. To this end, we first show that Theorem 4.4.6 holds for $m = 1$ and n arbitrary.

Lemma 4.4.7. *Let $a, b_1, \ldots, b_n$ be rational functions in $k(x, \mathbf{y})$ such that $b_1, \ldots, b_n$ satisfy the integrability condition (4.3), $b_1 \cdots b_n \neq 0$ and*

$$\frac{\delta(b_j)}{b_j} = \sigma_j(a) - a \quad \text{for all } j \text{ with } 1 \leq j \leq n, \tag{4.18}$$

where $\delta = d/dx$. *Then there exist* $f \in k(x, \mathbf{y})$, $\alpha_1, \ldots, \alpha_n \in k(\mathbf{y})$, $\beta_1, \ldots, \beta_n \in k(x)$, *and* $\gamma \in k(x)$ *such that*

$$a = \frac{\delta(f)}{f} + \sum_{j=1}^{n} y_j \frac{\delta(\beta_j)}{\beta_j} + \gamma \quad \text{and} \quad b_j = \frac{\sigma_j(f)}{f}\beta_j\alpha_j \quad \text{for } 1 \le j \le n. \tag{4.19}$$

Moreover, the α_i*'s also satisfy* (4.3).

Proof. We proceed by induction on n. If $n = 1$, then the lemma follows from Proposition 5 in [42]. Assume that n is greater than 1, and the claim holds for the values lower than n. Regard $a, b_1, \ldots, b_n$ as rational functions in $x, y_1, \ldots, y_{n-1}$ over $k(y_n)$. By the induction hypothesis, there exist $\tilde{f}$ in $k(x, \mathbf{y})$, $\tilde{\beta}_1, \ldots, \tilde{\beta}_{n-1}$ in $k(x, y_n)$ and $\tilde{\gamma}$ in $k(x, y_n)$ such that

$$a = \frac{\delta(\tilde{f})}{\tilde{f}} + y_1\frac{\delta(\tilde{\beta}_1)}{\tilde{\beta}_1} + \cdots + y_{n-1}\frac{\delta(\tilde{\beta}_{n-1})}{\tilde{\beta}_{n-1}} + \tilde{\gamma}.$$

Similarly, regarding $a, b_1, \ldots, b_n$ as rational functions in x, y_2, $\ldots$, y_n over $k(y_1)$, we have

$$a = \frac{\delta(f')}{f'} + y_2\frac{\delta(\beta'_2)}{\beta'_2} + \cdots + y_n\frac{\delta(\beta'_n)}{\beta'_n} + \gamma',$$

where f' is in $k(x, \mathbf{y})$, $\beta'_2, \ldots, \beta'_n$ are in $k(x, y_1)$, and γ' is in $k(x, y_1)$. It follows from Lemma 4.3.10 that there exist $f \in k(x, \mathbf{y})$ and $\beta_1, \ldots, \beta_n, \gamma \in k(x)$ such that

$$a = \frac{\delta(f)}{f} + \sum_{j=1}^{n} y_j\frac{\delta(\beta_j)}{\beta_j} + \gamma. \tag{4.20}$$

Assume that, for j with $1 \le j \le n$,

$$b_j = \frac{\sigma_j(f)}{f}\beta_j\alpha_j \tag{4.21}$$

for some $\alpha_j \in k(x, \mathbf{y})$. By (4.20),

$$\sigma_j(a) - a = \frac{\sigma_j(\delta(f))}{\sigma_j(f)} - \frac{\delta(f)}{f} + \frac{\delta_j(\beta)}{\beta},$$

and, by (4.21),

$$\frac{\delta(b_j)}{b_j} = \frac{\delta(\sigma_j(f))}{\sigma_j(f)} - \frac{\delta(f)}{f} + \frac{\delta(\beta_j)}{\beta_j} + \frac{\delta(\alpha_j)}{\alpha_j}.$$

These two equalities, the integrability condition (4.18), and the commutativity of δ and σ_j imply

$$\frac{\delta(\alpha_j)}{\alpha_j} = 0.$$

Consequently, α_j belongs to $k(\mathbf{y})$. By (4.3), the α_j's in the above proof satisfy

$$\frac{\sigma_q(\alpha_j)}{\alpha_j} = \frac{\sigma_j(\alpha_q)}{\alpha_q}$$

for all j, q with $1 \le j < q \le n$. This completes the proof. ∎

Now, we present a proof of Theorem 4.4.6.

Proof. We proceed by induction on m with Lemma 4.4.7 as our induction base. Let n be a fixed positive integer, and assume that the theorem holds for the values lower than m. Consider the continuous variables $x_1, \ldots, x_{m-1}$ and the discrete ones $y_1, \ldots, y_n$ over $k(x_m)$. We have, for all j with $1 \leq j \leq n$, there exist $\tilde{f}_j \in k(\mathbf{x}, \mathbf{y})$, $\tilde{\beta}_j \in k(\mathbf{x})$ and $\tilde{\alpha}_j \in k(x_m, \mathbf{y})$ such that

$$b_j = \frac{\sigma_j(\tilde{f}_j)}{\tilde{f}_j} \tilde{\beta}_j \tilde{\alpha}_j.$$

Consider the continuous variables $x_2, \ldots, x_m$ and the discrete ones $y_1, \ldots, y_n$ over $k(x_1)$. We have that for all j with $1 \leq j \leq n$, there exist $\hat{f}_j \in k(\mathbf{x}, \mathbf{y})$, $\hat{\beta}_j \in k(\mathbf{x})$ and $\hat{\alpha}_j \in k(x_1, \mathbf{y})$ such that

$$b_j = \frac{\sigma_j(\hat{f}_j)}{\hat{f}_j} \hat{\beta}_j \hat{\alpha}_j.$$

By the above two equalities and Lemma 4.3.11, there exist $f'_j \in k(\mathbf{x}, \mathbf{y})$, $\beta'_j \in k(\mathbf{x})$, $\alpha'_j \in k(\mathbf{y})$ such that

$$b_j = \frac{\sigma_j(f'_j)}{f'_j} \beta'_j \alpha'_j \quad \text{for all } j \text{ with } 1 \leq j \leq n. \tag{4.22}$$

Now, we claim that there exist $f \in k(\mathbf{x}, \mathbf{y})$, $\beta_1, \ldots, \beta_n \in k(\mathbf{x})$ and $\alpha_1, \ldots, \alpha_n \in k(\mathbf{y})$ such that

$$b_j = \frac{\sigma_j(f)}{f} \beta_j \alpha_j \quad \text{for all } j \text{ with } 1 \leq j \leq n.$$

We prove the claim by induction on n. If $n = 1$, then the claim is true by (4.22). Assume that the claim holds for $\ell < n$, and that

$$b_{\ell+1} = \frac{\sigma_{\ell+1}(f)}{f} u \tag{4.23}$$

for some $u \in k(\mathbf{x}, \mathbf{y})$. By (4.3),

$$\sigma_1(u) = \frac{\sigma_{\ell+1}(\alpha_1)}{\alpha_1} u, \ \ldots, \ \sigma_\ell(u) = \frac{\sigma_{\ell+1}(\alpha_\ell)}{\alpha_\ell} u.$$

Since $\alpha_1, \ldots, \alpha_\ell \in k(\mathbf{y})$,

$$u = vw,$$

where $v \in k(\mathbf{x}, y_{\ell+1}, \ldots, y_n)$ and $w \in k(\mathbf{y})$. By (4.22) and (4.23),

$$\frac{\sigma_{\ell+1}(f)}{f} vw = \frac{\sigma_{\ell+1}(f'_{\ell+1})}{f'_{\ell+1}} \beta'_{\ell+1} \alpha'_{\ell+1}. \tag{4.24}$$

From the equality (4.24), it follows that

$$v = \frac{\sigma_{\ell+1}(p)}{p}\beta'_{\ell+1}q, \tag{4.25}$$

where $p = f'/f$ and $q = \alpha'_{\ell+1}/w$. Since k is of characteristic zero, there exist $c_1, \ldots, c_\ell \in k$ such that the denominators and numerators of p and q evaluated at $y_1 = c_1, \ldots, y_\ell = c_\ell$ are all nonzero. Substituting $c_1, \ldots, c_\ell$ for $y_1, \ldots, y_\ell$, respectively, into (4.25), we see that

$$v = \frac{\sigma_{\ell+1}(g)}{g}\beta'_{\ell+1}r,$$

where $g \in k(\mathbf{x}, y_{\ell+1}, \ldots, y_n)$ and $r \in k(y_{\ell+1}, \ldots, y_n)$. Setting $\beta_{\ell+1} = \beta'_{\ell+1}$ and $\alpha_{\ell+1} = rw$, we see that

$$b_{\ell+1} = \frac{\sigma_{\ell+1}(fg)}{fg}\beta_{\ell+1}\alpha_{\ell+1}.$$

Moreover, since g is a constant with respect to $\sigma_1, \ldots, \sigma_\ell$,

$$b_j = \frac{\sigma_j(fg)}{fg}\beta_j\alpha_j \quad \text{for all } j \text{ with } 1 \le j \le \ell.$$

The claim holds for $\ell + 1$ when we replace f by fg.

It remains to obtain the form for the a_i's. Assume that, for all i with $1 \le i \le m$,

$$a_i = \frac{\delta_i(f)}{f} + \sum_{j=1}^{m} y_i \frac{\delta_i(\beta_j)}{\beta_j} + \gamma_i,$$

where $\gamma_i \in k(\mathbf{x}, \mathbf{y})$ is to be determined. Using the integrability conditions in (4.4), and performing a calculation similar to that below (4.21) in the proof of Lemma 4.4.7, we find that

$$\sigma_j(\gamma_i) = \gamma_i \quad \text{for all } j \text{ with } 1 \le j \le n,$$

which implies that γ_i belongs to $k(\mathbf{x})$. Moreover, for all $1 \le i < p \le m$,

$$\delta_i(\gamma_p) = \delta_p(\gamma_i).$$

By the integrability conditions (4.3)

$$\frac{\sigma_j(\alpha_q)}{\alpha_q} = \frac{\sigma_q(\alpha_j)}{\alpha_j}$$

for all j, q with $1 \le j < q \le n$. This completes the proof of Theorem 4.4.6. □

Remark 4.4.8. *Applying Theorem 4.4.2 and Theorem 4.4.4 to the γ_i's and α_j's in Theorem 4.4.6, respectively, we readily see that the structure Theorem 4.4.1 holds.*

4.5 Multiplicative structure

In this section, we assume that k is algebraically closed. We derive a multiplicative form of a hyperexponential-hypergeometric function from the structure of its certificates.

Let $h(\mathbf{x}, \mathbf{y})$ be a hyperexponential-hypergeometric function over $k(\mathbf{x}, \mathbf{y})$ with $m+n$ certificates $a_1, \ldots, a_m, b_1, \ldots, b_n$ such that the product $b_1 \cdots b_n$ is nonzero. Then (4.2), (4.3) and (4.4) hold. By Theorem 4.4.6, there exist $f \in k(\mathbf{x}, \mathbf{y})$, $\beta_1, \ldots, \beta_n \in k(\mathbf{x}) \setminus \{0\}$, $\gamma_1, \ldots, \gamma_m \in k(\mathbf{x})$ and $\alpha_1, \ldots, \alpha_n \in k(\mathbf{y})$ such that

$$a_i = \frac{\delta_i(f)}{f} + \sum_{j=1}^{n} y_j \frac{\delta_i(\beta_j)}{\beta_j} + \gamma_i$$

for all i with $1 \leq i \leq m$, and

$$b_j = \frac{\sigma_j(f)}{f} \beta_j \alpha_j$$

for all j with $1 \leq j \leq n$. Moreover,

$$\gamma_1, \ldots, \gamma_m,\ \alpha_1, \ldots, \alpha_n$$

satisfy the integrability conditions (4.2), (4.3) and (4.4). By Theorem 2 in [22], there exists a simple differential-difference extension R of $k(\mathbf{x}, \mathbf{y})$ containing functions h, β^{y_i}, and any $h' \in \mathcal{H}(\gamma_1, \ldots, \gamma_m,\ \alpha_1, \ldots, \alpha_n)$. Every hyperexponential-hypergeometric element in R is invertible, since R is simple. Moreover, two hyperexponential-hypergeometric elements having the same certificates differ by a multiplicative constant in k. It is straightforward to verify that h and $f\beta_1^{y_1} \cdots \beta_n^{y_n} h'$ have the same certificates. Therefore,

$$h = cf\beta_1^{y_1} \cdots \beta_n^{y_n} h',$$

where $c \in k$ and the certificates of h' are $\gamma_1, \ldots, \gamma_m, \alpha_1, \ldots, \alpha_n$. Note that the γ_i's are functions in $\mathbf{x}$, and the α_j's are functions in $\mathbf{y}$. Then we have the factorization

$$\mathcal{H}(\gamma_1, \ldots, \gamma_m,\ \alpha_1, \ldots, \alpha_n) = \mathcal{H}(\gamma_1, \ldots, \gamma_m, 1, \ldots, 1)\mathcal{H}(0, \ldots, 0,\ \alpha_1, \ldots, \alpha_n).$$

By Theorem 4.4.4, any element $h_e \in \mathcal{H}(\gamma_1, \ldots, \gamma_m, 1, \ldots, 1)$ is conjugate to

$$\exp(g_0) \prod_{\ell=1}^{L} g_\ell^{c_\ell},$$

where $c_1, \ldots, c_L \in k$ and $g_0, g_1, \ldots, g_L \in k(\mathbf{x})$. By Corollary 3.7.3 in [73] or Corollary 4 in [14], any element $h_g \in \mathcal{H}(0, \ldots, 0, \alpha_1, \ldots, \alpha_n)$ is conjugate to

$$\tilde{f}(\mathbf{y})T(\mathbf{y}),$$

where $\tilde{f} \in k(\mathbf{y})$ and $T(\mathbf{y})$ is a factorial term in $\mathbf{y}$ (see [73, Definition 3.5.1] or [14, Definition 5]). We summarize the above discussion by the following theorem.

Theorem 4.5.1. *If k is algebraically closed, then any hyperexponential-hypergeometric function h over $k(\mathbf{x}, \mathbf{y})$ is conjugate to a function*

$$r(\mathbf{x}, \mathbf{y}) \exp(g_0(\mathbf{x})) \prod_{\ell=1}^{L} g_\ell(\mathbf{x})^{c_\ell} \prod_{j=1}^{n} \beta_j(\mathbf{x})^{y_j} T(\mathbf{y}) \tag{4.26}$$

where $r \in k(\mathbf{x}, \mathbf{y})$, $g_0, g_1, \ldots, g_L, \beta_1, \ldots, \beta_n \in k(\mathbf{x})$, $c_l, \ldots, c_L \in k$, and $T(\mathbf{y})$ is a factorial term.

Example 4.5.2 (Jacobi polynomials)**.**

$$P_n^{(m_1,m_2)}(x) = \sum_{\ell} h(x, \ell, n, m_1, m_2),$$

where h is a hyperexponential-hypergeometric function over $k(x, \ell, n, m_1, m_2)$ with certificates:

$$a = \frac{2\ell + nx - n}{x^2 - 1}, \quad b_1 = \frac{(x-1)(n-\ell)(n+m_2-\ell)}{(m_1+\ell+1)(\ell+1)(x+1)},$$
$$b_2 = \frac{(x+1)(n+m_2+1)(n+m_1+1)}{2(n-\ell+1)(n+m_2-\ell+1)}, \quad b_3 = \frac{n+m_1+1}{m_1+\ell+1}, \quad b_4 = \frac{n+m_2+1}{n+m_2-\ell+1}$$

According to Theorem 4.4.6, we write (a, b_1, b_2, b_3) as

$$a = \ell \frac{\delta(\frac{x-1}{x+1})}{\frac{x-1}{x+1}} + n \frac{\delta(x+1)}{x+1}, \quad b_1 = \frac{x-1}{x+1} \cdot \frac{(n-\ell)(n+m_2-\ell)}{(m_1+\ell+1)(\ell+1)},$$
$$b_2 = (x+1) \cdot \frac{1}{2} \cdot \frac{(n+m_2+1)(n+m_1+1)}{(n-\ell+1)(n+m_2-\ell+1)}, \quad b_3 = \frac{n+m_1+1}{m_1+\ell+1}, \quad b_4 = \frac{n+m_2+1}{n+m_2-\ell+1}$$

Then h is a hyperexponential-hypergeometric function with a multiplicative form

$$h = \left(\frac{x-1}{x+1}\right)^{\ell} (x+1)^n h', \quad \textit{where} \quad h' = \frac{1}{2^n} \binom{n+m_1}{n-\ell} \binom{n+m_2}{\ell}.$$

Definition 4.5.3 (Properness)**.** *A hyperexponential-hypergeometric function over $k(\mathbf{x}, \mathbf{y})$ is said to be* proper *if it is conjugate to a function*

$$p(\mathbf{x}, \mathbf{y}) \exp(g_0(\mathbf{x})) \prod_{\ell=1}^{L} g_\ell(\mathbf{x})^{c_\ell} \prod_{j=1}^{n} \beta_j(\mathbf{x})^{y_j} T(\mathbf{y}) \tag{4.27}$$

where p is a polynomial in $k[\mathbf{x}, \mathbf{y}]$, $g_0, g_1, \ldots, g_L, \beta_1, \ldots, \beta_n \in k(\mathbf{x})$, $c_1, \ldots, c_L \in k$, and $T(\mathbf{y})$ is a factorial term over $k(\mathbf{y})$.

By Definition 4.5.3 and Theorem 4.4.4, we obtain the following result.

Corollary 4.5.4. *Any multivariate hyperexponential function $H(\mathbf{x})$ over $k(\mathbf{x})$ is proper.*

In their paper [92], Wilf and Zeilberger have listed several familiar special functions which are proper hyperexponential-hypergeometric functions.

Chapter 5

Existence of Telescopers for Hyperexponential-Hypergeometric Functions

5.1 Introduction

The termination problem of Zeilberger's algorithms for creative telescoping has been extensively studied in the last two decades. The algorithms terminate if the existence of telescopers is guaranteed. Zeilberger [95] has shown that holonomicity guarantees the success of his algorithms. In particular, Wilf and Zeilberger have provided an elementary proof, based on the ideas of Fasenmyer [40] and Verbaeten [89], that telescopers always exist for proper hyperexponential-hypergeometric functions [95, 92]. However, holonomicity is only a sufficient condition, i.e., there are cases in which the input functions are not holonomic but Zeilberger's algorithm still terminates [34]. Therefore, one challenge is to find theoretical criteria which enable us to algorithmically detect the existence of telescopers.

In view of the theoretical difficulty, special attention has been focused on the subclass of hyperexponential-hypergeometric functions. In the continuous case, the work by Bernstein [17] and Lipshitz [67] shows that every hyperexponential function has a telescoper. This implies that Zeilberger's algorithm always succeeds on those inputs. However, the situation in other cases turns out to be more involved. In the discrete case and its q-analogue, the first complete solution to the termination problem is Abramov and Le's criterion [64, 9], which decides whether telescopers exist for a given bivariate rational function in the discrete variables m and n. According to their criterion, the rational function

$$f = \frac{1}{m^2 + n^2}$$

has no telescoper. Soon, the criterion was extended to the general case of bivariate hypergeometric terms by Abramov [5, 6]. Basically, Abramov proves that a hypergeometric term can be written as a sum of a hypergeometric-summable term and a proper one if it has a telescoper [6, Theorem 10]. Similar results have been obtained in the general q-shift case by Chen, Hou and Mu [28]. These results are fundamental for detecting the termination of Zeilberger's algorithm.

The continuous-discrete analogue of Zeilberger's algorithm is presented by Almkvist and Zeilberger [15]. This analogue has been shown to be useful in the study of orthogonal polynomials [60, Chapters 10–13].

Motivation. In the bivariate mixed case, not all hyperexponential-hypergeometric functions have telescopers. For example, we will show in this chapter that the simple rational function

$$f = \frac{1}{x + n}$$

has no telescoper with respect to either the continuous variable x or the discrete variable n. Therefore, an Abramov-like criterion is also needed in this mixed case.

Our contribution. The main contribution in this chapter is two criteria for the existence of telescopers for bivariate hyperexponential-hypergeometric functions. Our criteria are of the same type as Abramov's, which says that a hyperexponential-hypergeometric function can be written as a sum of a hypergeometric-summable (integrable) function and a proper one if it has a telescoper with respect to the discrete (continuous) variable. The key ingredients of our proof are: standard representations of bivariate hyperexponential-hypergeometric functions in [42], and two additive decompositions: one is the Abramov-Petkovšek algorithm [11, 13] with respect to the discrete variable; and the other is the Hermite-like reduction [47] with respect to the continuous one.

The rest of this chapter is organized as follows. We describe an algebraic setting for bivariate hyperexponential-hypergeometric functions in Section 5.2. We discuss on the existence of telescopers in Section 5.3. A standard representation of hyperexponential-hypergeometric functions is introduced in Section 5.4. The Abramov-Petkovšek decomposition and the Hermite-like reduction are adapted to the continuous-discrete setting in Section 5.5. Our criteria for the existence of telescopers are presented in Section 5.6.

An earlier version of this chapter can be found in [26], which is a joint work (in progress) with Frédéric Chyzak, Ruyong Feng, and Ziming Li.

5.2 Preliminaries

In his papers [77, 78], Risch based his algorithm on Ritt's theory of differential fields and their extensions. From then on, differential algebra [79, 56, 61] has been the algebraic foundation of symbolic integration [21]. The analog investigation [57, 58] has been carried out for summation problems via difference algebra [36]. In this section, we describe an algebraic setting for hyperexponential-hypergeometric functions, which permits us to deal with the telescoping problem in an algebraic way.

In the rest of this chapter, let k be an algebraically closed field of characteristic zero, and $k(x, n)$ be the field of rational functions in x and n over k. We view x as the continuous variable and n as the discrete variable. On the field $k(x, n)$, the derivation δ and the shift operator σ are defined as

$$\delta(f(x, n)) = \frac{\partial f}{\partial x} \quad \text{and} \quad \sigma(f(x, n)) = f(x, n + 1)$$

for all $f \in k(x, n)$, respectively.

5.2.1 Ring of sequences

We recall the ring of sequences defined in [41, 43] and regard a hyperexponential-hypergeometric function as an element in that ring.

Let $\mathcal{K}$ be the universal Picard-Vessiot field [88, Chapter 10] of the field $k(x)$. Then $(\mathcal{K}, \delta)$ is a differential field extension of $(k(x), \delta)$. Furthermore, the constants in $\mathcal{K}$ with respect to δ are exactly the elements of k, since k is algebraically closed. Let $\mathcal{K}^{\mathbb{N}}$ be the ring of infinite sequences over $\mathcal{K}$, where the addition and multiplication of sequences are defined termwise. For a sequence

$$s = (s_0, s_1, s_2, \ldots)$$

in $\mathcal{K}^{\mathbb{N}}$, define

$$\delta'(s) = (\delta(s_0), \delta(s_1), \delta(s_2), \ldots).$$

Then δ' is a derivation operator on $\mathcal{K}^{\mathbb{N}}$. Let $I \subset \mathcal{K}^{\mathbb{N}}$ be the ideal consisting of sequences with finitely many nonzero terms. Since I is closed under δ', the quotient ring $\mathcal{S} = \mathcal{K}^{\mathbb{N}}/I$ is a differential ring whose derivation $\bar{\delta}$ is induced by δ'. In other words:

$$\bar{\delta}(s + I) = \delta'(s) + I.$$

We define a map $\bar{\sigma} : \mathcal{S} \to \mathcal{S}$ by sending

$$s + I \mapsto (s_1, s_2, s_3, \ldots) + I.$$

It is easy to verify that $\bar{\sigma}$ is a well-defined automorphism, and $\bar{\delta}$ and $\bar{\sigma}$ commute with each other.

We now embed the field $k(x, n)$ of rational functions into $\mathcal{S}$. For an element $f \in k(x, n)$, there exists $m \in \mathbb{N}$ such that $f(x, i)$ is well-defined when $i \geq m$. Define a map $\phi : k(x, n) \to \mathcal{S}$ by sending

$$f(x, n) \mapsto (\underbrace{0, \ldots, 0}_{m}, f(x, m), f(x, m + 1), f(x, m + 2), \ldots) + I.$$

By the definition of I, ϕ is a well-defined monomorphism such that $\phi \circ \delta = \bar{\delta} \circ \phi$ and $\phi \circ \sigma = \bar{\sigma} \circ \phi$. Let us identify $k(x, n)$ with $\phi(k(x, n))$. Then $\bar{\delta}$ and $\bar{\sigma}$ can be viewed as extensions of δ and σ, respectively. Hence, we may also identify $\bar{\delta}$ with δ, and $\bar{\sigma}$ with σ. Consequently, $\mathcal{S}$ is a differential-difference ring extension of $k(x, n)$, and k is the set of constants in $\mathcal{S}$.

Lemma 5.2.1. *Let a and b be in $k(x, n)$ with $b \neq 0$ and*

$$\frac{\delta(b)}{b} = \sigma(a) - a. \tag{5.1}$$

Then there exists an invertible hyperexponential-hypergeometric function in $\mathcal{S}$ such that its x-certificate is equal to a and its n-certificate is equal to b.

Proof. View a and b as rational functions in n. Let N be a sufficiently large integer such that for any $j \geq N$, $a(x,j)$ and $b(x,j)$ are well-defined and $b(x,j) \neq 0$. There exists a nonzero element $v \in \mathcal{K}$ such that

$$\delta(v) = a(x, N)v.$$

Let $h_i = 0$ for all i with $0 \leq i \leq N-1$, $h_N = v$, and $h_{i+1} = b(x,i)h_i$ for all $i > N$. We claim that

$$h = (h_0, h_1, \dots, h_N, h_{N+1}, \dots) + I$$

is an invertible element. Furthermore, its x-certificate and n-certificate are a and b, respectively.

Invertibility follows from the fact $h_i \neq 0$ for all $i \geq N$. We now verify that $\sigma(h) = bh$ by the following calculation:

$$\begin{aligned}
\sigma(h) &= (\underbrace{0, \dots, 0}_{N-1}, h_N, h_{N+1}, h_{N+2}, \cdots) + I \\
&= (\underbrace{0, \dots, 0}_{N-1}, h_N, b(x,N)h_N, b(x, N+1)h_{N+1}, \cdots) + I \\
&= (\underbrace{0, \dots, 1}_{N}, b(x,N), b(x,N+1), \cdots)(\underbrace{0, \dots, 0}_{N-1}, h_N, h_N, h_{N+1}, \cdots) + I \\
&= bh.
\end{aligned}$$

Next, we verify that $\delta(h_i) = a(x,i)h_i$ for $i \geq N$, which implies $\delta(h) = ah$. When $i = N$, the equality holds by the definition of h_N. Assume that the equality holds for all i with $N \leq i \leq \ell$. By (5.1), we have

$$\delta(b(x,\ell)) = a(x, \ell+1)b(x,\ell) - a(x,\ell)b(x,\ell).$$

It follows that

$$\begin{aligned}
\delta(h_{\ell+1}) = \delta(b(x,\ell)h_\ell) &= \delta(b(x,\ell))h_\ell + b(x,\ell)\delta(h_\ell) \\
&= (a(x,\ell+1)b(x,\ell) - a(x,\ell)b(x,\ell))h_\ell + b(x,\ell)a(x,\ell)h_\ell \\
&= a(x,\ell+1)b(x,\ell)h_\ell = a(x,\ell+1)h_{\ell+1}.
\end{aligned}$$

The claim is proved for $\ell + 1$, and so is the lemma by induction. $\square$

Example 5.2.2. *Assume that $\beta(x)$ is a nonzero element of $k(x)$. Then the pair $\left(n\frac{\delta(\beta)}{\beta}, \beta\right)$ satisfies* (5.1)*. It follows that*

$$\left(1, \beta, \beta^2, \ldots\right) + I$$

is a hyperexponential-hypergeometric function in $\mathcal{S}$ with certificates $n\delta(\beta)/\beta$ and β. For later convenience, this element is denoted as β^n.

Let a and b be the same as in Lemma 5.2.1, and assume that (5.1) holds. Recall that

$$\mathcal{H}(a, b) \triangleq \{h \in \mathcal{S} \mid \delta(h) = ah \text{ and } \sigma(h) = bh\}.$$

Proposition 5.2.3. *The set $\mathcal{H}(a, b)$ is a one-dimensional vector space over k, in which every nonzero element is invertible in $\mathcal{S}$.*

Proof. By Lemma 5.2.1, $\mathcal{H}(a, b)$ contains an invertible element, say h. Let g be another element of $\mathcal{H}(a, b)$. Then $\delta(gh^{-1}) = 0$ and $\sigma(gh^{-1}) = gh^{-1}$ by a straightforward calculation. Hence, gh^{-1} is a constant. The lemma follows from the fact that k is the set of constants in $\mathcal{S}$. □

This proposition implies that every first-order fully integrable system over $k(x, n)$ has a one-dimensional solution space over k. For later use, we specialize the three formulas in Lemma 4.2.3 for $\mathcal{H}(a, b)$.

Corollary 5.2.4. (*i*) *For any hyperexponential-hypergeometric function $g \in \mathcal{S}$, we have*

$$g\mathcal{H}(a, b) = \mathcal{H}\left(a + \frac{\delta(g)}{g}, b\frac{\sigma(g)}{g}\right);$$

(*ii*) *For all $a, b, a', b' \in k(x, n)$ with $bb' \neq 0$ such that both (a, b) and (a', b') satisfy* (5.1)*, we have*

$$\mathcal{H}(a, b)\mathcal{H}(a', b') = \mathcal{H}(a + a', bb');$$

(*iii*) *$\delta(\mathcal{H}(a, b)) = a\mathcal{H}(a, b)$ and $\sigma(\mathcal{H}(a, b)) = b\mathcal{H}(a, b)$.*

From now on, we assume that all hyperexponential-hypergeometric functions belong to $\mathcal{S}$, unless otherwise stated.

5.2.2 Ring of differential-recurrence operators

Let $k(x,n)\langle D_x, S_n\rangle$ be the ring of differential-recurrence operators whose commutation rules are

$$S_nD_x = D_xS_n, \quad D_xf = fD_x + \delta(f), \quad \text{and} \quad S_nf = \sigma(f)S_n,$$

for all $f \in k(x,n)$. In this ring, we denote Δ_n the forward difference operator $S_n - 1$. For every $s \in \mathcal{S}$, define two actions: $D_x(s) = \delta(s)$ and $S_n(s) = \sigma(s)$. Then $\mathcal{S}$ becomes a left module over $k(x,n)\langle D_x, S_n\rangle$. Assume that h is a hyperexponential-hypergeometric function with x-certificate a and n-certificate b. Then the integrability condition (5.1) can be rewritten as $S_nD_x(h) = D_xS_n(h)$.

Two hyperexponential-hypergeometric functions h_1 and h_2 are said to be *similar* if the ratio h_1/h_2 is in $k(x,n)$. It is easy to verify that similarity is an equivalence relation among hyperexponential-hypergeometric functions.

Lemma 5.2.5. *Let h_1 and h_2 be two hyperexponential-hypergeometric functions over $k(x,n)$. If h_1 and h_2 are similar, then*

(i) $h_1 + h_2$ is either equal to zero or similar to h_1;

(ii) $L(h_1)$ is either equal to zero or similar to h_1 for any $L \in k(x,n)\langle D_x, S_n\rangle$.

Proof. Let $r \in k(x,n)$ be the nonzero ratio h_1/h_2. Then the first assertion follows from the equality $h_1 + h_2 = (1 + 1/r)h_1$. Since h_1 is hyperexponential-hypergeometric over $k(x,n)$, successive derivatives and shifts of h_1 are all similar to h_1. By the first assertion, the second one follows. $\square$

5.2.3 Split polynomials

For a nonzero element $f \in k(x,n)$, the denominator and numerator of f are denoted by $\mathrm{den}(f)$ and $\mathrm{num}(f)$, respectively. $\mathrm{den}(f)$ and $\mathrm{num}(f)$ are two coprime polynomials in $k[x,n]$. We introduce the definition below to describe a standard form of rational functions in $k(x,n)$.

Definition 5.2.6. *A polynomial $p \in k[x,n]$ is said to be* split *if it is of the form $p_1(x)p_2(n)$ with $p_1 \in k[x]$ and $p_2 \in k[n]$.*

A rational function f can be decomposed as $f_1(x)f_2(n)f_3(x,n)$, where $f_1 \in k(x)$, $f_2 \in k(n)$ and both $\mathrm{num}(f_3)$ and $\mathrm{den}(f_3)$ have no split factors. The product $f_1 f_2$ is called the *split part* of f.

Algorithms for computing rational solutions of linear differential and difference equations with polynomial coefficients have been presented by Abramov [3, 8, 4]. The lemma below concerns the shape of denominators of rational solutions of linear differential and difference equations with the leading coefficients split.

Lemma 5.2.7. *Let L be an operator either in $k[x,n]\langle D_x\rangle$ or in $k[x,n]\langle S_n\rangle$ and let p be a polynomial in $k[x,n]$. If the leading coefficient of L is split, so is the denominator of rational solutions of $L(y(x,n)) = p$.*

Proof. Let $f \in k(x,n)$ be any rational solution of $L(y(x,n)) = p$ and $q \in k[x,n]$ be the leading coefficient of L. If L is in $k[x,n]\langle D_x\rangle$, the splitness of $\mathrm{den}(f)$ follows from the well-known fact that any root of $\mathrm{den}(f)$ in $\overline{k(n)}$ is also a root of q, see [3]. If L is in $k[x,n]\langle S_n\rangle$, the denominator of f is a divisor of the product $Q(x,n) := q(x,n)q(x,n-1)\cdots q(x,n-d)$ for some finite $d \in \mathbb{N}$ according to Abramov's algorithm for computing a denominator bound in [4]. If q is split, so is the product Q. Therefore, $\mathrm{den}(f)$ is split in both cases. □

5.3 Existence problems

The fact that differential and difference cases can be treated in similar-looking ways has been observed for a long time in mathematics. Once a result is obtained in one side, it can often be analogized in the other one. In the sequel, our presentation will reflect the symmetry between the two counterparts of differential and difference variables.

The method of differencing under the integral sign was first formulated by Almkvist and Zeilberger in [15], which can be used to find a linear recurrence equation for the integral (if exists)

$$H(n) := \int_0^{+\infty} h(x,n)\,dx,$$

where $h(x,n)$ is a hyperexponential-hypergeometric function over $k(x,n)$. Here, we suppose integrals are well-defined over k, say $\mathbb{C}$. The key step of Almkvist and Zeilberger's algorithm tries to find a nonzero linear recurrence operator $L(n, S_n)$ in $k(n)\langle S_n\rangle$ such that

$$L(n, S_n)(h) = D_x(g), \tag{5.2}$$

for some hyperexponential-hypergeometric function g over $k(x, n)$.

Definition 5.3.1. *Let h be a hyperexponential-hypergeometric function over $k(x, n)$. A nonzero linear recurrence operator $L(n, S_n) \in k(n)\langle S_n\rangle$ is called a* telescoper *for h* with respect to x *if there exists another hyperexponential-hypergeometric function g such that* (5.2) *holds.*

After that, applying the integral sign to both sides of (5.2) yields

$$L(n, S_n)(H(n)) = g(+\infty, n) - g(0, n).$$

This further implies that $L(n, S_n)$ is indeed the recurrence relation satisfied by $H(n)$ under certain nice boundary condition, say $g(+\infty, n) = g(0, n)$. For example, consider the integral

$$A(n) = \int_0^{+\infty} x^{n-1} \exp(-x)\, dx.$$

Almkvist and Zeilberger's algorithm returns a pair (L, g) with

$$L = S_n - n \quad \text{and} \quad g = -x^n \exp(-x).$$

Note that $g(+\infty, n) = g(0, n) = 0$, which implies that $L(A(n)) = A(n + 1) - nA(n) = 0$. So we recognize that $A(n) = (n - 1)!$ since the value $A(1)$ is equal to 1.

The differential analogue of Almkvist and Zeilberger's algorithm tries to construct a linear differential equation $L(x, D_x) \in k(x)\langle D_x\rangle$ such that

$$L(x, D_x)(h) = \Delta_n(g), \tag{5.3}$$

for some hyperexponential-hypergeometric function g over $k(x, n)$.

Definition 5.3.2. *Let h be a hyperexponential-hypergeometric function over $k(x, n)$. A nonzero linear differential operator $L(x, D_x) \in k(x)\langle D_x\rangle$ is called a* telescoper *for h* with respect to n *if there exists another hyperexponential-hypergeometric function g such that* (5.3) *holds.*

Both the g's in (5.2) and (5.3) are either zero or similar to h by Lemma 5.2.5. This analogue can be used to construct a linear differential equation for the sum

$$H(x) := \sum_{n=0}^{+\infty} h(x, n).$$

For more interesting examples, see the appendix of [15] or Koepf's book [60, Chapters 10–13].

As in the discrete case, not all hyperexponential-hypergeometric functions have telescopers with respect to x or n. That is, Almkvist and Zeilberger's algorithm does not terminate on all hyperexponential-hypergeometric inputs.

Example 5.3.3. *The non-existence phenomenon is well illustrated by the rational function*

$$f = \frac{1}{x+n}.$$

Let us show that f has no telescoper with respect to either x or n.

Differential case: *Suppose that f has a telescoper with respect to x, i.e., there exists a nonzero operator L in $k(n)\langle S_n \rangle$ such that $L(f) = D_x(g)$ for some $g \in k(x, n)$. Write $L = \sum_{i=0}^{\rho} \ell_i(n) S_n^i$ with $\rho \in \mathbb{N}$ and $\ell_i \in k(n)$. Then we have*

$$L(f) = \sum_{i=0}^{\rho} \ell_i(n) S_n^i \left(\frac{1}{x+n} \right) = \sum_{i=0}^{\rho} \left(\frac{\ell_i(n)}{x+n+i} \right) = \frac{A}{D},$$

where D divides the product $(x+n)(x+n+1)\cdots(x+n+\rho)$, which is squarefree with respect to x, and $A \in k(n)[x]$ with $\deg_x(A) < \deg_x(D)$ and $\gcd(A, D) = 1$ as polynomials in x. Since $A/D = D_x(g)$, $A = 0$ by Lemma 4.3.1 (i). So f is a rational solution of the linear difference equation $L(z) = 0$. By Lemma 5.2.7, the denominator of f should be split in $k[x, n]$ since the leading coefficient of L is free of x, but $x + n$ is not split, a contradiction.

Difference case: *Suppose that f has a telescoper with respect to n, i.e., there exists a nonzero operator L in $k(x)\langle D_x \rangle$ such that $L(f) = \Delta_n(g)$ for some $g \in k(x, n)$. Write $L = \sum_{i=0}^{\rho} \ell_i(x) D_x^i$ with $\rho \in \mathbb{N}$ and $\ell_i \in k(x)$. Then we have*

$$L(f) = \sum_{i=0}^{\rho} \ell_i(x) D_x^i \left(\frac{1}{x+n} \right) = \sum_{i=0}^{\rho} \left(\frac{(-1)^i \ell_i(x)}{(x+n)^{i+1}} \right) = \frac{A}{D},$$

where D is a factor of $(x+n)^{\rho+1}$, which is shift-free with respect to n, and $A \in k(x)[n]$ with $\deg_n(A) < \deg_n(D)$ *and* $\gcd(A, D) = 1$ *as polynomials in n. Since $A/D = \Delta_n(g)$, $A = 0$ by Lemma 4.3.1 (ii). So f is a rational solution of the linear differential equation $L(z) = 0$. By Lemma 5.2.7, the denominator of f should be split in $k[x, n]$ since the leading coefficient of L is free of n, but $x + n$ is not split, a contradiction.*

In this chapter, we solve the following problem, which is equivalent to the termination problem of Almkvist and Zeilberger's algorithm.

Problem 5.3.4. *Given a hyperexponential-hypergeometric function $h(x, n)$ over $k(x, n)$, decide whether h has a telescoper with respect to x or n.*

5.4 Standard representations

We derive a standard representation of hyperexponential-hypergeometric functions, based on [42, Proposition 5], i.e., the bivariate case of Theorem 4.4.6.

Lemma 5.4.1. *Let a and b be two nonzero rational functions in $k(x, n)$ such that the integrability condition* (5.1) *holds. Then there exist $f \in k(x, n)$ whose split part is trivial, $\alpha \in k(n)$, and $\beta, \gamma \in k(x)$ such that*

$$a = \frac{\delta(f)}{f} + \frac{\delta(\beta(x))}{\beta(x)} n + \gamma(x) \quad \textit{and} \quad b = \frac{\sigma(f)}{f} \alpha(n) \beta(x). \tag{5.4}$$

Proof. By Proposition 5 in [42], there exist $f \in k(x, n)$, $\alpha \in k(n)$, and $\beta, \gamma \in k(x)$ such that (5.4) holds. Write f as $f_1 f_2 f_3$ where $f_1 \in k(x)$, $f_2 \in k(n)$, and $f_3 \in k(x, n)$ whose split part is trivial. Replacing f by f_3, γ by $\gamma + \delta(f_1)/f_1$, and α by $\alpha\sigma(f_2)/f_2$ in (5.4) yields the lemma. $\square$

By Lemma 5.4.1, any hyperexponential-hypergeometric function can be written in a unique multiplicative form.

Proposition 5.4.2. *Let h be a hyperexponential-hypergeometric function. Then there exist $f \in k(x, n)$ with trivial split part, $\alpha \in k(n)$ with monic numerator and denominator, and $\beta, \gamma \in k(x)$ such that*

$$h \in f(x, n) \beta(x)^n \mathcal{H}(\gamma(x), \alpha(n)). \tag{5.5}$$

Moreover, if

$$h \in f'(x,n)\beta'(x)^n \mathcal{H}(\gamma'(x), \alpha'(n))$$

where $f' \in k(x,n)$ *with trivial split part,* $\alpha' \in k(n)$ *with monic numerator and denominator, and* $\beta', \gamma' \in k(x)$, *then* $f/f' \in k$, $\alpha = \alpha'$, $\beta = \beta'$, *and* $\gamma = \gamma'$.

Proof. Let a and b be the x-certificate and n-certificate of h, respectively. By Lemma 5.4.1, there exist $f \in k(x,n)$ with trivial split part, $\alpha \in k(n)$, and $\beta, \gamma \in k(x)$ such that (5.4). By a straightforward calculation, h and a nonzero element of $f\beta^n\mathcal{H}(\gamma, \alpha)$ have the same certificates. Hence, (5.5) holds by Proposition 5.2.3. Assume that h belongs to $f'(x,n)\beta'(x)^n\mathcal{H}(\gamma'(x), \alpha'(n))$, as given in the statement of the proposition. Then the x-certificate and n-certificate of every nonzero element in $f'(x,n)\beta'(x)^n\mathcal{H}(\gamma'(x), \alpha'(n))$ are a and b, respectively. It follows that

$$\frac{\delta(f)}{f} + \frac{\delta(\beta(x))}{\beta(x)} n + \gamma(x) = \frac{\delta(f')}{f'} + \frac{\delta(\beta'(x))}{\beta'(x)} n + \gamma'(x), \tag{5.6}$$

and

$$\frac{\sigma(f)}{f}\alpha(n)\beta(x) = \frac{\sigma(f')}{f'}\alpha'(n)\beta'(x). \tag{5.7}$$

Consequently,

$$\frac{\delta(f)}{f} = \frac{\delta(f')}{f'}, \quad \frac{\delta(\beta(x))}{\beta(x)} = \frac{\delta(\beta'(x))}{\beta'(x)}, \quad \text{and } \gamma(x) = \gamma'(x)$$

by (5.6), the uniqueness of the partial fraction decomposition of a, and the assumption that both f and f' have trivial split parts. Hence, $f = cf'$ and $\beta = c'\beta'$ for some $c, c' \in k(n)$. Again, by the assumption on f and f', c belongs to k. Since β and β' are free of n, so is c'. It follows from (5.7) that $\alpha(n) = c'\alpha(n)$. Thus, c' is equal to 1 since the numerators and denominators of $\alpha(n)$ and $\alpha'(n)$ are assumed to be monic. ☐

Definition 5.4.3. *Let* $h(x,n) \in \mathcal{H}(a,b)$ *be a hyperexponential-hypergeometric function. We say that a quadruple* $(f(x,n), \alpha(n), \beta(x), \gamma(x)) \in k(x,n)^4$ *is the* standard representation *of* h *if* $f \in k(x,n)$ *with trivial split part,* $\alpha \in k(n)$ *with monic numerator and denominator,* $\beta, \gamma \in k(x)$, *and* (5.5) *holds.*

By the definition in [92] or Definition 4.5.3 in Chapter 4, a bivariate hyperexponential-hypergeometric function is *proper* if its standard representation is

$$(p(x,n),\ \alpha(n),\ \beta(x),\ \gamma(x)), \tag{5.8}$$

where $p \in k[x,n]$, $\alpha \in k(n)$, and $\beta, \gamma \in k(x)$.

Proposition 5.4.4. *Let $h \in f(x,n)\beta(x)^n\mathcal{H}(\gamma(x), \alpha(n))$ with $f \in k(x,n)$, $\alpha \in k(n)$, and $\beta, \gamma \in k(x)$. If the denominator of f is split, then h is proper.*

Proof. Write $f = p/q$ with $p, q \in k[x,n]$ and $\gcd(p,q) = 1$. Since q is split, $q = A(x)B(n)$ for some $A \in k[x]$ and $B \in k[n]$. By Corollary 5.2.4 (i), we have

$$h \in p(x,n)\beta(x)^n\mathcal{H}\left(\gamma(x) - \frac{\delta(A(x))}{A(x)}, \alpha(n)\frac{B(n)}{\sigma(B(n))}\right).$$

Therefore, h is proper by definition. □

5.5 Two additive decompositions

For a bivariate hyperexponential-hypergeometric function $h(x,n)$ in the continuous variable x and the discrete one n, we can perform two kinds of *complete additive decompositions*: one is with respect to the continuous variable, obtaining $h = D_x(h_1) + h_2$, and the other is with respect to the discrete variable, obtaining $h = \Delta_n(h_1) + h_2$. The h_1's and h_2's above are hyperexponential-hypergeometric functions with h_2 minimal in some sense. Algorithms for complete additive decompositions generalize the capability of Gosper's algorithm [51] and its differential analogue [15]. That is, it returns $h_2 = 0$ if $h = D_x(g)$ or $h = \Delta_n(g)$ for some g hyperexponential-hypergeometric. In this section, we adapt the two additive decompositions to hyperexponential-hypergeometric functions described by their standard representations.

Notation. In what follows, let F be either $k(n)$ if the decomposition is with respect to x or $k(x)$ if it is with respect to n.

5.5.1 Additive decomposition with respect to x

In Chapter 3, Hermite reduction or Ostrogradsky and Horowitz's method decomposes a rational function $f \in F(x)$ into $f = D_x(g) + r$, where $r = a/b$ is minimal in the sense that $\deg_x(a) < \deg_x(b)$ and b is squarefree, that is, the factors of b have minimal multiplicity. Hermite reduction was later extended by Davenport [39] and by Geddes, Le and Li [47] with two different motivations: one is to solve the Risch differential equation

$$D_x(y) + fy = g, \quad \text{where } f \text{ and } g \text{ are in } F(x),$$

and the other is to solve the decomposition problem of hyperexponential functions. We will use the latter to compute additive decompositions of hyperexponential-hypergeometric functions with respect to x. To this end, we first recall some terminology from [47].

Definition 5.5.1 (GLL triple)**.** *Let $u, v \in F[x]$ and $w \in F(x)$. We call (u, v, w) a* Geddes–Le–Li triple*, in short* GLL triple*, if the following conditions are satisfied:*

(*i*) $\gcd(u, v) = 1$*;*

(*ii*) *u is squarefree with respect to x;*

(*iii*) *w is differential-reduced with respect to x;*

(*iv*) $\gcd(u, \mathrm{den}(w)) = 1$.

Definition 5.5.2 (Hyperexponential-integrable)**.** *A hyperexponential function h over $F(x)$ is said to be* hyperexponential-integrable *if there exists a hyperexponential function g such that $h = D_x(g)$.*

Note that the function g in the above definition is similar to h if it exists.

Definition 5.5.3 (Additive decomposition with respect to x)**.** *Let h be in $\mathcal{H}(a, b)$ with $a, b \in F(x)$. An additive decomposition of h with respect to x is of the form*

$$h = D_x(h_1) + h_2,$$

where h_1 is hyperexponential-hypergeometric over $F(x)$, and either h_2 is zero or

$$h_2 \in \frac{v}{u} \cdot \mathcal{H}(a_2, b_2), \quad \text{where } b_2 \in F(x) \text{ and } (u, v, a_2) \text{ is a GLL triple.}$$

Such a decomposition is said to be complete *if $h_2 = 0$ whenever h is hyperexponential-integrable.*

Remark 5.5.4. *In the sequel we never use complete additive decompositions. In fact, the h_1 and h_2 above are similar to h by the same argument as in [75, Proposition 5.6.2].*

Geddes–Le–Li's algorithm

We recall the algorithm from [47] to perform additive decompositions with respect to x. View a hyperexponential-hypergeometric function $h \in \mathcal{H}(a,b)$ as a hyperexponential function with x-certificate $a \in F(x)$. Let $(K,S) \in F(x) \times F(x)$ be a strict DRNF of a, that is, K is differential-reduced with respect to x and

$$a = K + \frac{\delta(S)}{S}, \quad \text{where } \gcd(\mathrm{den}(K), \mathrm{den}(S)) = 1.$$

Immediately, the additive form above for a leads to a multiplicative form for h, that is,

$$h = S \cdot \tilde{h}, \quad \text{where } \tilde{h} \in \mathcal{H}\left(K, b\frac{S}{\sigma(S)}\right).$$

Now comes the Hermite-like reduction: Theorem 2 in [47] shows that there exists $S_1 \in F(x)$ such that

$$S = \delta(S_1) + KS_1 + \frac{v}{u \cdot \mathrm{den}(K)^i}, \tag{5.9}$$

where $i \in \{0,1\}$ and (u,v,K) is a GLL triple. Since $D_x(\tilde{h}) = K\tilde{h}$, the decomposition for S in (5.9) further implies that

$$h = D_x(S_1\tilde{h}) + \frac{v}{u \cdot \mathrm{den}(K)^i}\,\tilde{h},$$

In order to get a complete additive decomposition for h, we still need to decide whether

$$h_2 = \frac{v}{u \cdot \mathrm{den}(K)^i}\,\tilde{h} \tag{5.10}$$

is hyperexponential-integrable or not. A necessary condition is as follows.

Lemma 5.5.5 (Geddes–Le–Li, 2004)**.** *Let h be a hyperexponential function over $F(x)$. If h is hyperexponential-integrable, and*

$$\frac{\delta(h)}{h} = w + \frac{\delta(v/u)}{v/u},$$

where $v, u \in F[x]$, $w \in F(x)$, and (u,v,w) is a GLL triple, then u is in F.

Proof. For the proof, see [47, Theorem 4]. ☐

Additive decompositions with respect to x from standard representations

Let $(f(x,n), \alpha(n), \beta(x), \gamma(x))$ be the standard representation of $h \in \mathcal{H}(a,b)$, that is,

$$h \in f(x,n)\beta(x)^n \mathcal{H}(\gamma(x), \alpha(n)),$$

where $f \in k(x,n)$ with trivial split part, $\alpha \in k(n)$ with monic numerator and denominator, and $\beta, \gamma \in k(x)$. We show how to obtain an additive decomposition of h with respect to x from its standard representation.

First, we compute a strict DRNF for a from the standard representation of h. The following lemma helps us decide whether certain rational functions are differential-reduced.

Lemma 5.5.6. *Let*

$$g = \frac{p}{q} + n\frac{s}{t} + \frac{u}{v},$$

where $p, q, s, t, u, v \in k[x]$ *with* $\gcd(p,q) = \gcd(s,t) = \gcd(u,v) = 1$*. Assume that* $v \mid t$*. Then*

(*i*) *the denominator of* g *is the least common multiple of* q *and* t*;*

(*ii*) *if* p/q *is differential-reduced, so is* g*;*

(*iii*) *if* $p/q + ns/t$ *is differential-reduced, so is* g*.*

Proof. Let $M = \operatorname{lcm}(q,t)$, $w_1 = M/q$ and $w_2 = M/t$. We first show the following claim.

Claim. If w is a nontrivial irreducible factor of

$$\gcd(M, sw_2 n + r), \quad \text{where } r \text{ is an element in } k[x],$$

then w divides w_2, r and q, but does not divide w_1.

Proof of the Claim. Since n is an indeterminate over k, w is a factor of sw_2. Suppose that w is not a factor of w_2. Then it is a factor of s. Since w is a factor of M, it is a factor of $w_2 t$, and, thus, a factor of t, a contradiction to the assumption that $\gcd(s,t) = 1$. We have concluded that w is a factor of w_2. Consequently, it is a factor of r because n is an indeterminate, but it is not a factor of w_1 because $\gcd(w_1, w_2) = 1$. Since w divides qw_1, it divides q. The claim is proved.

To prove the first assertion, we let $w_3 = M/v$, which is a multiple of w_2 since t is a multiple of v. Put $W = pw_1 + sw_2n + uw_3$. We have $g = W/M$. It remains to show that $\gcd(M, W) = 1$. Set $r = pw_1 + uw_3$ and assume that w is an irreducible factor of $\gcd(M, W)$. By the claim, w divides r and w_2, hence, it divides pw_1. The claim then implies that w divides p. But w also divides q by the same claim. Thus, $w \in k$ since $\gcd(p, q) = 1$.

To prove the second assertion, we only need to show that $\gcd(q, p - \ell\delta(q)) = 1$ implies that $\gcd(M, W - \ell\delta(M)) = 1$. Setting $r = -pw_1 - uw_2 - \ell\delta(M)$, we have

$$\gcd(M, W - \ell\delta(M)) = \gcd(M, sw_2n + r).$$

Assume that w is an irreducible factor of $\gcd(M, W - \ell\delta(M))$. By the claim, w divides r and w_2, therefore, it divides $pw_1 - \ell\delta(M)$, which is equal to $pw_1 - \ell w_1\delta(q) - \ell\delta(w_1)q$. Since w divides q, it divides $(p - \ell\delta(q))w_1$. Again, by the claim, w divides $p - \ell\delta(q)$. On the other hand, w divides q because $w \mid M$ and $w \nmid w_1$. Hence, w is a factor of $\gcd(q, p - \ell\delta(q))$, which is equal to 1, because p/q is differential-reduced. Consequently, w is in k.

To prove the third assertion, it suffices to show that $\gcd(M, pw_1 + sw_2n - \ell\delta(M)) = 1$ implies $\gcd(M, W - \ell\delta(M)) = 1$. Let r and w be defined as above. By the claim, w divides w_3 since w_2 divides w_3. It follows from the definition of r that w divides $pw_1 + sw_2n - \ell\delta(M)$, and, hence, w is in k. $\square$

For later use, we recall the definition of *pumps*, given in [13, page 533].

Definition 5.5.7 (Pump)**.** *Let $p, q \in F[x]$ be such that $p \mid q$. A factor $\widetilde{p}$ of q is said to be the* pump *of p in q if* $\gcd(q/\tilde{p}, \widetilde{p}) = 1$, *$p \mid \widetilde{p}$, and any irreducible factor of $\widetilde{p}$ divides p.*

Example 5.5.8. *Let p_1, p_2, p_3 be in $F[x]$ with positive degrees. Assume that they are pairwise coprime and $p = p_1p_2^2p_3^3$. Then the pump of $p_1p_3^2$ in p is equal to $p_1p_3^3$.*

In other words, a pump of a factor p of q has the same squarefree part as p's, but each of its irreducible factors has the maximal multiplicity in q. For a rational function $f \in k(x, n)$, denote by f^* the squarefree part of $\mathrm{num}(f) \cdot \mathrm{den}(f)$. Now, we show how to compute a strict DRNF of a from that of γ.

Lemma 5.5.9. *Let $(f(x,n), \alpha(n), \beta(x), \gamma(x))$ be a standard representation of a nonzero element in $\mathcal{H}(a,b)$. Let $(\widetilde{K}, \widetilde{S})$ be a strict DRNF of γ. Let $u = \gcd(\operatorname{den}(\widetilde{S}), \beta^*)$ and T the pump of u in $\operatorname{den}(\widetilde{S})$. If*

$$K = \widetilde{K} + n\frac{\delta(\beta)}{\beta} - \frac{\delta(T)}{T} \quad \text{and} \quad S = f\widetilde{S}T, \tag{5.11}$$

then (K, S) is a strict DRNF of a with respect to x.

Proof. Let $(\tilde{K}, \tilde{S})$ be a strict DRNF of $\gamma(x)$. Then by definition, we have

$$a = \frac{\delta(f)}{f} + n\frac{\delta(\beta)}{\beta} + \gamma = \frac{\delta(f)}{f} + n\frac{\delta(\beta)}{\beta} + \frac{\delta(\tilde{S})}{\tilde{S}} + \tilde{K}.$$

By Lemma 2.3.3, the denominators of $\delta(\beta)/\beta$ and $\delta(T)/T$ are β^* and T^*, respectively. By the definition of pump and $u \mid \beta^*$, u is the squarefree part of T, which implies $T^* \mid \beta^*$. Hence, K is differential-reduced by Lemma 5.5.6 (*ii*).

It remains to show $\gcd(\operatorname{den}(K), \operatorname{den}(S)) = 1$. By the definition of pumps, $\operatorname{den}(\widetilde{S}) = Tw$, where $w \in k[x]$ and $\gcd(T, w) = 1$. On the other hand,

$$\operatorname{den}(S) = \operatorname{den}(f)\operatorname{den}(\widetilde{S}T) = \operatorname{den}(f)w,$$

since $\operatorname{num}(f)\operatorname{den}(f)$ has only trivial factors in $k[x]$. It follows that

$$\gcd(\operatorname{den}(K), \operatorname{den}(S)) = \gcd(\operatorname{den}(K), \operatorname{den}(f)w),$$

which is equal to $\gcd(\operatorname{den}(K), w)$ since $\operatorname{den}(K) \in k[x]$. So it suffices to show that $\operatorname{den}(K)$ and w are coprime. By Lemma 5.5.6 (*i*), $\operatorname{den}(K) = \operatorname{lcm}(\operatorname{den}(\widetilde{K}), \beta^*)$. So it suffices to show that

$$\gcd(\operatorname{den}(\widetilde{K}), w) = 1 \quad \text{and} \quad \gcd(\beta^*, w) = 1.$$

The first equality is immediate from the assumptions that $(\widetilde{K}, \widetilde{S})$ is a strict DRNF and that w is a factor of $\operatorname{den}(\widetilde{S})$. To prove the second, we assume that p is an irreducible factor of $\gcd(\beta^*, w)$. Then p is a factor of u by the definitions of u and w. Hence, it is a factor of T. Consequently, p is a common factor of T and w. We conclude that p is in k. $\square$

We apply Geddes–Le–Li's algorithm to a hyperexponential-hypergeometric function described by its standard representation and obtain a special form of its additive decomposition with respect to x.

Theorem 5.5.10. *Let $(f(x,n),\alpha(n),\beta(x),\gamma(x))$ be the standard representation of a nonzero element h in $\mathcal{H}(a,b)$. Then one can find a hyperexponential-hypergeometric function h_1 and $\tilde{\gamma}(x)\in k(x)$ such that*

$$h - D_x(h_1) \in \frac{v}{u}\cdot\mathcal{H}\left(\tilde{\gamma}+n\frac{\delta(\beta)}{\beta},\,\beta\alpha\right) \tag{5.12}$$

where u,v are in $k(n)[x]$ and $(u,v,\tilde{\gamma}+n\delta(\beta)/\beta)$ is a GLL triple.

Proof. By Lemma 5.5.9, the x-certificate a of h has a strict DRNF (K,S) of the form

$$\left(\tilde{K}+n\cdot\frac{\delta(\beta)}{\beta}-\frac{\delta(T)}{T},\; f\cdot\tilde{S}\cdot T\right),$$

where $(\tilde{K},\tilde{S})$ is a strict DRNF of $\gamma(x)$ and $T\in k[x]$. A direct calculation verifies that

$$h = S\tilde{h} \quad \text{for some } \tilde{h}\in\mathcal{H}\,(K,\beta\alpha).$$

By (5.9), Hermite-like reduction rewrites S into

$$S = \delta(S_1) + S_1K + \frac{v}{u\mathrm{den}(K)^i},$$

where $i\in\{0,1\}$, $S_1\in k(x,n)$ and (u,v,K) is a GLL triple. It follows that

$$h = D_x(S_1\tilde{h}) + \frac{v}{u\mathrm{den}(K)^i}\tilde{h}.$$

Since $\mathrm{den}(K)$ is in $k[x]$ and by Corollary 5.2.4

$$\begin{aligned}\frac{v}{u\mathrm{den}(K)^i}\mathcal{H}(K,\beta\alpha) &= \frac{v}{u}\mathcal{H}\left(K-\frac{\delta(\mathrm{den}(K)^i)}{\mathrm{den}(K)^i},\beta\alpha\frac{\mathrm{den}(K)^i}{\sigma(\mathrm{den}(K)^i)}\right)\\ &= \frac{v}{u}\mathcal{H}\left(K-\frac{\delta(\mathrm{den}(K)^i)}{\mathrm{den}(K)^i},\beta\alpha\right).\end{aligned}$$

Setting $h_1 = S_1\tilde{h}$ and $\tilde{\gamma} = \tilde{K} - \frac{\delta(T)}{T} - \frac{\delta(\mathrm{den}(K)^i}{\mathrm{den}(K)^i}$ yields (5.12). It remains to show that the triple $\left(u,v,\tilde{\gamma}+n\frac{\delta(\beta)}{\beta}\right)$ is a GLL triple. Note that

$$g := \tilde{\gamma}+n\frac{\delta(\beta)}{\beta} = K - \frac{\delta(\mathrm{den}(K)^i)}{\mathrm{den}(K)^i}.$$

Since K is differential-reduced, so is g by definition. Moreover, u and $\mathrm{den}(g)$ are coprime with respect to x, as u and $\mathrm{den}(K)$ are coprime. It follows that (u,v,g) is a GLL triple. □

Remark 5.5.11. *An additive decomposition with respect to x can be computed by the algorithm* `ReducedHyperexp` *in [47].*

5.5.2 Additive decomposition with respect to n

In the discrete case, the algorithms in [2, 4, 76, 72] decompose a rational function $f \in F(n)$ into $f = \Delta_n(g) + r$, where $r = a/b$ is minimal in the sense that $\deg_n(a) < \deg_n(b)$ and b is shift-free, that is b has minimal dispersion. Those decomposition algorithms are extended to the case of hypergeometric terms by Abramov and Petkovšek [11, 13]. This extension also generalizes the capability of the well-known Gosper algorithm [51], which decides whether the indefinite sum of a hypergeometric term is hypergeometric. More precisely, for a given hypergeometric term $H(n)$ over $F(n)$, Abramov and Petkovšek's algorithm computes two hypergeometric terms $H_1(n)$ and $H_2(n)$ such that $H = \Delta_n(H_1) + H_2$, where H_2 is minimal in some sense. In particular, H_2 is identically zero if $H(n)$ has a hypergeometric indefinite sum. Here, we will use Abramov and Petkovšek's algorithm to compute additive decompositions of hyperexponential-hypergeometric functions with respect to the discrete variable n. To this end, we recall some terminology from [13].

Definition 5.5.12 (AP triple)**.** *Let $u, v \in F[n]$ and $w \in F(n)$. We call (u, v, w) an* Abramov–Petkovšek triple*, in short* AP triple*, if the following conditions are satisfied:*

(i) $\gcd(u, v) = 1$*;*

(ii) u is shift-free with respect to n;

(iii) w is shift-reduced with respect to n;

(iv) $\gcd(u, \sigma^{-\ell}(\mathrm{num}(w))) = \gcd(u, \sigma^{\ell}(\mathrm{den}(w))) = 1$ *for all* $\ell \in \mathbb{N}$.

Definition 5.5.13 (Hypergeometric-summable)**.** *A hypergeometric term h over $F(n)$ is said to be* hypergeometric-summable *if there exists a hypergeometric function g such that $h = \Delta_n(g)$.*

Note that the g in the above definition is similar to h if it exists.

Definition 5.5.14 (Additive decomposition with respect to n)**.** *Let h be in $\mathcal{H}(a, b)$ with $a, b \in F(n)$. An additive decomposition of h with respect to n is of the form*

$$h = \Delta_n(h_1) + h_2,$$

where h_1 is hyperexponential-hypergeometric over $F(n)$, and either h_2 is zero or

$$h_2 \in \frac{v}{u} \cdot \mathcal{H}(a_2, b_2), \quad \text{where } a_2 \in F(n) \text{ and } (u, v, b_2) \text{ is an AP triple.}$$

Such a decomposition is said to be complete *if $h_2 = 0$ whenever h is hypergeometric-summable.*

Remark 5.5.15. *In the sequel we never use complete additive decompositions. In fact, the h_1 and h_2 above are similar to h by the same argument in [75, Proposition 5.6.2].*

Abramov and Petkovšek's algorithm

We recall the algorithm from [11, 13] to perform additive decomposition with respect to n. View a hyperexponential-hypergeometric function $h \in \mathcal{H}(a, b)$ as a hypergeometric term with n-certificate $b \in F(n)$. Let $(K, S) \in F(n) \times F(n)$ be a strict SRNF of b, that is, K is shift-reduced with respect to n and

$$b = K \cdot \frac{\sigma(S)}{S},$$

where $\gcd(\mathrm{num}(K), \sigma(\mathrm{den}(S)) \cdot \mathrm{num}(S)) = \gcd(\mathrm{den}(K), \sigma(\mathrm{num}(S)) \cdot \mathrm{den}(S)) = 1$. Immediately, the multiplicative form above for b leads to a multiplicative form for h, that is,

$$h = S \cdot \tilde{h}, \quad \text{where } \tilde{h} \in \mathcal{H}\left(a - \frac{\delta(S)}{S}, K\right).$$

Now comes Abramov and Petkovšek's reduction: Lemma 9 in [13] implies that there exists $S_1 \in F(n)$ such that

$$S = \sigma(S_1)K - S_1 + \frac{v}{u \cdot (\sigma^{-1}(k_1))^i \cdot k_2^j}, \tag{5.13}$$

where $i, j \in \{0, 1\}$, k_1 and k_2 are the numerator and denominator of K, respectively, and (u, v, K) is an AP triple. Since $S_n(\tilde{h}) = K\tilde{h}$, the decomposition for S in (5.13) further implies that

$$h = \Delta_n(S_1\tilde{h}) + \frac{v}{u \cdot (\sigma^{-1}(k_1))^i \cdot k_2^j}\,\tilde{h}.$$

In order to get a complete additive decomposition of h, we still need to decide whether

$$h_2 = \frac{v}{u \cdot (\sigma^{-1}(k_1))^i \cdot k_2^j}\,\tilde{h} \tag{5.14}$$

is hypergeometric-summable or not. A necessary condition is as follows.

Lemma 5.5.16 (Abramov–Petkovšek, 2002). *Let h be a hypergeometric function over $F(n)$. If h is hypergeometric-summable, and*

$$\frac{\sigma(h)}{h} = \frac{\sigma(v/u)}{v/u} w,$$

where $v, u \in F[n]$, $w \in F(n)$, and (u, v, w) is an AP triple, then u is in F.

Proof. For the proof, see [13, Theorem 11]. □

Additive decompositions with respect to n from standard representations

Let $(f(x,n), \alpha(n), \beta(x), \gamma(x))$ be the standard representation of $h \in \mathcal{H}(a,b)$. We show how to obtain an additive decomposition of h with respect to n from its standard representation.

The next lemma relates a strict SRNF of the n-certificate b of h to its standard representation.

Lemma 5.5.17. *Let $(f(x,n), \alpha(n), \beta(x), \gamma(x))$ be the standard representation of a nonzero element in $h \in \mathcal{H}(a,b)$. If $(\tilde{K}, \tilde{S})$ is a strict SRNF of $\alpha(n)$ with respect to n, then $(\beta\tilde{K}, f\tilde{S})$ is a strict SRNF of the n-certificate a of h.*

Proof. Let $K = \beta\tilde{K}$ and $S = f\tilde{S}$. Since $(\tilde{K}, \tilde{S})$ is a strict SRNF of $\alpha(n)$, $\alpha = \tilde{K}\sigma(\tilde{S})/\tilde{S}$. It follows that

$$b = \frac{\sigma(h)}{h} = \beta\alpha\frac{\sigma(f)}{f} = \beta\left(\tilde{K}\frac{\sigma(\tilde{S})}{\tilde{S}}\right)\frac{\sigma(f)}{f} = \beta\tilde{K} \cdot \frac{\sigma(f\tilde{S})}{f\tilde{S}}.$$

The product $\beta\tilde{K}$ is shift-reduced with respect to n, because $\tilde{K}$ is shift-reduced and β is in $k(x)$. Write $K = k_1/k_2$ and $S = s_1/s_2$ with $\gcd(k_1, k_2) = \gcd(s_1, s_2) = 1$ in $k[x, n]$. It remains to verify the GCD condition

$$\gcd(k_1, \sigma(s_2)s_1) = \gcd(k_2, \sigma(s_1)s_2) = 1 \tag{5.15}$$

for (K, S). Since $\beta \in k(x)$ and $\tilde{K} \in k(n)$, we have

$$k_1 = \mathrm{num}(\beta) \cdot \mathrm{num}(\tilde{K}) \quad \text{and} \quad k_2 = \mathrm{den}(\beta) \cdot \mathrm{den}(\tilde{K}).$$

Similarly, since $\tilde{S} \in k(n)$ and f has a trivial split part, we have

$$s_1 = \mathrm{num}(f) \cdot \mathrm{num}(\tilde{S}), \quad \text{and} \quad s_2 = \mathrm{den}(f) \cdot \mathrm{den}(\tilde{S}).$$

Hence, the condition (5.15) holds, because $(\tilde{K}, \tilde{S})$ is a strict SRNF, and every nonzero polynomial in $k[n]$ is coprime with both $\mathrm{num}(f)$ and $\mathrm{den}(f)$. □

We apply Abramov–Petkovšek's algorithm to a hyperexponential-hypergeometric function described by standard representation and obtain a special form of its additive decomposition with respect to n.

Theorem 5.5.18. *Let $(f(x,n), \alpha(n), \beta(x), \gamma(x))$ be the standard representation of a nonzero element in $h \in \mathcal{H}(a,b)$. Then one can find $\tilde{\gamma}(x) \in k(x)$, $\tilde{\alpha} \in k(n)$, and a hyperexponential-hypergeometric function h_1 such that*

$$h - \Delta_n(h_1) \in \frac{v}{u} \cdot \mathcal{H}\left(\tilde{\gamma} + n\frac{\delta(\beta)}{\beta}, \beta\tilde{\alpha}\right)$$

where $u, v \in k(x)[n]$ and $(u, v, \beta\tilde{\alpha})$ is an AP triple.

Proof. By Lemma 5.5.17, the n-certificate b of h has a strict SRNF (K, S) of the form

$$\left(\beta\tilde{K}, f\tilde{S}\right),$$

where $(\tilde{K}, \tilde{S})$ is a strict SRNF of $\alpha(n)$. A direct calculation verifies that

$$h = S\tilde{h} \quad \text{for some } \tilde{h} \in \mathcal{H}\left(\gamma + n\frac{\delta(\beta)}{\beta}, K\right).$$

Write $K = k_1/k_2$ and $S = s_1/s_2$ with $\gcd(k_1, k_2) = \gcd(s_1, s_2) = 1$ in $k[x, n]$. Now, Abramov–Petkovšek's reduction [13] decomposes S into

$$S = \delta(S_1)K - S_1 + \frac{v}{u(\sigma^{-1}(k_1))^i k_2^j},$$

where $i, j \in \{0, 1\}$, $S_1 \in k(x, n)$ and (u, v, K) is an AP triple. It follows that

$$h = \Delta_n(S_1\tilde{h}) + \frac{v}{u(\sigma^{-1}(k_1))^i k_2^j}\tilde{h}.$$

Set $T = (\sigma^{-1}(k_1))^i k_2^j \in k[x, n]$. Since K is split in $k[x, n]$, T is split in $k[x, n]$. By Corollary 5.2.4, we have

$$\frac{v}{u \cdot T}\tilde{h} \in \frac{v}{u \cdot T}\mathcal{H}\left(\gamma + n\frac{\delta(\beta)}{\beta}, K\right) = \frac{v}{u}\mathcal{H}\left(\gamma + n\frac{\delta(\beta)}{\beta} - \frac{\delta(T)}{T}, K\frac{T}{\sigma(T)}\right).$$

Set $h_1 = S_1\tilde{h}$, $\tilde{\gamma} = \gamma - \frac{\delta(T)}{T}$, and $\tilde{\alpha} = \tilde{K}\frac{T}{\sigma(T)}$. Since T is split, $\gamma \in k(x)$, and $\tilde{K} \in k(n)$, we have that $\tilde{\gamma} \in k(x)$ and $\tilde{\alpha} \in k(n)$ hold. It remains to show that $(u, v, \beta\tilde{\alpha})$ is an AP triple.

We only need to verify the last two conditions in Definition 5.5.12. Since $i, j \in \{0, 1\}$, $\beta\tilde{\alpha}$ is in the set

$$\left\{\frac{k_1}{k_2}, \frac{\sigma^{-1}(k_1)}{k_2}, \frac{k_1}{\sigma(k_2)}, \frac{\sigma^{-1}(k_1)}{\sigma(k_2)}\right\}.$$

Then $\beta\tilde{\alpha}$ is shift-reduced, because $K = k_1/k_2$ is. Note that the numerator and denominator of $\beta\tilde{\alpha}$ are just some shifts of that of K, respectively. Then the last gcd condition follows from the assumptions on u and k_1, k_2. $\square$

Remark 5.5.19. *An additive decomposition with respect to n can be computed by the algorithm* `dterm` *in [13].*

5.5.3 Applying operators to additive decompositions

In this section, we apply an operator in $k(x)\langle D_x\rangle$, resp. in $k(n)\langle S_n\rangle$, to an additive decomposition with respect to n, resp. x, and prove that the application results in another additive decomposition. To this end, we need a simple lemma.

Lemma 5.5.20. *Let u and v be in $k(x)[n]$ and let w be any factor of u^m with $m \in \mathbb{N}$. Then*

(i) if u is shift-free with respect to n, so is w;

(ii) for all $i \in \mathbb{Z}$, if $\gcd(u, \sigma^i(v)) = 1$, then $\gcd(w, \sigma^i(v)) = 1$.

Proof. A polynomial in $k(x)[n]$ is shift-free with respect to n if and only if the largest integer distance of its roots is equal to zero [2]. This fact implies the first assertion. For any $i \in \mathbb{Z}$, if $\gcd(u, \sigma^i(v)) = 1$, then $\gcd(u^m, \sigma^i(v)) = 1$ for all $m \in \mathbb{N}$, which implies $\gcd(w, \sigma^i(v)) = 1$. The second assertion holds. $\square$

Proposition 5.5.21. *With the assumptions and notation introduced in Theorem 5.5.10, assume further that $h_2 = h - D_x h_1$, and that L belongs to $k(n)\langle S_n\rangle$ such that $L(h_2)$ is nonzero. Set*

$$A_2 = \tilde{\gamma} + n\frac{\delta(\beta)}{\beta} - \rho\frac{\delta(\mathrm{den}(\beta))}{\mathrm{den}(\beta)} \quad \textit{and} \quad B_2 = \beta\alpha,$$

where ρ denotes the order of L. Then

$$L(h_2) \in \frac{V}{U}\mathcal{H}(A_2, B_2)$$

for some $U, V \in k(n)[x]$, and (U, V, A_2) is a GLL triple.

Proof. Setting $a_2 = \tilde{\gamma} + n\delta(\beta)/\beta$. we get $A_2 = a_2 - \rho\delta(\mathrm{den}(\beta))/\mathrm{den}(\beta)$.

By Theorem 5.5.10, $h_2 \in \frac{v}{u} \cdot \mathcal{H}\left(a_2,\, B_2\right)$. An easy calculation implies

$$L(h_2) = \frac{v'}{u'\mathrm{den}(\beta)^{\rho}} \cdot \mathcal{H}\left(a_2,\, B_2\right),$$

where $v' \in k(n)[x]$ and u' is the least common multiple of u, $S_n(u)$, $\ldots$, $S_n^{\rho}(u)$ in $k(n)[x]$. Since u is squarefree with respect to x, so are its shifts. Consequently, u' is squarefree with respect to x. Set

$$U = \mathrm{den}(v'/u') \quad \text{and} \quad V = \mathrm{num}(v'/u').$$

Then $L(h_2) = (V/U) \cdot \mathcal{H}(A_2, B_2)$.

It remains to show that (U, V, A_2) is a GLL triple. First, U and V are coprime by their definition. Second, U is squarefree with respect to x, as u' is squarefree with respect to x. To verify the third condition in Definition 5.5.1, we observe that the squarefree part of $\mathrm{den}(\beta)\mathrm{num}(\beta)$ is the denominator of $\delta(\beta)/\beta$, and that $\mathrm{den}(\beta)$ is the denominator of $\rho\delta(\mathrm{den}(\beta))/\mathrm{den}(\beta)$, which divides the former denominator. This observation enables us to apply Lemma 5.5.6. We conclude that A_2 is differential-reduced by Lemma 5.5.6 (iii) and since $\tilde{\gamma} + n\frac{\delta(\beta)}{\beta}$ is differential-reduced. At last, we verify that U and $\mathrm{den}(A_2)$ are coprime over $k(n)$. By Lemma 5.5.6 (i), $\mathrm{den}(a_2) = \mathrm{den}(A_2)$. Since (u, v, a_2) is a GLL triple, $\gcd(u, \mathrm{den}(a_2)) = 1$. It follows from the fact $\mathrm{den}(a_2) \in k[x]$ that

$$\gcd\left(S_n^i(u), \mathrm{den}(a_2)\right) = 1$$

for all $i \in \mathbb{N}$, and, consequently, $\gcd(u', \mathrm{den}(a_2)) = 1$. This implies that $\gcd(U, \mathrm{den}(a_2)) = 1$, that is, $\gcd(U, \mathrm{den}(A_2)) = 1$. $\square$

Proposition 5.5.22. *With the assumptions and notation introduced in Theorem 5.5.18, assume further that $h_2 = h - \Delta_n(h_1)$, and that L belongs to $k(x)\langle D_x \rangle$ such that $L(h_2)$ is nonzero. Set*

$$A_2 = \tilde{\gamma} + n\frac{\delta(\beta)}{\beta} \quad \text{and} \quad B_2 = \beta\tilde{\alpha}.$$

Then

$$L(x, D_x)(h_2) \in \frac{V}{U} \cdot \mathcal{H}\left(A_2,\, B_2\right)$$

for some $U, V \in k(x)[n]$, and (U, V, B_2) is an AP triple.

Proof. By Theorem 5.5.18,

$$h_2 \in \frac{v}{u}\mathcal{H}(A_2, B_2)$$

for some $u, v \in k(x)[n]$ and (u, v, B_2) is an AP triple. A straightforward calculation leads to

$$L(x, D_x)(h_2) \in (V/U) \cdot \mathcal{H}(A_2, B_2)$$

for some $U, V \in k(x)[n]$ with $\gcd(U, V) = 1$. By Theorem 5.5.18, $\mathrm{den}(A_2)$ is in $k[x]$. It follows that U is a factor of a power of u in $k(x)[n]$. Thus, (U, V, B_2) is an AP triple by Lemma 5.5.20 and since (u, v, B_2) is an AP triple, as stated in Theorem 5.5.18. □

5.6 Two existence criteria

The Fundamental Theorem in [92] shows that telescopers exist for proper hyperexponential-hypergeometric functions. However, properness is just a sufficient condition. For instance, the rational function $f = 1/(x+n)^2$ is not proper, but it does have a telescoper with respect to x since $f = D_x(-1/(x+n))$. On the other hand, Example 5.3.3 indicates that not all hyperexponential-hypergeometric functions have telescopers. In this section, we present sufficient and necessary conditions in the following theorems for the existence of telescopers.

Theorem 5.6.1. *Let h be a hyperexponential-hypergeometric function over $k(x, n)$ and $h = D_x(h_1) + h_2$ be an additive decomposition of h with respect to x. Then h has a telescoper with respect to x if and only if h_2 is either zero or proper.*

Theorem 5.6.2. *Let h be a hyperexponential-hypergeometric function over $k(x, n)$ and $h = \Delta_n(h_1) + h_2$ be an additive decomposition of h with respect to n. Then h has a telescoper with respect to n if and only if h_2 is either zero or proper.*

Our proofs of these two theorems will be divided into several steps.

5.6.1 Existence of telescopers implies properness

The lemma below shows that the existence problem for h is equivalent to that for h_2. The proofs take the same argument as in [9, page 3].

Lemma 5.6.3. *Let h, h_1 and h_2 be three hyperexponential-hypergeometric functions in the ring $\mathcal{S}$ of sequences.*

(i) *Assume that $h = D_x(h_1) + h_2$. Then L is a telescoper for h with respect to x if and only if it is a telescoper for h_2 with respect to x.*

(ii) *Assume that $h = \Delta_n(h_1) + h_2$. Then L is a telescoper for h with respect to n if and only if it is a telescoper for h_2 with respect to n.*

Proof. By the commutativity between $L(n, S_n) \in k(n)\langle S_n\rangle$ and D_x, we have

$$L(n, S_n)(h) = D_x(g) \quad \Leftrightarrow \quad L(n, S_n)(h_2) = D_x(g - L(h_1)).$$

This implies the first assertion. Similarly, the second one follows from the commutativity between $L(x, D_x) \in k(x)\langle D_x\rangle$ and Δ_n. $\square$

Proof of the necessity in Theorem 5.6.1

Proof. Assume that h has a telescoper with respect to x, and that h_2 is nonzero. Our goal is to show that h_2 is proper. By Theorem 5.5.10,

$$h_2 = \frac{v}{u} H \quad \text{for some } H \in \mathcal{H}\left(a_2, b_2\right),$$

where $b_2 = \beta\alpha$ with $\beta \in k(x)$ and $\alpha \in k(n)$, and (u, v, a_2) is a GLL triple. Without loss of generality, we further assume that u is in $k[x, n]$. To prove that h_2 is proper, it suffices to prove that u is split by Proposition 5.4.4. Let $L \in k(n)\langle S_n\rangle$ be a telescoper for h with respect to x. By Lemma 5.6.3, L is also a telescoper for h_2 with respect to x. Then

$$L(h_2) = L\left(\frac{v}{u} H\right) = D_x(g), \tag{5.16}$$

for some hyperexponential-hypergeometric function g. Since b_2 is split, a direct calculation yields

$$L(h_2) = M\left(\frac{v}{u}\right) H \tag{5.17}$$

for some nonzero element $M \in k(n, x)\langle S_n\rangle$ whose coefficients are split. On the other hand, Proposition 5.5.21 implies

$$L(h_2) \in \frac{V}{U}\mathcal{H}(A_2, B_2). \tag{5.18}$$

where $U, V \in k(n)[x]$ and (U, V, A_2) is a GLL triple. Since $L(h_2)$ is hyperexponential-integrable by (5.16), it follows from Lemma 5.5.5 that U is free of x.

By Proposition 5.5.21, $A_2 = a_2 - \rho\delta(\mathrm{den}(\beta))/\mathrm{den}(\beta)$ and $B_2 = b_2$, where ρ denotes the order of L. It follows from (5.18) and $\mathrm{den}(\beta) \in k[x]$ that

$$L(h_2) \in \frac{V}{U\mathrm{den}(\beta)^\rho}\mathcal{H}(a_2, b_2),$$

which, together with (5.17) and $\mathcal{H}(a_2, b_2) = \{cH \mid c \in k\}$, implies that

$$M\left(\frac{v}{u}\right) = \frac{cV}{U\mathrm{den}(\beta)^\rho} \quad \text{for some } c \in k.$$

Hence, there exist a nonzero operator $M' \in k[x, n]\langle S_n\rangle$ with leading coefficient in $k[n]$ and a rational function $g \in k(x)$ such that $M'(gv/u)$ belongs to $k[x, n]$. Consequently, u is split by Lemma 5.2.7, and, hence, h_2 is proper. □

Proof of the necessity in Theorem 5.6.2

Proof. Assume that h has a telescoper with respect to n, and that h_2 is nonzero. Our goal is to show that h_2 is proper. By Theorem 5.5.18,

$$h_2 = \frac{v}{u}H \quad \text{for some } H \in \mathcal{H}\left(a_2, b_2\right),$$

where $a_2 = \tilde{\gamma} + n\frac{\delta(\beta)}{\beta}$ for some $\tilde{\gamma} \in k(x)$, and (u, v, b_2) is an AP triple. Without loss of generality, we further assume that u is in $k[x, n]$. To prove that h_2 is proper, it suffices to prove that u is split by Proposition 5.4.4.

Let $L \in k(x)\langle D_x\rangle$ be a telescoper for h with respect to n. By Lemma 5.6.3, L is also a telescoper for h_2 with respect to n. Then

$$L(h_2) = L\left(\frac{v}{u}H\right) = \Delta_n(g), \tag{5.19}$$

for some hyperexponential-hypergeometric function g. For all $i \in \mathbb{N}$, let f_i be the rational function in $k(x, n)$ such that

$$D_x^i(H) = f_i H.$$

Then $\mathrm{den}(f_i)$ is in $k[x]$, because $\mathrm{den}(a_2)$ is in $k[x]$. It follows from Leibniz's formula that there exists $M_i \in k(x)[n]\langle D_x\rangle$ such that

$$D_x^i(h_2) = M_i\left(\frac{v}{u}\right)H.$$

Moreover, the leading coefficient of M_i is in k. Consequently, there exists $M \in k(x)[n]\langle D_x \rangle$ with leading coefficient in $k(x)$ such that

$$L(h_2) = M\left(\frac{v}{u}\right) H. \tag{5.20}$$

On the other hand, Proposition 5.5.22 and $\mathcal{H}(a_2, b_2) = \{cH \mid c \in k\}$ imply,

$$L(h_2) = \frac{V}{U} H. \tag{5.21}$$

where U and V are in $k(x)[n]$, and (U, V, b_2) is an AP triple. Since $L(h_2)$ is hypergeometric-summable by (5.19), it follows from Lemma 5.5.16 that U is free of n.

Comparing (5.20) with (5.21), we have that $M(v/u) = V/U$, which is in $k(x)[n]$. It follows that there exists a nonzero operator $M' \in k[x, n]\langle D_x \rangle$ with leading coefficient in $k[x]$ such that $M'(v/u)$ belongs to $k[x, n]$. By Lemma 5.2.7, u is split, and then h_2 is proper. □

5.6.2 Properness implies the existence of telescopers

Wilf and Zeilberger present an elementary proof of the existence of telescopers for proper multivariate hypergeometric terms [92, Theorem 3.1], and indicate that their argument should be also valid in the continuous-discrete setting. For the sake of completeness, we elaborate on the proof of the following theorem.

Theorem 5.6.4. *If h is a proper hyperexponential-hypergeometric function over $k(x, n)$, then h has a telescoper with respect to n and a telescoper with respect to x.*

Preparation lemmas

Before the proof of the theorem above, we shall show some lemmas.

Lemma 5.6.5. *Let $h \in \mathcal{H}(a, b)$ be a hyperexponential-hypergeometric function over $k(x, n)$.*

(i) If h has a telescoper with respect to n, so does the product Hh for any hyperexponential-hypergeometric function H with $\Delta_n(H) = 0$.

(ii) If h has a telescoper with respect to x, so does the product Hh for any hyperexponential-hypergeometric function H with $D_x(H) = 0$.

Proof. Assume that H is a nonzero element of $\mathcal{H}(A,B)$. If $\Delta_n(H)=0$, then $B=1$ and $a\in k(x)$ by the integrability condition $\sigma(A)-A=\delta(B)/B$. It is straightforward to verify that $HD_x(h)=(D_x-A)Hh$, which, together with an easy induction, implies that

$$HD_x^i(h)=(D_x-A)^i(Hh)\quad\text{for all } i\in\mathbb{N}.$$

Let $L\in k(x)\langle D_x\rangle$ be a telescoper of h. By the equality above,

$$HL(h)=M(Hh)$$

where M is obtained by replacing D_x^i in L by $(D_x-A)^i$. Since there exists a hypergeometric function g such that $L(h)=\Delta_n(g)$ and $\Delta_n(H)=0$, the equality above implies that

$$M(Hh)=H\Delta_n(g)=\Delta_n(Hg).$$

The first assertion is proved.

To prove the second assertion, assume that H is a nonzero element in $\mathcal{H}(A,B)$. If $D_x(H)$ is equal to 0, then $A=0$ and $B\in k(n)$ by the integrability condition $\sigma(A)-A=\delta(B)/B$. Since $HS_n(h)=S_n(Hh)/B$, we have that, for all $i\in\mathbb{N}$,

$$HS_n^i(h)=B_iS_n^i(Hh)\quad\text{for some } B_i\in k(n).$$

Let $L\in k(n)\langle S_n\rangle$ be a telescoper of h. By the equality above,

$$HL(h)=M(Hh)$$

where M is obtained by replacing S_n^i in L by $B_iS_n^i$. Since there exists a hypergeometric function g such that $L(h)=D_x(g)$ and $D_x(H)=0$, the equality above implies that

$$M(Hh)=HD_x(g)=D_x(Hg).$$

The second assertion is proved. □

Lemma 5.6.6. *Let h_1 and h_2 be two similar hyperexponential-hypergeometric functions over the field $k(x,n)$. Let $\mathbb{V}_n$ be the vector space spanned by $\left\{\Delta_n^i(h_j)\,|\,i\in\mathbb{N}, j\in\{1,2\}\right\}$ over $k(x)$, and $\mathbb{V}_x$ be the vector space spanned by $\left\{D_x^i(h_j)\,|\,i\in\mathbb{N}, j\in\{1,2\}\right\}$ over $k(n)$.*

1. *If both h_1 and h_2 have telescopers with respect to n, so does any nonzero element in $\mathbb{V}_n$.*

2. If both h_1 and h_2 have telescopers with respect to x, so does any nonzero element in $\mathbb{V}_x$.

Proof. Assume that there exists a nonzero $L_j \in k(x)\langle D_x\rangle$ such that $L_j(h_j) = \Delta_n(g_j)$ for some hyperexponential-hypergeometric function g_j, and for $j = 1, 2$. Then

$$L_j(\Delta_n^i(h_j)) = \Delta_n(\Delta_n^i(g_j)).$$

Hence, every element of $\left\{\Delta_n^i(h_j) \,|\, i \in \mathbb{N}, j \in \{1, 2\}\right\}$ has a telescoper. If two hypergeometric functions have telescopers with respect to n, say T_1 and T_2, then their sum, assumed to be nonzero, also has a telescoper with respect to n, which can be taken as a common left multiple of T_1 and T_2. The first assertion then follows from Lemma 5.6.5 (*ii*). The second assertion can be proved in the same manner. □

In the proof of the Fundamental Theorem [92], Wilf and Zeilberger neglected the case that the remainder of a nonzero m-free operator $P(n, S_n, S_m) \in k(n)\langle S_n, S_m\rangle$ by $S_m - 1$ may be zero. Wegschaider used a "noncommutative trick" to deal with this case. Here, we will apply Wegschaider's trick and its differential analogue to transform an x-free or n-free operator to a telescoper with respect to n or x.

Lemma 5.6.7 (Wegschaider's trick)**.** *Let h be a hypergeometric function over $k(x, n)$.*

(i) If there exists a nonzero operator $A \in k(x)\langle S_n, D_x\rangle$ such that $A(h) = 0$, then h has a telescoper with respect to n.

(ii) If there exists a nonzero operator $A \in k(n)\langle S_n, D_x\rangle$ such that $A(h) = 0$, then h has a telescoper with respect to x.

Proof. Assume that there exists a nonzero operator A in $k(x)\langle S_n, D_x\rangle$ such that $A(h) = 0$. Since $\Delta_n = S_n - 1$, we can write $A = \Delta_n^m(L(x, D_x) + \Delta_n M)$, where m is in $\mathbb{N}$, L is a nonzero operator in $k(x)\langle D_x\rangle$ and M is in $k(x)\langle S_n, D_x\rangle$. By Wegschaider's trick in [91, Theorem 3.2], there exist $w \in k(n)$ and $r \in k$ with $r \neq 0$ such that

$$w\Delta_n^m = \Delta_n Q + r \tag{5.22}$$

for some $Q \in k(n)\langle S_n\rangle$. In particular, $r = (-1)^m m!$ if we take $w = n^m$. Using the fact $r\Delta_n = \Delta_n r$ and (5.22), we find

$$\frac{w}{r}A = L + \Delta_n N \quad \text{for some } N \in k(x, n)\langle D_x, S_n\rangle.$$

Hence, L is a telescoper for h with respect to n.

The second assertion can be proved in a similar way. Instead of (5.22), we need to find $w \in k(x)$ and $r \in k \setminus \{0\}$ such that

$$wD_x^m = D_x Q + r$$

for some $Q \in k(x)\langle D_x \rangle$. In particular, $r = (-1)^m m!$ if we take $w = x^m$. $\square$

Proof of Theorem 5.6.4

Now, we present the proof of Theorem 5.6.4.

Proof. Assume that h is a proper hyperexponential-hypergeometric function. Then the standard representation of h is of the form

$$(p(x, n),\ \beta(x),\ \gamma(x),\ \alpha(n)),$$

where $p \in k[x, n]$, $\beta, \gamma \in k(x)$, and $\alpha \in k(n)$. Writing p as $\sum_{i=0}^m p_i(x) n^i$ with $p_i \in k[x]$ yields

$$p\beta^n \mathcal{H}(\gamma, \alpha) = \sum_{i=0}^m p_i n^i \beta^n \mathcal{H}(\gamma, \alpha) = \sum_{i=0}^m \beta^n \mathcal{H}\left(\gamma + \frac{\delta(p_i)}{p_i}, \alpha(n)\frac{\sigma(n^i)}{n^i}\right).$$

It follows that

$$h = \sum_{i=0}^m \beta(x)^n G_i H_i \tag{5.23}$$

for some $G_i \in \mathcal{H}\left(\gamma(x) + \frac{\delta(p_i)}{p_i}, 1\right)$ and $H_i \in \mathcal{H}\left(0, \alpha(n)\frac{\sigma(n^i)}{n^i}\right)$.

First, we show that h has a telescoper with respect to n. Note that each G_i in (5.23) has n-certificate equal to 1. By Lemmas 5.6.5 and 5.6.6, it suffices to show that a hypergeometric function $\hat{h} \in \beta(x)^n \cdot \mathcal{H}(0, g(n))$ with $g \in k(n)$ has a telescoper with respect to n. Let $s = \mathrm{num}(\beta)$, $t = \mathrm{den}(\beta)$, $a = \mathrm{num}(g)$ and $b = \mathrm{den}(g)$. Moreover, let $v = \mathrm{num}(\delta(\beta)/\beta)$ and $w = \mathrm{den}(\delta(\beta)/\beta)$. A straightforward calculation yields that the x-certificate and n-certificate of $\hat{h}$ are, respectively,

$$\frac{nv}{w} \quad \text{and} \quad \frac{sa}{tb}.$$

Note that s, t, v, w are in $k[x]$, and a, b in $k[n]$.

We claim that there exists a nonzero operator A in $k(x)\langle S_n, D_x \rangle$ such that $A(\hat{h}) = 0$. The claim will be proved by a well-known argument used in [67]. The first assertion will then follow from Lemma 5.6.7.

Let $\mathcal{F}_N$ be the linear space spanned by $\{S_n^i D_x^j \mid i+j \leq N\}$ over $k(x)$. Let μ be the maximum of degrees of a and b in n, and let

$$\mathcal{W}_N = \operatorname{span}_{k(x)} \left\{ \frac{n^i \hat{h}}{b(n+N-1)\cdots b(n)} \mid i \leq (\mu+1)N \right\}.$$

An easy induction on i and j yields

$$S_n^i D_x^j(\hat{h}) = \frac{q(n,x)\hat{h}}{b(n+i-1)\cdots b(n)}, \text{ where } \deg_n(q) \leq i\mu + j.$$

Hence, $S_n^i D_x^j(\hat{h})$ belongs to $\mathcal{W}_N$ if $i+j \leq N$. Accordingly, there is a $k(x)$-linear map ϕ_N from $\mathcal{F}_N$ to $\mathcal{W}_N$ that sends L to $L(\hat{h})$ for all $L \in \mathcal{F}_N$. Since the dimension of $\mathcal{F}_N$ over $k(x)$ is $\binom{N+2}{2}$, while that of $\mathcal{W}_N$ is $(\mu+1)N+1$, the kernel of ϕ_N is nontrivial when N is sufficiently large. Any nonzero element in the kernel annihilates $\hat{h}$. The claim is established.

Second, we show that h has a telescoper with respect to x. Note that each nonzero element in $\mathcal{H}\left(0, \alpha(n)\frac{\sigma(n^i)}{n^i}\right)$ in (5.23) has x-certificate equal to zero. By Lemmas 5.6.5 and 5.6.6, it suffices to show that any hypergeometric function $\hat{h}(x,n)$ in the set

$$\beta(x)^n \cdot \mathcal{H}(g(x), 1) \quad \text{with } g \in k(x)$$

has a telescoper with respect to x. Let $s = \operatorname{num}(\beta)$, $t = \operatorname{den}(\beta)$,

$$v = \operatorname{num}\left(n\frac{\delta(\beta)}{\beta} + g\right) \quad \text{and} \quad w = \operatorname{den}\left(n\frac{\delta(\beta)}{\beta} + g\right).$$

A straightforward calculation yields that the x-certificate and n-certificate of $\hat{h}$ are, respectively,

$$\frac{v(x,n)}{w(x)} \quad \text{and} \quad \frac{s(x)}{t(x)}.$$

Note that s, t, and w are in $k[x]$ and v is in $k[x,n]$ with $\deg_n(v) = 1$.

We claim that there exists a nonzero operator A in $k(n)\langle S_n, D_x\rangle$ such that $A(\hat{h}) = 0$. The second assertion then follows from Lemma 5.6.7.

Following an argument similar to the one above, we now consider a linear space $\mathcal{F}_N$ spanned by $\{S_n^i D_x^j \mid i+j \leq N\}$ over $k(n)$. Let μ be the maximum of the degrees in x of s, t, v and w, and let

$$\mathcal{W}_N = \operatorname{span}_{k(n)} \left\{ \frac{x^i \hat{h}}{(tw)^N} \mid i \leq 2\mu N \right\}.$$

Algorithm ExistenceContinuous

Input: A hyperexponential-hypergeometric function $h \in \mathcal{H}(a, b)$ with $a, b \in k(x, n)$.

Output: True, if h has a telescoper with respect to x; false, otherwise.

1. Compute a strict DRNF (K, S) of a with respect to x;
2. Apply **ReduceCert** in [47] to (K, S) to get the decomposition
$$S = \delta(S_1) + S_1 K + \frac{v}{u\,\mathrm{den}(K)^i}.$$
3. If u is split, then return true, otherwise, return false.

Figure 5.1: Algorithm for deciding the existence of telescopers with respect to x.

An easy induction on i and j yields

$$S_n^i D_x^j(\hat{h}) = \frac{q(n,x)}{t^i w^j}\hat{h}, \text{ where } \deg_x(q) \leq (i+j)\mu.$$

Hence, $S_n^i D_x^j(\hat{h})$ belongs to $\mathcal{W}_N$ if $i+j \leq N$. Accordingly, there is a $k(n)$-linear map ϕ_N from $\mathcal{F}_N$ to $\mathcal{W}_N$ that sends L to $L(\hat{h})$ for all $L \in \mathcal{F}_N$. Since the dimension of $\mathcal{F}_N$ over $k(n)$ is $\binom{N+2}{2}$, while that of $\mathcal{W}_N$ is $2\mu N + 1$, the kernel of ϕ_N is nontrivial when N is sufficiently large. Any nonzero element in the kernel annihilates $\hat{h}$. The claim is established. $\square$

Remark 5.6.8. *Theorems 5.6.1 and 5.6.2 hold by the proofs of necessity in Theorems 5.6.1 and 5.6.2, and by Theorem 5.6.4*

5.7 Algorithms and examples

5.7.1 Algorithms

According to our criteria, we only use the continuous, resp. discrete, certificate for deciding the existence of telescopers with respect to the continuous, resp. discrete, variable. Algorithms in Figures 5.1 and 5.2 are based on strict DRNF and strict SRNF computation, two additive decompositions, and our criteria.

Algorithm ExistenceDiscrete

Input: A hyperexponential-hypergeometric function $h \in \mathcal{H}(a, b)$ with $a, b \in k(x, n)$.

Output: True, if h has a telescoper with respect to n; false, otherwise.

1. Compute a strict SRNF (K, S) of b with respect to n;

2. Apply **dcert** in [13] to (K, S) to get the decomposition

$$S = \sigma(S_1)K - S_1 + \frac{v}{u\,(\sigma^{-1}(\mathrm{num}(K)))^i \mathrm{den}(K)^j}.$$

3. If u is split, then return true, otherwise, return false.

Figure 5.2: Algorithm for deciding the existence of telescopers with respect to n.

5.7.2 Examples

Example 5.7.1. *It is possible that a hyperexponential-hypergeometric function has a telescoper with respect to one variable but no telescoper with respect to the other variable. Consider the rational function*

$$h = \frac{1}{(x+n)^2}.$$

Applying Hermite reduction to h with respect to x yields

$$h = D_x\left(\frac{-1}{x+n}\right),$$

which implies that 1 is a telescoper for h with respect to x. However, h has no telescoper with respect to n, because $(x+n)^2$ is shift-free with respect to n and it is not split. Similarly, consider the rational function

$$h = \frac{1}{(x+n)(x+n+1)}.$$

Since $h = \Delta_n(-1/(x+n))$, 1 is a telescoper for h with respect to n. However, h has no telescoper with respect to x, because $(x+n)(x+n+1)$ is squarefree with respect to x and it is not split.

Example 5.7.2. *As we know, the properness is only a sufficient condition for the existence of telescopers. We show a concrete example to illustrate this fact. Consider the hyperexponential-hypergeometric function*

$$h(x, n) = \frac{-x + 2nx + 2n^2}{(x+n)^2 x} \cdot x^n \cdot e^{-x}.$$

Though h is not proper, it still has a telescoper with respect to x since h can be decomposed into

$$h = D_x \left(\frac{1}{x+n} \cdot x^n \cdot e^{-x} \right) + x^{n-1} \cdot e^{-x},$$

where $x^{n-1} \cdot e^{-x}$ is proper. In fact, our criteria indicate that all nonproper examples are of this form.

Chapter 6

Conclusion and Perspectives

In this thesis, we have focused on the practical efficiency, theoretical complexity, and the termination of creative-telescoping algorithms. For bivariate rational functions, we have managed to blend the general method of creative telescoping with the Hermite reduction. According to our complexity analyses and experiments, the Hermite reduction based method has been proved to outperform the existing methods. In order to study the general case of Wilf and Zeilberger's conjecture, we have presented a structure theorem for multivariate hyperexponential-hypergeometric functions. With the help of standard representations and two additive decompositions, we have derived two existence criteria for deciding whether a bivariate hyperexponential-hypergeometric function has a telescoper with respect to the continuous variable or the discrete one. Our criteria solve the termination problem of Zeilberger-style algorithms for bivariate hyperexponential-hypergeometric functions. Still, there are many further studies to be carried out. In what follows, we would like to mention a few directions for future work.

6.1 Extensions of the Hermite-reduction based method

For the moment, the Hermite-reduction based method only works for bivariate rational-function inputs. As we have already seen in Chapter 5, Hermite reduction can be extended to hyperexponential or hypergeometric cases. So it is natural to wonder whether the approach of the Hermite-reduction based method can be used to compute the minimal telescopers for bivariate hyperexponential functions or hypergeometric terms. We have considered two possible extensions: one is for hyperexponential functions, especially in simple radical extensions; and the other is for multivariate rational functions. In the multivariate rational case, the corresponding telescoping problem is as follows.

Problem 6.1.1. *Given a rational function $f \in k(x, y_1, \ldots, y_n)$, construct a nonzero operator $L \in k(x)\langle D_x \rangle$ such that*

$$L(x, D_x)(f) = \sum_{i=1}^{n} D_{y_i}(g_i) \quad \textit{for some } g_i \in k(x, y_1, \ldots, y_n).$$

In order to solve the above problem, we have tried to extend the Hermite reduction to the multivariate case. That is, we would like to decompose a multivariate rational function f in $k(y_1, \ldots, y_n)$ into

$$f = D_{y_1}(g_1) + \cdots + D_{y_n}(g_n) + r,$$

where r is minimal in some sense. For a special class of rational functions, we show this can be obtained.

Let $f = P/Q^m$ be a rational function in $k(y_1, \ldots, y_n)$ with $m > 1$, $P, Q \in k[y_1, \ldots, y_n]$, and $\gcd(P, Q) = 1$. Let $\mathcal{I}$ be the ideal generated by $Q, D_{y_1}(Q), \ldots, D_{y_n}(Q)$ in $k[y_1, \ldots, y_n]$. We observe that the ideal $\mathcal{I}$ is equal to the ring $k[y_1, \ldots, y_n]$ in certain applications, see examples gathered in [74]. In the sequel, we proceed the discussion under the above constraints on Q. Then we can always express P in the form

$$P = AQ + \sum_{i=1}^{n} B_i D_{y_i}(Q),$$

where $A, B_1, \ldots, B_n$ are polynomials in $k(y_1, \ldots, y_n)$. Now, integration by parts yields

$$\frac{P}{Q^m} = \frac{AQ + \sum_{i=1}^{n} B_i D_{y_i}(Q)}{Q^m} = \sum_{i=1}^{n} D_{y_i}\left(\frac{-B_i}{mQ^{m-1}}\right) + \frac{A + m^{-1}\sum_{i=1}^{n} D_{y_i}(B_i)}{Q^{m-1}}.$$

Repeating the above process at most $m-1$ times, we get

$$\frac{P}{Q^m} = \sum_{i=1}^{n} D_{y_i}(g_i) + \frac{\widetilde{P}}{Q}, \tag{6.1}$$

where $g_i \in k(y_1, \ldots, y_n)$ and $\widetilde{P} \in k[y_1, \ldots, y_n]$. In the univariate case when Q is squarefree, this is exactly Hermite reduction. When $n > 1$, the essential difference from the univariate case is that one cannot make the remaining fraction $\widetilde{P}/Q$ proper with respect to all the variables. The uniqueness of the decomposition in (6.1) is also not well studied. In the future, we would like to understand this decomposition better and expect it to be useful to solve the telescoping problem for multivariate rational functions.

6.2 Existence criteria for telescopers in the q-setting

There are nine termination problems of Zeilberger's method for bivariate hyperexponential-hypergeometric functions if telescopers could involve the q-shift operator, denoted by Q_n. In Table 6.1, we mark the solved instances by ✓ and unsolved by ?. In order to solve the remaining termination problems, we would like to extend Theorem 4.4.1 to the q-shift setting. Moreover, we expect a unified way to deal with all of the nine cases.

(L, g)	D_y	$S_n - 1$	$Q_n - 1$
$L(x, D_x)$	✓	✓	?
$L(n, S_n)$	✓	✓	?
$L(n, Q_n)$	?	?	✓

Table 6.1: Solved and unsolved termination problems

6.3 Wilf and Zeilberger's conjecture

We present some remarks on the general case of Wilf and Zeilberger's conjecture. First, we recall basic notation about holonomic functions and modules following the presentation in Coutinho's book [38]. Let $\mathcal{A}_m$ denote the m-th Weyl algebra $k[x_1, \ldots, x_m]\langle D_1, \ldots, D_m\rangle$, in which we have the following multiplication rule: for all $1 \leq i, j \leq m$,

$$x_i x_j = x_j x_i, \quad D_i D_j = D_j D_i, \quad \text{and} \quad D_i x_j - x_j D_i = \delta_{ij},$$

where δ_{ij} is the Kronecker delta symbol: it equals one if $i = j$ and zero otherwise. A finitely generated left $\mathcal{A}_m$-module is *holonomic* if it is zero, or if it has Hilbert dimension m [38, Chapter 5]. By Bernstein's inequality, any finitely generated nonzero left $\mathcal{A}_m$-module has dimension at least m. So holonomicity indicates the minimality of dimension for nonzero $\mathcal{A}_m$-modules.

Let $H(\mathbf{x})$ be a function in a left $\mathcal{A}_m$-module of functions. Then one can define the annihilating ideal of $H(\mathbf{x})$ in $\mathcal{A}_m$ by

$$\operatorname{ann}_{\mathcal{A}_m}(H) = \{p \in \mathcal{A}_m \mid p \cdot H = 0\}.$$

The function $H(\mathbf{x})$ is said to be *holonomic* if its annihilating ideal is holonomic as a left $\mathcal{A}_m$-module. When the function $H(\mathbf{x})$ can be viewed as an element in a left module over the ring $k(x_1, \ldots, x_m)\langle D_1, \ldots, D_m\rangle$ of differential operators, $H(\mathbf{x})$ is said to be *D-finite* if the vector space generated by all derivatives $D_1^{i_1} \cdots D_m^{i_m}(H)$ over $k(\mathbf{x})$ is finite dimensional over $k(\mathbf{x})$ [67, 68]. In the continuous case, an elementary proof of Kashiwara's equivalence [59] between holonomicity and D-finiteness was presented by Takayama in [85]. For this reason, we choose the following definition, used in [24, page 59], for holonomicity of continuous-discrete functions.

Definition 6.3.1. *A function $h(\mathbf{x}, \mathbf{y})$ is said to be holonomic if its associated generating function*

$$H(\mathbf{x}, \mathbf{z}) = \sum_{y_1, \ldots, y_n \in \mathbb{N}} h(\mathbf{x}, \mathbf{y}) z_1^{y_1} \cdots z_n^{y_n}$$

is a holonomic function in a left module over the ring $k(\mathbf{x}, \mathbf{z})\langle D_{x_1}, \ldots, D_{x_m}, D_{z_1}, \ldots, D_{z_n}\rangle$.

Now, we can describe the conjecture by Wilf and Zeilberger [92, page 585] on holonomic hyperexponential-hypergeometric functions.

Conjecture 6.3.2 (Wilf and Zeilberger)**.** *Let $h(\mathbf{x}, \mathbf{y})$ be a hyperexponential-hypergeometric function over $k(\mathbf{x}, \mathbf{y})$. Then h is holonomic if and only if it is proper.*

In the case of several discrete variables, a slightly modified version of the conjecture was proved independently by Payne in his Ph.D. thesis [73] and by Abramov and Petkovšek [14]. In particular, the case of two variables has also been shown by Hou [54, 55] and by Abramov and Petkovšek [12]. In the continuous case, any multivariate hyperexponential function is D-finite, and then holonomic by Kashiwara's equivalence. By Corollary 4.5.4, it is also proper. Thus, Wilf and Zeilberger's conjecture holds naturally in this case.

In the discrete case, the structure theorem on multivariate hypergeometric terms helps us reduce Wilf and Zeilberger's conjecture to rational case [73, 14], where one only need to show that any holonomic rational functions are proper. For the moment, we have not obtained any complete proof of the general continuous-discrete case of the conjecture. In the future, we would like to prove the following conjecture on holonomic rational functions in the mixed setting.

Conjecture 6.3.3. *Let $f(\mathbf{x},\mathbf{y})$ be a rational function in $k(\mathbf{x},\mathbf{y})$. Then f is holonomic if and only if the denominator of f splits into the form*

$$A(\mathbf{x})\prod_{\ell=1}^{L}(v_\ell \cdot \mathbf{y} + \lambda_\ell)$$

where $A \in k[\mathbf{x}]$, $\lambda_1, \ldots, \lambda_L \in \bar{k}$, $v_\ell \in \mathbb{Z}^n$ for all ℓ with $1 \leq \ell \leq L$, and the $v_\ell \cdot \mathbf{y}$ means the inner product of vectors v_ℓ and $\mathbf{y}$.

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Published Papers

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