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# qMultiSum — A Package for Proving $q$-Hypergeometric Multiple Summation Identities

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**Abstract**

A Mathematica package for finding recurrences for $q$-hypergeometric multiple sums is introduced. Together with a detailed description of the theoretical background, we present several examples to illustrate its usage and range of applicability. In particular, various computer proofs of recently discovered identities are exhibited.

## 1. Introduction

In recent years Doron Zeilberger's (18) algorithm has gained more and more attention as a valuable tool for automatically proving hypergeometric and $q$-hypergeometric single summation identities. Due to constantly improved implementations we are now able to settle almost all problems to which it is applicable within reasonable time. Nevertheless, the situation concerning multiple summation identities is fairly different. Although Wilf and Zeilberger (17) showed nearly 10 years ago that in principle also ($q$-)hypergeometric multi-sums can be handled algorithmically, until recently their multivariate generalization of the so-called Sister Celine's method in practice could be applied only to relatively simple examples. It was actually Kurt Wegschaider's (16) package MultiSum† that changed the situation drastically. With his significant improvements of Sister Celine's technique he was the first to attack multiple binomial sums efficiently with a computer.

Based on Wegschaider's ideas we have developed a new package, qMultiSum‡,

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†available at *http://www.risc.uni-linz.ac.at/research/combinat/risc/software/MultiSum*

‡available at *http://www.risc.uni-linz.ac.at/research/combinat/risc/software/qMultiSum*

which can be viewed as a $q$-version of his implementation. The object of this paper is twofold. First, we examine the theoretical background of ($q$-) Sister Celine's technique and its extensions. Second, we want to provide a manual for the package. Therefore we present both a rigorous description of all available functions and several examples that shall illustrate the usage and also the limitations of the software.

The paper is organized as follows. In Section 2 we give a detailed account on the algorithmic backbone of the package, the concept of so-called **k**-free recurrences (or Sister Celine's technique). We show under which conditions such recurrences exist in theory and present a method for improving the performance that allows to handle also more sophisticated examples. In Section 3 we explain how **k**-free recurrences can be transformed into certificate recurrences which are multi-dimensional analogues of recurrences computed with Zeilberger's algorithm. In Section 4 we switch from **k**-free recurrences to a more general domain to further increase efficiency. In Section 5 we describe in detail the functions contained in our package. Finally, in Section 6 we present several computer generated proofs, among them many of recently discovered summation identities.

*Notation.* Throughout this paper we will frequently use vector notation. Vectors are always denoted by bold symbols. For the element at position $i$ of a vector $\mathbf{j}$ we write $j_i$. For a family of vectors $\{\mathbf{j}_s\}_{s=1}^{n}$, the $i$-th element of $\mathbf{j}_m$ is denoted by $j_{m,i}$. For $\mathbf{j} = (j_1, \dots, j_r)$ and $\mathbf{k} = (k_1, \dots, k_r)$ we define

$$\begin{aligned}
\mathbf{j} + \mathbf{k} &:= (j_1 + k_1, \dots, j_r + k_r),\\
\mathbf{j} \cdot \mathbf{k} &:= j_1 k_1 + \dots + j_r k_r,\\
i \cdot \mathbf{k} &:= (i\, k_1, \dots, i\, k_r),\\
\mathbf{j}^{\mathbf{k}} &:= (j_1^{k_1}, \dots, j_r^{k_r}),\\
i^{\mathbf{k}} &:= (i^{k_1}, \dots, i^{k_r}).
\end{aligned}$$

Concatenation of a scalar and a vector is sloppily abbreviated by $(i, \mathbf{j}) := (i, j_1, \dots, j_r)$. For functions we write $F(n, \mathbf{k})$ for $F(n, k_1, \dots, k_r)$ and $\sum_{\mathbf{k}} F(n, \mathbf{k})$ for $\sum_{k_1} \cdots \sum_{k_r} F(n, \mathbf{k})$.

## 2. k-Free Recurrences

Our implementation is based on the method of **k**-free recurrences, also known as the multivariate Sister Celine's technique, which we will sketch now briefly before going into the details below. First of all we need some basic definitions. Let $\mathbb{K} = \mathbb{C}(q, \tau_1, \dots, \tau_m)$ denote the transcendental extension of the complex numbers $\mathbb{C}$ by the indeterminates $q, \tau_1, \dots, \tau_m$. From now on we will assume $n$ to be a variable and $\mathbf{k} = (k_1, \dots, k_r)$ to be a non-empty vector of variables all ranging over the integers. For reasons of convenience, in most applications $n$ will actually range only over $\mathbb{N}_0 := \{0, 1, 2, \dots\}$.

We say that $P(n,\mathbf{k})$ is a polynomial in $q^n$ and $q^{\mathbf{k}}$ over $\mathbb{K}$, written as $P(n,\mathbf{k}) \in \mathbb{K}[q^n,q^{\mathbf{k}}]$, if there exists a polynomial $P^* \in \mathbb{K}[x_0,x_1,\ldots,x_r]$ such that $P(n,\mathbf{k}) = P^*(q^n,q^{k_1},\ldots,q^{k_r})$. Analogously, $R(n,\mathbf{k})$ is said to be a rational function in $q^n$ and $q^{\mathbf{k}}$ over $\mathbb{K}$, written as $R(n,\mathbf{k}) \in \mathbb{K}(q^n,q^{\mathbf{k}})$, if there exists a rational function $R^* \in \mathbb{K}(x_0,x_1,\ldots,x_r)$ such that $R(n,\mathbf{k}) = R^*(q^n,q^{k_1},\ldots,q^{k_r})$.

A function $F(n,\mathbf{k})$ is called *q-hypergeometric* in $n$ and $\mathbf{k}$ over $\mathbb{K}$, if the quotients

$$\frac{F(n+1,k_1,\ldots,k_r)}{F(n,k_1,\ldots,k_r)}, \frac{F(n,k_1+1,\ldots,k_r)}{F(n,k_1,\ldots,k_r)}, \ldots, \frac{F(n,k_1,\ldots,k_r+1)}{F(n,k_1,\ldots,k_r)}$$

are rational functions in $q^n$ and $q^{\mathbf{k}}$ over $\mathbb{K}$.

The central concept of (the $q$-version of) Sister Celine's technique is the computation of recurrences for multiple sums $\sum_{\mathbf{k}} F(n,\mathbf{k})$, where $F(n,\mathbf{k})$ is $q$-hypergeometric. For this we proceed by computing a so-called $\mathbf{k}$-free recurrence for the summand first.

Definition 2.1: A $q$-hypergeometric function $F(n,\mathbf{k})$ over $\mathbb{K}$ satisfies a $\mathbf{k}$-*free recurrence*, if there exist a finite set $S$ of integer tuples of length $r+1$ and polynomials $\sigma_{i,\mathbf{j}}(n) \in \mathbb{K}[q^n]$ not all zero, such that

$$\sum_{(i,\mathbf{j})\in S} \sigma_{i,\mathbf{j}}(n)\, F(n-i,\mathbf{k}-\mathbf{j}) = 0 \tag{1}$$

holds at every point $(n,\mathbf{k})$ where all values of $F$ occurring in (1) are well-defined. The set $S$ is called a *structure set.*

If we define $N$ and $K_h$ as usual to be the *forward shift operators* w.r.t. $n$ and $k_h$, respectively, i.e., $NF(n,\mathbf{k}) = F(n+1,\mathbf{k})$ and $K_hF(n,\mathbf{k}) = F(n,k_1,\ldots,k_{h-1},k_h+1,k_{h+1},\ldots,k_r)$, recurrence (1) can be written in operator notation as

$$\left(\sum_{(i,\mathbf{j})\in S} \sigma_{i,\mathbf{j}}(n)\, N^{-i}\mathbf{K}^{-\mathbf{j}}\right) F(n,\mathbf{k}) = 0.$$

The computation of a $\mathbf{k}$-free recurrence is done by making an Ansatz of the form (1) for some structure set $S$ and undetermined $\sigma_{i,\mathbf{j}}$. Dividing equation (1) by $F(n,\mathbf{k})$ leads to the rational equation

$$\sum_{(i,\mathbf{j})\in S} \sigma_{i,\mathbf{j}}(n)\, R_{F,i,\mathbf{j}}(n,\mathbf{k}) = 0, \tag{2}$$

which after clearing denominators turns into the polynomial equation

$$\sum_{(i,\mathbf{j})\in S} \sigma_{i,\mathbf{j}}(n)\, P_{F,i,\mathbf{j}}(n,\mathbf{k}) = 0. \tag{3}$$

Next we compare the coefficients of all power products $q^{k_1 l_1} \cdots q^{k_r l_r}$ in equation (3) with zero to get a homogeneous system of linear equations for the $\sigma_{i,\mathbf{j}}(n)$. Every non-trivial solution of this equation system gives rise to a **k**-free recurrence.

The main problem concerning algorithmic efficiency — as in the $q = 1$ case — is the choice of the structure set $S$ for which a **k**-free recurrence exists. Again it turns out that "rectangular" structure sets are in general not usable. Therefore we also generalized the concept of $P$-maximal structure sets to the $q$-case, leading to more satisfactory results. Furthermore we incorporated Wegschaider's idea of dealing with special types of **k**-dependent recurrences to decrease once more the size of the structure set.

### 2.1. *q*-Proper Hypergeometric Functions

In this subsection we will define the notion of $q$-proper hypergeometric functions for which it can be shown that a **k**-free recurrence always exists. Essentially we will follow Wegschaider's (16) rigorous presentation, however with omitting some of the details, such as distinguishing between terms and functions.

Let the *q-shifted factorial* (or *q-Pochhammer symbol*) of $A \in \mathbb{K}$ be defined as usual (see, e.g. Gasper and Rahman (8)) by

$$(A;q)_k := \begin{cases} (1-A)(1-Aq)\cdots(1-Aq^{k-1}), & \text{if } k > 0, \\ 1, & \text{if } k = 0, \\ \left[(1-Aq^{-1})(1-Aq^{-2})\cdots(1-Aq^{k})\right]^{-1}, & \text{if } k < 0, \end{cases}$$

and

$$(A;q)_\infty := \prod_{k=0}^{\infty}(1-Aq^k)$$

with the common abbreviation

$$(A_1, \ldots, A_m; q)_k := (A_1;q)_k \cdots (A_m;q)_k.$$

The *Gaussian polynomials* (or *q-binomial coefficients*) are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2: We call a function $F(n, \mathbf{k})$ *q-proper hypergeometric* over $\mathbb{K}$, if it is of the form

$$F(n,\mathbf{k}) = P(n,\mathbf{k})\,\frac{\prod_s (A_s;q)_{a_s n + \mathbf{b}_s \mathbf{k} + c_s}}{\prod_t (B_t;q)_{u_t n + \mathbf{v}_t \mathbf{k} + w_t}}\; x_0^n x_1^{k_1} \cdots x_r^{k_r}\; q^{\alpha_0\binom{n}{2} + \alpha_1\binom{k_1}{2} + \cdots + \alpha_r\binom{k_r}{2} + \beta(n,\mathbf{k})},$$

where

- $P(n, \mathbf{k}) \in \mathbb{K}[q^n, q^{\mathbf{k}}]$,
- $A_s \in \mathbb{K}$, $B_t \in \mathbb{K}$,
- $a_s \in \mathbb{Z}$, $u_t \in \mathbb{Z}$,
- $\mathbf{b}_s \in \mathbb{Z}^r$, $\mathbf{v}_t \in \mathbb{Z}^r$,
- $c_s$ and $w_t$ are integers (possibly depending on parameters different from $n$ and $k_1, \ldots, k_r$) ,
- $x_0, x_1, \ldots, x_r \in \mathbb{K}$,
- $\alpha_0, \alpha_1, \ldots, \alpha_r \in \mathbb{Z}$, and
- $\beta$ is an integer quadratic form in $n$ and $\mathbf{k}$, i.e., $\beta(y_0, \ldots, y_r) = \sum_{i,j=0}^{r} \beta_{i,j} y_i y_j$, where $\beta_{i,j} \in \mathbb{Z}$.

*Remark:* Note that this definition also includes $q$-shifted factorials of the form

$$(A_s q^{(i_s d_s)n+(i_s \mathbf{e}_s)\mathbf{k}}; q^{i_s})_{a_s n+\mathbf{b}_s\mathbf{k}+c_s} \quad \text{and} \quad (B_t q^{(j_t f_t)n+(j_t \mathbf{g}_t)\mathbf{k}}; q^{j_t})_{u_t n+\mathbf{v}_t\mathbf{k}+w_t}$$

with $d_s, f_t \in \mathbb{Z}$, $\mathbf{e}_s, \mathbf{g}_t \in \mathbb{Z}^r$ and $i_s, j_t \in \mathbb{Z}\setminus\{0\}$, since those terms can be rewritten by using the rules

$$(Aq^d; q)_c = \frac{(A;q)_{d+c}}{(A;q)_d}, \tag{4}$$
$$(A; q^{-1})_c = (A^{-1}; q)_c \, (-A)^c \, q^{-\binom{c}{2}}$$

and

$$(A; q^i)_c = (A_1, A_2, \ldots, A_i; q)_c, \quad i > 1,$$

where the $A_1, A_2, \ldots, A_i$ are the $i$-th complex roots of $A$.

### 2.2. The Rational Equation

Now we will look at the fundamental quotients $F(n-i, \mathbf{k}-\mathbf{j})/F(n, \mathbf{k})$ in the rational equation (2). For this, relation (4) comes in handy.

DEFINITION 2.3: Let $F(n, \mathbf{k})$ be $q$-proper hypergeometric as in Definition 2.2 and let $(i, \mathbf{j}) \in \mathbb{Z}^{r+1}$. We define

$$R_{F,i,\mathbf{j}} := \frac{P(n-i, \mathbf{k}-\mathbf{j})}{P(n, \mathbf{k})} \, \frac{\prod_s (A_s q^{a_s n+\mathbf{b}_s\mathbf{k}+c_s}; q)_{-ia_s-\mathbf{j}\mathbf{b}_s}}{\prod_t (B_t q^{u_t n+\mathbf{v}_t\mathbf{k}+w_t}; q)_{-iu_t-\mathbf{j}\mathbf{v}_t}} \, x_0^{-i} x_1^{-j_1} \cdots x_r^{-j_r}$$
$$\cdot \, q^{\alpha_0\left(\binom{i+1}{2}-in\right)+\alpha_1\left(\binom{j_1+1}{2}-j_1k_1\right)+\cdots+\alpha_r\left(\binom{j_r+1}{2}-j_rk_r\right)+\beta(n-i,\mathbf{k}-\mathbf{j})-\beta(n,\mathbf{k})}.$$

Clearly, $R_{F,i,\mathbf{j}}$ is a rational function in $q^n$ and $q^{\mathbf{k}}$. Note that the $q$-shifted factorials in the numerator of $R_{F,i,\mathbf{j}}$, for which $-ia_s - \mathbf{j}\mathbf{b}_s < 0$ actually contribute to the denominator. Conversely, $q$-shifted factorials in the denominator may go into the numerator. The following result is obvious.

Lemma 2.1: *For $q$-proper hypergeometric $F(n,\mathbf{k})$ as in Definition 2.2 we have*

$$\frac{F(n-i,\mathbf{k}-\mathbf{j})}{F(n,\mathbf{k})} = R_{F,i,\mathbf{j}}(n,\mathbf{k}),$$

*for all $n,\mathbf{k},i,\mathbf{j}$ where the quotient on the left-hand side is well-defined.*

It has been shown by Wegschaider (16) that once the rational equation (2) holds formally, i.e., $\sum \sigma_{i,\mathbf{j}}\, R_{F,i,\mathbf{j}}$ is identically zero in $\mathbb{K}(q^n,q^{\mathbf{k}})$, equation (1) is valid also at points $(n,\mathbf{k})$ where $F(n,\mathbf{k})=0$, a fact that had been neglected in previous investigations. His argumentation of course applies in the $q$-case, too.

### 2.3. The Polynomial Equation

Finally we will now transform the rational equation (2) into the polynomial equation (3). While in the $q=1$ case the corresponding rational functions $R_{F,i,\mathbf{j}}$ are simply quotients of two polynomials, the situation in the $q$-case is different, since here we are faced with quotients of Laurent-polynomials. As an illustrating example consider the expression $t(n,k) := (q;q)_{n-k}$. Then for $j>0$ the quotient

$$\frac{t(n,k-j)}{t(n,k)} = \prod_{i=0}^{j-1}(1-q^{n-k+i})$$

is a Laurent-polynomial in $q^n$ and $q^k$, whereas the corresponding term in the $q=1$ case, $t^*(n,k) := (n-k)!$, leads to

$$\frac{t^*(n,k-j)}{t^*(n,k)} = \prod_{i=0}^{j-1}(n-k+i),$$

which is a polynomial in $n$ and $k$.

As a consequence the degree analysis of the associated polynomial turns out to be much more difficult. Therefore we will first transform equation (2) into a Laurent-polynomial equation by multiplying it with $P(n,\mathbf{k})$ and the least common multiple of the $q$-shifted factorials in the denominators of the rational functions, and canceling the greatest common divisor of the $q$-shifted factorials in the numerators.

As in the $q=1$ case, the points $(i,\mathbf{j})\in S$ for which the numbers $-ia_s-\mathbf{j}\mathbf{b}_s$ and $-iu_t-\mathbf{j}\mathbf{v}_t$ are minimal, respectively maximal, play a special role.

Definition 2.4: Let $F(n,\mathbf{k})$ be $q$-proper hypergeometric as in Definition 2.2 and let $S$ be a structure set.

(i) For fixed $s$, a point $(I,\mathbf{J})\in S$ is called a *numerator boundary point* denoted by $(I_s^{num},\mathbf{J}_s^{num})$, if

$$Ia_s+\mathbf{J}\mathbf{b}_s \geq ia_s+\mathbf{j}\mathbf{b}_s \quad \text{for all } (i,\mathbf{j})\in S.$$

(ii) For fixed $t$, a point $(I, \mathbf{J}) \in S$ is called a *denominator boundary point* denoted by $(I_t^{den}, \mathbf{J}_t^{den})$, if

$$Iu_t + \mathbf{J}\mathbf{v}_t \leq iu_t + \mathbf{j}\mathbf{v}_t \quad \text{for all } (i, \mathbf{j}) \in S.$$

The corresponding Laurent-polynomial can now be explicitly given as follows.

DEFINITION 2.5: Let $F(n, \mathbf{k})$ be $q$-proper hypergeometric as in Definition 2.2 and let $S$ be a structure set. The Laurent-polynomial

$$\begin{aligned} L_{F,S} := \sum_{(i,\mathbf{j})\in S} & \sigma_{i,\mathbf{j}}(n)\, P(n-i, \mathbf{k}-\mathbf{j})\, x_0^{-i} x_1^{-j_1} \cdots x_r^{-j_r} \\ & \cdot \frac{\prod_s (A_s q^{a_s n + \mathbf{b}_s \mathbf{k} + c_s - I_s^{num} a_s - \mathbf{J}_s^{num} \mathbf{b}_s}; q)_{(I_s^{num} - i)a_s + (\mathbf{J}_s^{num} - \mathbf{j})\mathbf{b}_s}}{\prod_t (B_t q^{u_t n + \mathbf{v}_t \mathbf{k} + w_t - I_t^{den} u_t - \mathbf{J}_t^{den} \mathbf{v}_t}; q)_{(I_t^{den} - i)u_t + (\mathbf{J}_t^{den} - \mathbf{j})\mathbf{v}_t}} \\ & \cdot q^{\alpha_0\left(\binom{i+1}{2} - in\right) + \alpha_1\left(\binom{j_1+1}{2} - j_1 k_1\right) + \cdots + \alpha_r\left(\binom{j_r+1}{2} - j_r k_r\right) + \beta(n-i, \mathbf{k}-\mathbf{j}) - \beta(n, \mathbf{k})} \end{aligned} \tag{5}$$

is called the *associated Laurent-polynomial* of $F$ and $S$.

It is easily seen that $L_{F,S}$ is indeed a Laurent-polynomial, because each $q$-shifted factorial in the denominator is actually the reciprocal of a Laurent-polynomial.

THEOREM 2.1: *Let $F(n, \mathbf{k})$ be $q$-proper hypergeometric and let $S$ be a structure set. The rational equation $\sum_{(i,\mathbf{j})\in S} \sigma_{i,\mathbf{j}}(n)\, R_{F,i,\mathbf{j}} = 0$ is equivalent to the Laurent-polynomial equation $L_{F,S} = 0$.*

*Proof:* First we multiply every $R_{F,i,\mathbf{j}}$ with $P(n, \mathbf{k})$. For $q$-shifted factorials in general observe that $(A; q)_{c_1}$ divides $(A; q)_{c_2}$ if $0 \leq c_1 \leq c_2$. Now we fix $s$. To identify the factors of $(A_s q^{a_s n + \mathbf{b}_s \mathbf{k} + c_s}; q)_{-ia_s - \mathbf{j}\mathbf{b}_s}$ that are in all numerators or in the common denominator of the $R_{F,i,\mathbf{j}}$ we distinguish two cases:

- If $-I_s^{num} a_s - \mathbf{J}_s^{num} \mathbf{b}_s \geq 0$ then each $(A_s q^{a_s n + \mathbf{b}_s \mathbf{k} + c_s}; q)_{-ia_s - \mathbf{j}\mathbf{b}_s}$ is in the numerator of the corresponding $R_{F,i,\mathbf{j}}$. Hence, the greatest common factor is $(A_s q^{a_s n + \mathbf{b}_s \mathbf{k} + c_s}; q)_{-I_s^{num} a_s - \mathbf{J}_s^{num} \mathbf{b}_s}$ and can be canceled.
- If $-I_s^{num} a_s - \mathbf{J}_s^{num} \mathbf{b}_s < 0$ then some of the $(A_s q^{a_s n + \mathbf{b}_s \mathbf{k} + c_s}; q)_{-ia_s - \mathbf{j}\mathbf{b}_s}$ contribute to the denominator of $R_{F,i,\mathbf{j}}$. The least common multiple of these Laurent-polynomials is $1/(A_s q^{a_s n + \mathbf{b}_s \mathbf{k} + c_s}; q)_{-I_s^{num} a_s - \mathbf{J}_s^{num} \mathbf{b}_s}$ by which we multiply each $R_{F,i,\mathbf{j}}$.

In both cases the remaining factors of $(A_s q^{a_s n + \mathbf{b}_s \mathbf{k} + c_s}; q)_{-ia_s - \mathbf{j}\mathbf{b}_s}$ in $R_{F,i,\mathbf{j}}$ are

$$(A_s q^{a_s n + \mathbf{b}_s \mathbf{k} + c_s - I_s^{num} a_s - \mathbf{J}_s^{num} \mathbf{b}_s}; q)_{(I_s^{num} - i)a_s + (\mathbf{J}_s^{num} - \mathbf{j})\mathbf{b}_s}.$$

Similar reasoning for the $q$-shifted factorials in the denominator of $F(n, \mathbf{k})$ shows that the remaining factors of $1/(B_t q^{u_t n + \mathbf{v}_t \mathbf{k} + w_t}; q)_{-iu_t - \mathbf{j}\mathbf{v}_t}$ are

$$\frac{1}{(B_t q^{u_t n + \mathbf{v}_t \mathbf{k} + w_t - I_t^{den} u_t - \mathbf{J}_t^{den} \mathbf{v}_t}; q)_{(I_t^{den} - i)u_t + (\mathbf{J}_t^{den} - \mathbf{j})\mathbf{v}_t}}.$$

□

For the final step of transforming the Laurent-polynomial equation $L_{F,S} = 0$ into a polynomial equation, we simply multiply $L$ by powers of $q^{k_1}, \ldots, q^{k_r}$ to eliminate negative exponents.

Definition 2.6: Let $F(n, \mathbf{k})$ be $q$-proper hypergeometric as in Definition 2.2 and let $S$ be a structure set. For $1 \le h \le r$ denote by $m_h$ the minimal power of $q^{k_h}$ occurring in $L_{F,S}$. The polynomial

$$P_{F,S} := L_{F,S} \cdot q^{-m_1 k_1} \cdots q^{-m_r k_r} \tag{6}$$

is called the *associated polynomial* of $F$ and $S$.

The following holds trivially.

Corollary 2.1: *Let $F(n, \mathbf{k})$ be $q$-proper hypergeometric and let $S$ be a structure set. The Laurent-polynomial equation $L_{F,S} = 0$ is equivalent to the polynomial equation $P_{F,S} = 0$.*

As mentioned above, determining the degree of the associated polynomial $P_{F,S}$ is more difficult than in the $q = 1$ case. However, a careful study is needed both for proving the existence of $\mathbf{k}$-free recurrences for $q$-proper hypergeometric functions and for the concept of $P$-maximal structure sets which will be introduced later.

Definition 2.7: For $x \in \mathbb{Z}$ we define $x^+ := \max(x, 0)$ and $x^- := \max(-x, 0)$.

The total degree of the polynomial $P_{F,S}$ is clearly the maximal degree of each of its summands. For the $q$-shifted factorials note that the degree is the sum over the subscripts multiplied with the sum of the positive components of $\mathbf{b}_s$ and $\mathbf{v}_t$, respectively. Hence we find that

$$\begin{aligned}
\deg_{\mathbf{k}} P_{F,S} = \deg_{\mathbf{k}} P(n, \mathbf{k}) \\
&+ \max_{(i,\mathbf{j}) \in S} \Bigg\{ \sum_s \left[ (I_s^{num} - i) a_s + (\mathbf{J}_s^{num} - \mathbf{j}) \mathbf{b}_s \right] \cdot \sum_{h=1}^r b_{s,h}^+ \\
&\qquad - \sum_t \left[ (I_t^{den} - i) u_t + (\mathbf{J}_t^{den} - \mathbf{j}) \mathbf{v}_t \right] \cdot \sum_{h=1}^r v_{t,h}^+ \\
&\qquad + \sum_{h=1}^r \left[ (-\alpha_h j_h) + \beta_{i,\mathbf{j},h} \right] \Bigg\} \\
&+ \sum_{h=1}^r \max_{(i,\mathbf{j}) \in S} \Bigg\{ \sum_s \left[ (I_s^{num} - i) a_s + (\mathbf{J}_s^{num} - \mathbf{j}) \mathbf{b}_s \right] \cdot b_{s,h}^- \\
&\qquad - \sum_t \left[ (I_t^{den} - i) u_t + (\mathbf{J}_t^{den} - \mathbf{j}) \mathbf{v}_t \right] \cdot v_{t,h}^- \\
&\qquad + (\alpha_h j_h)^+ + \beta_{i,\mathbf{j},h}^- \Bigg\},
\end{aligned} \tag{7}$$

where $\beta_{i,\mathbf{j},h}$ denotes the coefficient of $k_h$ in $\beta(n-i, \mathbf{k}-\mathbf{j}) - \beta(n, \mathbf{k})$. Note that each $\beta_{i,\mathbf{j},h}$ is a linear function in $i$ and $\mathbf{j}$ with integer coefficients.

With this degree formula in hands, we can immediately show that for every $q$-proper hypergeometric function there exists a $\mathbf{k}$-free recurrence. We prove the result for rectangular structure sets.

Definition 2.8: For $I \in \mathbb{N}_0$ and $\mathbf{J} \in \mathbb{N}_0^r$ we denote by $S_{I,\mathbf{J}}$ the structure set $\{(i, \mathbf{j}) \in \mathbb{N}_0^{r+1} \mid 0 \le i \le I, 0 \le j_h \le J_h\}$.

Theorem 2.2: *Every $q$-proper hypergeometric function $F(n, \mathbf{k})$ satisfies a $\mathbf{k}$-free recurrence.*

*Proof:* We will show that for any $q$-proper hypergeometric $F(n, \mathbf{k})$ there exist $I \in \mathbb{N}_0$ and $\mathbf{J} \in \mathbb{N}_0^r$, such that the polynomial equation (3) has a solution for the structure set $S = S_{I,\mathbf{J}}$. For that it suffices to show that the number of variables in the corresponding equation system exceeds the number of equations. Without loss of generality we assume that all components of $\mathbf{J}$ are equal to some $J \in \mathbb{N}_0$. Clearly, the number of variables then is equal to $(I+1)(J+1)^r$. On the other hand, the number of equations equals the number of power products $q^{k_1 l_1} \cdots q^{k_r l_r}$ in $P_{F,S}$. But it is well known that the number of power products in $r$ variables of total degree less or equal to $d$ is $\binom{d+r}{r}$. Hence, by observing that for the structure set $S_{I,\mathbf{J}}$ the degree bound (7) is a linear function in $I$ and $J$, say $\gamma I + \delta J + \epsilon$, we only need to show that there exist $I$ and $J$ such that

$$\binom{\gamma I + \delta J + \epsilon + r}{r} < (I+1)(J+1)^r.$$

But this follows immediately from the asymptotic behavior of both functions. □

### 2.4. *P*-Maximal Structure Sets

In this subsection we will deal with the problem of finding minimal structure sets for computing $\mathbf{k}$-free recurrences. As in the $q = 1$ case, the rectangular sets $S_{I,\mathbf{J}}$ usually do not have the right shape, i.e., for many $(i, \mathbf{j}) \in S_{I,\mathbf{J}}$ the $\sigma_{i,\mathbf{j}}$ vanish in the result. As an example we consider a special case of the $q$-Vandermonde identity

$$\sum_{k=0}^{2n} (-1)^k \, q^{(n-k)^2} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 = (-1)^n \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}. \tag{8}$$

The reason for investigating a single-sum identity here, which could be much faster proved by the $q$-analogue of Zeilberger's (18) algorithm (see also Koornwinder (10) or Paule and Riese (13)), is that in this case structure sets can be drawn easily.

For our example it turns out that the smallest rectangular structure set for

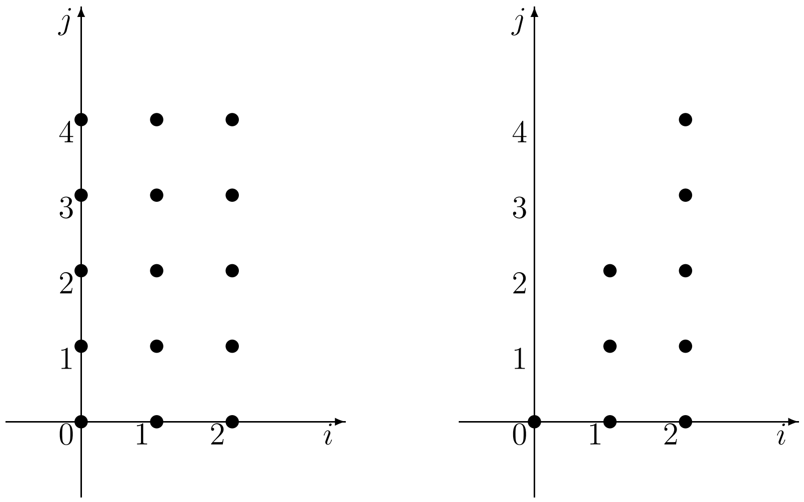


**Figure 1:** The structure sets $S_{2,4}$ and $S_V$.

which a **k**-free recurrence exists, is the set $S_{2,4}$ (cf. Figure 1). Our program outputs the following:

$$\begin{aligned}
&q^{4n+5}(q-q^n)(q+q^n)(q^3-q^{2n})(q-q^{4n})\,F(n-2,k-4)+\\
&q^{2n+2}(1+q)(q-q^n)(q+q^n)(q^3-q^{2n})(q-q^{4n})(q^5+q^{4n})\,F(n-2,k-3)+\\
&(q-q^n)(q+q^n)(q^3-q^{2n})(q-q^{4n})(q^{10}+q^{4n+4}+2\,q^{4n+5}+q^{4n+6}+q^{8n})\\
&\quad F(n-2,k-2)+\\
&q^{2n+2}(1+q)(q-q^n)(q+q^n)(q^3-q^{2n})(q-q^{4n})(q^5+q^{4n})F(n-2,k-1)+\\
&q^{4n+5}(q-q^n)(q+q^n)\,(q^3-q^{2n})(q-q^{4n})\,F(n-2,k)-\\
&q^{2n+8}(q^3-q^{4n})(-2q^3+q^{2n}+q^{2n+1}+q^{2n+2}+q^{2n+3}-2q^{4n})\,F(n-1,k-2)-\\
&q^6(1+q)(q^3-q^{4n})(q^6+q^{2n+4}+q^{2n+5}-q^{4n+1}-2q^{4n+2}-2q^{4n+4}-\\
&\qquad q^{4n+5}+q^{6n+1}+q^{6n+2}+q^{8n})\,F(n-1,k-1)-\\
&q^{2n+8}(q^3-q^{4n})(-2q^3+q^{2n}+q^{2n+1}+q^{2n+2}+q^{2n+3}-2q^{4n})\,F(n-1,k)+\\
&q^9(1-q^n)(1+q^n)(q-q^{2n})(q^5-q^{4n})\,F(n,k)=0.
\end{aligned}$$

Note that in this recurrence for some of the $(i,j)\in S_{2,4}$ we have that $\sigma_{i,j}(n)=0$. If we delete those elements from $S_{2,4}$ we are led to a smaller structure set $S_V$ which is shown on the right of Figure 1. From algorithmic point of view it is clear that starting with $S_V$ instead of $S_{2,4}$ results in a significant speedup, since $S_{2,4}$ contains 15 points whereas $S_V$ consists of only 9 points. The corresponding degree of the associated polynomial drops from 16 to 8 and consequently we solve a $9\times 9$ system in 8 seconds instead of a $17\times 15$ system in 90 seconds.

Fortunately Wegschaider's approach to computing structure sets of this type, also called *P-maximal structure sets*, can be carried over to the $q$-case. The underlying existence theory was originally introduced by Verbaeten (14) (see also Hornegger (9)) for single-sums in the $q = 1$ case. Since it is based on arguments from plane geometry, there is no direct generalization to multi-sums. Nevertheless, as Wegschaider (16) pointed out, $P$-maximal structure sets can be computed also in this situation.

The basic idea is to start with a small rectangular structure set $S_{I,\mathbf{J}}$ and then to add all those points $(i, \mathbf{j})$ that do not increase the degree of the associated polynomial. This way the number of equations in the underlying linear system remains the same, whereas we maximize the number of unknowns. This procedure is also known as *Verbaeten completion*, which in the $q = 1$ case amounts to solving one system of linear inequalities over the integers. However, it will become clear in the following that $P$-maximal structure sets in the $q$-case are the union of many such solution sets. More precisely, in the worst case we have to solve $2^{2r}$ systems of linear inequalities.

We will now construct our inequalities for fixed $S_{I,\mathbf{J}}$. For this we look at the first maximum in (7) which can be rewritten as

$$\sum_s \left(I_s^{num} a_s + \mathbf{J}_s^{num}\mathbf{b}_s\right) \cdot \sum_{h=1}^r b_{s,h}^+ - \sum_t \left(I_t^{den} u_t + \mathbf{J}_t^{den}\mathbf{v}_t\right) \cdot \sum_{h=1}^r v_{t,h}^+$$
$$+ \max_{(i,\mathbf{j})\in S} \Bigg\{ - \sum_s \left(ia_s + \mathbf{j}\mathbf{b}_s\right) \cdot \sum_{h=1}^r b_{s,h}^+ + \sum_t \left(iu_t + \mathbf{j}\mathbf{v}_t\right) \cdot \sum_{h=1}^r v_{t,h}^+ + \sum_{h=1}^r \left(-\alpha_h j_h + \beta_{i,\mathbf{j},h}\right) \Bigg\}.$$

From the first two sums we obtain for each $s$ and $t$ with $\sum_h b_{s,h}^+ \neq 0$ and $\sum_h v_{t,h}^+ \neq 0$,

$$ia_s + \mathbf{j}\mathbf{b}_s \le \max_{(i,\mathbf{j})\in S_{I,\mathbf{J}}} ia_s + \mathbf{j}\mathbf{b}_s \quad \text{and} \quad -iu_t - \mathbf{j}\mathbf{v}_t \le \max_{(i,\mathbf{j})\in S_{I,\mathbf{J}}} -iu_t - \mathbf{j}\mathbf{v}_t,$$

respectively, and from the maximum we get

$$i\Big( - \sum_s a_s \cdot \sum_{h=1}^r b_{s,h}^+ + \sum_t u_t \cdot \sum_{h=1}^r v_{t,h}^+ + \beta_0\Big)$$
$$+ \mathbf{j}\Big( - \sum_s \mathbf{b}_s \cdot \sum_{h=1}^r b_{s,h}^+ + \sum_t \mathbf{v}_t \cdot \sum_{h=1}^r v_{t,h}^+ - \boldsymbol{\alpha} + \boldsymbol{\beta}\Big) \le \max_{(i,\mathbf{j})\in S_{I,\mathbf{J}}} \text{idem}, \qquad (9)$$

where $\beta_0$ is the coefficient of $i$ in $\sum_h \beta_{i,\mathbf{j},h}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r)$ with $\beta_l$ denoting the coefficient of $j_l$ in $\sum_h \beta_{i,\mathbf{j},h}$, and idem is an abbreviation for the expression on the left-hand side.

Up to here the situation is quite analogous to the $q = 1$ case. The difference comes with the last $r$ maxima in (7). After rewriting them as we did above, we obtain for each $s$ and $t$ with $\sum_h b_{s,h}^- \neq 0$ and $\sum_h v_{t,h}^- \neq 0$,

$$ia_s + \mathbf{j}\mathbf{b}_s \le \max_{(i,\mathbf{j})\in S_{I,\mathbf{J}}} ia_s + \mathbf{j}\mathbf{b}_s \quad \text{and} \quad -iu_t - \mathbf{j}\mathbf{v}_t \le \max_{(i,\mathbf{j})\in S_{I,\mathbf{J}}} -iu_t - \mathbf{j}\mathbf{v}_t,$$

respectively. The remaining inequalities are for each $h$

$$i\Big(-\sum_s a_s\, b^-_{s,h} + \sum_t u_t\, v^-_{t,h} - \beta_0^{(h)}\Big) \\ +\mathbf{j}\Big(-\sum_s \mathbf{b}_s\, b^-_{s,h} + \sum_t \mathbf{v}_t\, v^-_{t,h} + \boldsymbol{\alpha}^{(h)} - \boldsymbol{\beta}^{(h)}\Big) \le \max_{(i,\mathbf{j})\in S_{I,\mathbf{J}}} \text{idem}, \qquad (10)$$

where $\boldsymbol{\alpha}^{(h)} = (0,\dots,0,\alpha_h^{(h)},0,\dots,0)$ with $\alpha_h^{(h)} = \alpha_h$ if $\alpha_h j_h > 0$, and $\alpha_h^{(h)} = 0$ otherwise. Similarly, $\beta_0^{(h)}$ is the coefficient of $i$ in $\beta_{i,\mathbf{j},h}$ if $\beta_{i,\mathbf{j},h} < 0$ and $\beta_0^{(h)} = 0$ otherwise, and $\boldsymbol{\beta}^{(h)} = (\beta_1^{(h)},\dots,\beta_r^{(h)})$, where $\beta_l^{(h)}$ is the coefficient of $j_l$ in $\beta_{i,\mathbf{j},h}$ if $\beta_{i,\mathbf{j},h} < 0$ and $\beta_l^{(h)} = 0$ otherwise. From this it is clear that we have to distinguish $2^r \cdot 2^r = 2^{2r}$ cases depending on the signs of all $j_h$ and $\beta_{i,\mathbf{j},h}$, if all $\alpha_h$ and $\beta_{i,\mathbf{j},h}$ are non-zero.

For the $q$-Vandermonde identity (8) above, the Verbaeten completion of the structure set $S_{2,0}$ yields the following. First of all observe that we have $\alpha_1 = -1$, $\beta(n,k) = -2nk$, and $\beta(n-i,k-j) - \beta(n,k) = 2jn + 2ik - 2ij$. From the $q$-shifted factorials in the summand and relation (9) we obtain the inequalities

$$I_0 = \{-j \le 0, -2i + j \le 0\} \cup \{i \le 2\}.$$

Depending on the sign of $j$ and $\beta_{i,j,1} = 2i$ we obtain four systems of inequalities from (10), namely

$$\begin{aligned} I_1 &= I_0 \cup \{j \le -1, 2i \le -1, i - j \le 2\},\\ I_2 &= I_0 \cup \{j \le -1, -2i \le 0, 2i - j \le 4\},\\ I_3 &= I_0 \cup \{-j \le 0, 2i \le -1, i \le 2\},\\ I_4 &= I_0 \cup \{-j \le 0, -2i \le 0, i \le 2\}. \end{aligned}$$

The $P$-maximal structure set containing $S_{2,0}$ is then the union of all solutions of $I_1$, $I_2$, $I_3$ and $I_4$, which in this case is just the solution of $I_4$ (see Figure 2, where the original points of $S_{2,0}$ are the black ones and the points added by the Verbaeten completion are the white ones). Of course we actually do not have to consider $I_1$ and $I_2$, since $-j \le 0$ from $I_0$ and $j < 0$ will always lead to an empty solution set. Note that in this example we have exactly found the set $S_V$. In general $P$-maximal structure sets may contain superfluous points.

Summarizing, the advantages of Sister Celine's technique together with Verbaeten completion are evident. On the one hand the size of the equation system to be solved is much smaller and on the other hand the number of structure sets that we have to try until we find a solution is also smaller.

Nevertheless, **k**-free recurrences computed this way are still very large and in many applications we cannot find them in reasonable time. Therefore we will improve Sister Celine's technique once more in Section 4.

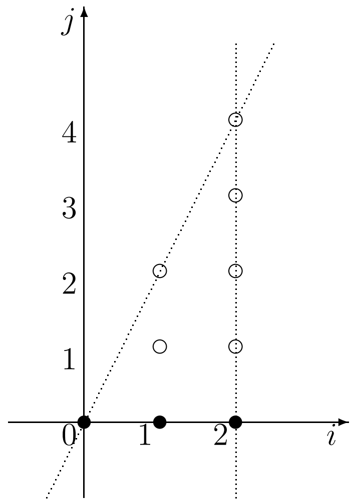


**Figure 2:** The Verbaeten completion of the structure set $S_{2,0}$.

## 3. Certificate Recurrences

So far we have seen how to compute **k**-free recurrences for $q$-proper hypergeometric summands $F(n, \mathbf{k})$. However, our final goal is to compute a recurrence for the sum itself, i.e., we have to transform a **k**-free recurrence into an appropriate form for summation.

For this we define the *forward difference operators* as

$$\Delta_n := (N - 1) \quad \text{and} \quad \Delta_{k_h} := (K_h - 1),$$

where 1 denotes the identity operator.

Let us return to the operator notation for **k**-free recurrences,

$$\left( \sum_{(i,\mathbf{j}) \in S} \sigma_{i,\mathbf{j}}(n)\, N^{-i} \mathbf{K}^{-\mathbf{j}} \right) F(n, \mathbf{k}) = 0. \tag{11}$$

To eliminate negative exponents of the shift operators, we multiply equation (11) by suitable powers of $N$ and $\mathbf{K}$ to obtain

$$P(n, N, \mathbf{K})\, F(n, \mathbf{k}) = 0,$$

where $P(n, N, \mathbf{K})$ is from the (non-commutative) ring of the **k**-free polynomial recurrence operators

$$\mathbb{K}[q^n]\langle N, \mathbf{K} \rangle := \left\{ \sum_{(i,\mathbf{j}) \in S} \sigma_{i,\mathbf{j}}(n)\, N^i\, \mathbf{K}^{\mathbf{j}} \;\middle|\; \sigma_{i,\mathbf{j}}(n) \in \mathbb{K}[q^n] \text{ and } S \subset \mathbb{N}_0^{r+1} \text{ finite} \right\}.$$

A certificate recurrence operator (see Wegschaider (16)) is then defined as follows.

Definition 3.1: A *certificate recurrence operator* over $\mathbb{K}$ is an element from $\mathbb{K}[q^n, q^{\mathbf{k}}]\langle N, \mathbf{K}\rangle$ of the form

$$P(n,N) + \sum_{h=1}^{r} \Delta_{k_h} S_h(n, \mathbf{k}, N, \mathbf{K}),$$

where $P \in \mathbb{K}[q^n]\langle N\rangle$ and $S_h \in \mathbb{K}[q^n, q^{\mathbf{k}}]\langle N, \mathbf{K}\rangle$. We call $P$ the *principal part* of the operator.

Note that a certificate recurrence for $F(n, \mathbf{k})$ has the appropriate form for summation. Suppose that the summand has finite support, i.e., for all $n$ there exist finite integer intervals $I_{n,h}$ such that $F(n, \mathbf{k}) = 0$ for $k_h \notin I_{n,h}$. Then by summing over the certificate recurrence, the $\Delta$-parts telescope and the principal part yields a recurrence for the sum $\mathrm{SUM}(n) := \sum_{\mathbf{k}} F(n, \mathbf{k})$,

$$P(n,N)\ \mathrm{SUM}(n) = 0.$$

Wilf and Zeilberger (17) proved that any $\mathbf{k}$-free recurrence can be transformed into a certificate recurrence by first dividing the recurrence operator by $(K_1 - 1)$, the remainder by $(K_2 - 1)$, and so on. We omit the proof here.

Theorem 3.1: *Let $P(n, N, \mathbf{K})$ be a $\mathbf{k}$-free recurrence operator in $\mathbb{K}[q^n]\langle N, \mathbf{K}\rangle$ that annihilates $F(n, \mathbf{k})$. Then there exists a non-zero certificate recurrence operator that annihilates $F(n, \mathbf{k})$.*

Wegschaider (16, Thm. 3.2) introduced a "non-commutative trick" to always end up with certificate recurrence operators that have non-trivial principal parts. This trick could be carried over to the $q$-case, however, since in the following section we will present a more efficient generalization of Sister Celine's technique, for which this trick is no longer applicable, we do not go into the details here.

To illustrate the process of transforming a $\mathbf{k}$-free recurrence into a certificate recurrence, we consider a special case of the $q$-binomial theorem,

$$\sum_{k=-n}^{n} (-1)^k\, q^{\binom{k}{2}} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q = \delta_{n,0}.$$

Our program computes the following $\mathbf{k}$-free recurrence for the summand (where backward shifts have already been transformed into forward shifts):

$$-q^n\, F(n,k) + (1 + q^{2n+1})\, F(n, k+1) - q^{n+1}\, F(n, k+2) - F(n+1, k+1) = 0.$$

Hence, the corresponding recurrence operator is given by

$$P(n, N, K) = -q^n + (1 + q^{2n+1})K - q^{n+1}K^2 - NK.$$

Dividing the recurrence operator by $(K-1)$ we obtain

$$P(n,N,K) = (1-q^n)(1-q^{n+1}) - N + (K-1)\big[(1-q^{n+1}+q^{2n+1}) - q^{n+1}K - N\big].$$

Finally we apply this operator to $F$ and sum over all $k \in \mathbb{Z}$ to find that $\mathrm{SUM}(n)$ satisfies the recurrence

$$(1-q^n)(1-q^{n+1})\ \mathrm{SUM}(n) - \mathrm{SUM}(n+1) = 0.$$

Algorithmically, dividing a recurrence operator by $(K-1)$ can be achieved by additions only, since

$$\sum_{j=0}^{J} a_j K^j = (K-1)\Bigg(\sum_{j=0}^{J-1}\sum_{i=j+1}^{J} a_i K^j\Bigg) + \sum_{j=0}^{J} a_j.$$

From this we see that, if no remainder vanishes, the principal part of the certificate recurrence operator equals $\sum_i \sum_{\mathbf{j}} \sigma_{i,\mathbf{j}}\, N^i$, or in other words

$$P(n,N) = P(n,N,\mathbf{1}).$$

## 4. A Generalization of Sister Celine's Technique

Certificate recurrences computed from **k**-free recurrences have the property that not only the principal part is **k**-free but also the $\Delta$-parts, which is not necessary at all. Wegschaider (16) made the important observation that looking for **k**-dependent recurrences that yield certificate recurrences with **k**-free principal parts only, dramatically improves the performance of Sister Celine's technique. His approach also works in the $q$-case.

To see this let

$$\sum_{(i,\mathbf{j})\in S} \sigma_{i,\mathbf{j}}(n,\mathbf{k})\, N^{-i}\, \mathbf{K}^{-\mathbf{j}} \tag{12}$$

be a **k**-dependent recurrence operator, where the $\sigma_{i,j}$ are now polynomials in $q^n$ and $q^{\mathbf{k}}$. In the process of transforming this operator into a certificate recurrence operator, the last remainder equals (see Wegschaider (16))

$$\sum_{(i,\mathbf{j})\in S} \sigma_{i,\mathbf{j}}(n,\mathbf{k}+\mathbf{j})\, N^i.$$

Therefore, we have to guarantee that for all $i$

$$\sum_{\mathbf{j}\in S(i)} \sigma_{i,\mathbf{j}}(n,\mathbf{k}+\mathbf{j}) \in \mathbb{K}[q^n], \tag{13}$$

where $S(i) = \{\mathbf{j} \mid (i,\mathbf{j}) \in S\}$. Algorithmically we proceed by making an Ansatz of the form (12), where

$$\sigma_{i,\mathbf{j}}(n,\mathbf{k}) = \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{d}_{i,\mathbf{j}}} \sigma_{i,\mathbf{j},\mathbf{l}}(n)\, q^{\mathbf{lk}}.$$

Here the $\mathbf{d}_{i,\mathbf{j}}$ are the degree bounds for the undetermined polynomials that have to be specified as additional input. Usually we set all of them to a single constant. To fulfill condition (13) we have to guarantee that for every $i$

$$\sum_{\mathbf{j}\in S(i)} \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{d}_{i,\mathbf{j}}} \sigma_{i,\mathbf{j},\mathbf{l}}(n)\, q^{\mathbf{l}(\mathbf{k}+\mathbf{j})} \in \mathbb{K}[q^n].$$

By comparing the coefficients of every non-trivial power product $q^{\mathbf{lk}}$ with zero, we obtain several dependencies between certain $\sigma_{i,\mathbf{j},\mathbf{l}}(n)$. This means that some $\sigma_{i,\mathbf{j},\mathbf{l}}(n)$, the so-called *reducible* unknowns, can be expressed as linear combinations of the remaining unknowns. Finally we replace the reducible unknowns by these linear combinations and solve the reduced Ansatz. From this we obtain a **k**-dependent recurrence, which can be transformed into a certificate recurrence with **k**-free principal part. However, we have no guarantee that this principal part is non-trivial.

To illustrate the fact that recurrences found this way can be significantly simpler, we again consider the $q$-Vandermonde identity (8) from Subsection 2.4. A **k**-dependent recurrence of degree 1 is, for instance,

$$\begin{aligned}
&q\,(q^{2n+1} + q^{4n} + q^{4n+1} - 3q^{6n} - 2q^{k+1} + 2q^{4n+k})\, F(n-1, k-2) + \\
&(q^3 + q^{2n+2} + q^{2n+3} - 3q^{4n+1} - 3q^{4n+2} + q^{6n} + q^{6n+1} + \\
&\qquad q^{8n} + 2q^{k+3} - 2q^{2n+k+1} - 2q^{4n+k+2} + 2q^{6n+k})\, F(n-1, k-1) + \\
&q^{2n+1}(-3q + q^{2n} + q^{2n+1} + q^{4n} + 2q^{k+1} - 2q^{4n+k})\, F(n-1, k) - \\
&q^2(1-q^n)(1+q^n)(q-q^{2n})\, F(n,k) = 0.
\end{aligned}$$

Note that this recurrence is now of order 1 with respect to $n$. The corresponding certificate recurrence equals

$$\begin{aligned}
&(1-q^{2n+1})^2(1+q^{2n+1})(1+q^{2n+2})\, F(n,k) - \\
&(1-q^{n+1})(1+q^{n+1})(1-q^{2n+1})\, F(n+1,k) + \\
&\Delta_k\big(p_1(n,k)\, F(n,k) + p_2(n,k)\, F(n,k+1) + p_3(n,k)\, F(n+1,k) + \\
&\qquad p_4(n,k)\, F(n+1,k+1)\big) = 0,
\end{aligned}$$

where we did not spell out the polynomials $p_1(n,k), \ldots, p_4(n,k)$ which all have degree 1 w.r.t. $q^k$. Finally the whole sum $\mathrm{SUM}(n)$ satisfies the recurrence

$$(1-q^{2n+1})(1+q^{2n+1})(1+q^{2n+2})\,\mathrm{SUM}(n) - (1-q^{n+1})(1+q^{n+1})\,\mathrm{SUM}(n+1) = 0.$$

It is easily seen that also $(-1)^n \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}$ satisfies this recurrence. Since the initial values of both sides agree for $n = 0$, the proof of identity (8) is complete.

# 5. The Mathematica Implementation

In this section we will describe the usage of the author's Mathematica package qMultiSum for proving $q$-hypergeometric multi-sum identities. The syntax is very close to Wegschaider's package MultiSum, in particular we decided to use the same names for the basic functions prefixed with a "q". The source is contained in the file qMultiSum.m, which can be read in by typing

```
In[1]:= <<qMultiSum.m

Out[1]= Axel Riese's qMultiSum implementation version 2.1 loaded
```

### 5.1. The Function qFindRecurrence

The most important function is qFindRecurrence which computes recurrences for $q$-proper hypergeometric functions. The calling syntax is

qFindRecurrence[*summand*, *recvars*, *sumvars*, *recdims*, *sumdims*, *degbounds*, *opts*],

where the parameters *degbounds* and *opts* are optional. As with all basic functions, there exists an abbreviation for qFindRecurrence, namely qFR.

Here *summand* is a $q$-proper hypergeometric function $F(n_1, \ldots, n_s, k_1, \ldots, k_r)$ over $\mathbb{Q}(q, \tau_1, \ldots, \tau_m)$ as in Definition 2.2 and the remark below, where in addition we admit more than one recurrence variable, which sometimes is of advantage as will be shown in the following section. The $q$-shifted factorial $(A; q^i)_a$ has to be typed as qPochhammer[$A$,q^$i$,$a$]. In addition we allow terms of the form qBrackets[$a$,q] for $[a]_q := (1 - q^a)/(1 - q)$, qFactorial[$a$,q] for $[a]_q! := [1]_q[2]_q \cdots [a]_q$, and qBinomial[$a$,$b$,q] for $\begin{bmatrix} a \\ b \end{bmatrix}_q$, where also for these terms powers of $q$ are admitted.

The parameters *recvars* and *sumvars* (both lists of Mathematica symbols) denote the recurrence variables $n_1, \ldots, n_s$ and the summation variables $k_1, \ldots, k_r$, respectively. In case of only one recurrence variable, *recvars* may be specified as a scalar.

By *recdims* and *sumdims* (lists of $s$, respectively $r$ non-negative integers) the dimensions of a rectangular structure set have to be specified. Recurrences are then computed over the Verbaeten completion of this set. In case of only one recurrence variable, *recdims* may be specified as a scalar.

If the optional parameter *degbounds* is omitted then $\mathbf{k}$-free recurrences are computed. Otherwise, if *degbounds* is a list of $r$ non-negative integers, then the generalization of Sister Celine's technique as described in Section 4 is invoked, where the degree of the coefficients in the recurrences w.r.t. $q^{k_h}$ is bounded by *degbound*$_h$. If all degree bounds should be the same, *degbounds* may simply be set to a non-negative integer.

By the parameter *opts* additional options can be specified. With qWZ->True recurrences are computed over the full rectangular structure set given by *recdims* and *sumdims* instead of the Verbaeten completion of this set. With

`OnlyStructSet->True`, only the structure set is computed and returned, which is useful for experimenting with the parameters. With `StructSet->`$S$, a structure set $S$ can be specified explicitly, where $S$ is a list of $s+r$ dimensional lists of integers. Note that in this case no Verbaeten completion is performed and the values of *recdims* and *sumdims* are ignored. With `qProtocol->True`, the program prints additional debugging information. Finally, with `EquationSolver->`$NS$, the function $NS$ is used for computing the nullspace of a matrix instead of the function provided with the package.

`qFindRecurrence` returns a recurrence or a list of recurrences for the input function *summand*, which is referred to as `F[`$n_1$`,...,`$n_s$`,`$k_1$`,...,`$k_r$`]` in the output.

As an introductory example we consider the double-sum case of the $q$-multinomial theorem (see, for instance, Gasper and Rahman (8))

$$\sum_i \sum_j \frac{(x;q)_i\,(y;q)_j\,(z;q)_{n-i-j}}{(q;q)_i\,(q;q)_j\,(q;q)_{n-i-j}}\,y^i\,z^{i+j} = \frac{(xyz;q)_n}{(q;q)_n}.$$

The smallest **k**-free recurrence for the summand we can find with our program is the following (actually we find two recurrences, but since they only differ by a shift in $n$ we show only one here).

```
In[2]:= qFindRecurrence[qPochhammer[x,q,i] qPochhammer[y,q,j] *
                        qPochhammer[z,q,n-i-j] y^i z^(i+j) /
                        (qPochhammer[q,q,i] qPochhammer[q,q,j] *
                         qPochhammer[q,q,n-i-j]),
                        n, {i,j}, 0, {1,1}] // First
```

Out[2]=

$$-\left(y\,z^2\,(-q^4+q^n\,x\,y\,z)\,F(-4+n,-1+i,-1+j)\right) -$$
$$q\,y\,(q^3-q^n\,x\,y)\,z^2\,F(-3+n,-1+i,-1+j) -$$
$$q\,y\,z\,(q^3-q^n\,x\,z)\,F(-3+n,-1+i,j) - q\,z\,(q^3-q^n\,y\,z)\,F(-3+n,i,-1+j) +$$
$$q^2\,(q^2-q^n\,x)\,y\,z\,F(-2+n,-1+i,j) + q^2\,(q^2-q^n\,y)\,z\,F(-2+n,i,-1+j) +$$
$$q^2\,(q^2-q^n\,z)\,F(-2+n,i,j) - q^3\,(q-q^n)\,F(-1+n,i,j) = 0$$

By computing **k**-dependent recurrences we are able to decrease the order of the recurrence with respect to $n$ but not the size of the initial structure set.

```
In[3]:= qFindRecurrence[qPochhammer[x,q,i] qPochhammer[y,q,j] *
                        qPochhammer[z,q,n-i-j] y^i z^(i+j) /
                        (qPochhammer[q,q,i] qPochhammer[q,q,j] *
                         qPochhammer[q,q,n-i-j]),
                        n, {i,j}, 0, {1,1}, {0,1}] // First
```

Out[3]=

$$-\big(q^n\,x\,y\,(-q+q^j\,y)\,z^2\,F(-2+n,-1+i,-1+j)\big)-$$
$$y\,z\,(-q^{2+j}+q^n\,x\,z)\,F(-2+n,-1+i,j)-$$
$$(q-q^j\,y)\,z\,(-q+q^n\,x\,y\,z)\,F(-2+n,i,-1+j)-$$
$$q^{1+j}\,(q-q^n\,x)\,y\,z\,F(-1+n,-1+i,j)-q\,(q-q^j\,y)\,z\,F(-1+n,i,-1+j)-$$
$$q\,(q-q^n\,z-q^n\,x\,y\,z+q^{j+n}\,x\,y\,z)\,F(-1+n,i,j)-q^2\,(-1+q^n)\,F(n,i,j)=0$$

### 5.2. Miscellaneous Functions

The function `qRecurrenceToCertificate` transforms a recurrence (or a list of recurrences) computed by `qFindRecurrence` into the corresponding certificate recurrence(s). The calling syntax is

`qRecurrenceToCertificate[`*rec*`,` *s*`]`,

where $s$, the number of recurrence variables, is optional with default value 1. The abbreviation for `qRecurrenceToCertificate` is `qRC`.

For the $q$-multinomial theorem above we obtain the following:

In[4]:= `qRecurrenceToCertificate[%]`

Out[4]=

$$\Delta_i\big(q^2\,(-1+q^j\,y)\,z\,(-1+q^{1+n}\,x\,y\,z)\,F(n,i,j)+q^2\,(-1+q^j\,y)\,z\,F(1+n,i,j)-$$
$$q^2\,(1-q^{1+n}\,z-q^{1+n}\,x\,y\,z+q^{2+j+n}\,x\,y\,z)\,F(1+n,i,1+j)-$$
$$(-1+q^{2+n})\,F(2+n,i,1+j)\big)+$$
$$\Delta_j\big(-(q^2\,y\,z\,(-q^j+q^n\,x\,z)\,F(n,i,j))+q^2\,(-1+q^{1+n}\,z-q^j\,y\,z+q^{1+n}\,x\,y\,z)$$
$$F(1+n,i,j)-q^2\,(-1+q^{2+n})\,F(2+n,i,j)\big)-$$
$$q^2\,z\,(-1+q^n\,x\,y\,z)\,F(n,i,j)+q^2\,(-1-z+q^{1+n}\,z+q^{1+n}\,x\,y\,z)\,F(1+n,i,j)-$$
$$q^2\,(-1+q^{2+n})\,F(2+n,i,j)=0$$

The function

`qSumCertificate[`*certrec*`,` *s*`]`

computes a recurrence for the multi-sum $\sum_{\mathbf{k}} F(\mathbf{n},\mathbf{k})$ from a certificate recurrence (or a list of certificate recurrences) for $F$ computed by `qRecurrence-ToCertificate` under the assumption that $F$ has finite support. Again $s$ is optional with default value 1. In the output `SUM[`$n_1$`,...,`$n_s$`]` denotes the sum. The abbreviation for `qSumCertificate` is `qSC`.

In[5]:= `qSumCertificate[%]`

Out[5]=

$$-\big(z\,(-1+q^n\,x\,y\,z)\,\mathrm{SUM}(n)\big)+(-1-z+q^{1+n}\,z+q^{1+n}\,x\,y\,z)\,\mathrm{SUM}(1+n)+$$
$$(1-q^{2+n})\,\mathrm{SUM}(2+n)=0$$

If one is not interested in the certificate recurrence but only in the recurrence for the sum, the last two steps can be computed faster by calling

qSumRecurrence[*rec*, *s*],

where $s$ is optional with default value 1. The abbreviation for qSumRecurrence is qSR.

```
In[6]:= qSumRecurrence[%3]
```

Out[6]=

$$-\big(z\,(-1+q^{n}\,x\,y\,z)\,\mathrm{SUM}(n)\big)+(-1-z+q^{1+n}\,z+q^{1+n}\,x\,y\,z)\,\mathrm{SUM}(1+n)+(1-q^{2+n})\,\mathrm{SUM}(2+n)=0$$

The functions

BackwardShifts[*rec*]

and

ForwardShifts[*rec*]

transform a recurrence (or a list of recurrences) computed by one of the previously described functions into recurrences involving backward, respectively forward shifts only, for instance:

```
In[7]:= BackwardShifts[%]
```

Out[7]=

$$-\big(z\,(-1+q^{-2+n}\,x\,y\,z)\,\mathrm{SUM}(-2+n)\big)+(-1-z+q^{-1+n}\,z+q^{-1+n}\,x\,y\,z)\,\mathrm{SUM}(-1+n)+(1-q^{n})\,\mathrm{SUM}(n)=0$$

Finally,

qCheckRecurrence[*rec*, *F*]

checks whether the function $F$ satisfies the recurrence *rec*, where *rec* is a recurrence computed by one of the previously described functions. Again, *rec* may be a list of recurrences. The abbreviation for qCheckRecurrence is qCR.

```
In[8]:= qCheckRecurrence[%, qPochhammer[x y z,q,n] / qPochhammer[q,q,n]]

Out[8]= True
```

## 6. Applications

In this section we shall present several computer proofs derived with our package. Due to limitation of space we will omit the recurrences for the summands and only show the recurrences for the sums. Also checking the initial values for the identities is left to the reader in most cases. The timings refer to tests on an SGI Octane using Mathematica 3.0.1.

### 6.1. Two Summation Theorems for $U(n)$ Basic Hypergeometric Series

We begin with the double-sum case of Milne's (11, Thm. 5.52) fourth terminating $U(n+1)$ refinement of the $q$-binomial theorem,

$$\sum_{\substack{y_1,\dots,y_n\geq 0\\ y_1+\dots+y_n\leq N}} \prod_{1\leq r<s\leq n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} (q^{-N};q)_{y_1+\dots+y_n} \prod_{r,s=1}^{n} \left(q\,\frac{x_r}{x_s};q\right)_{y_r}^{-1} q^{y_2+2y_3+\dots+(n-1)y_n}\, z^{y_1+\dots+y_n} = (z\,q^{-N};q)_N.$$

```
In[2]:= qFR[(1 - x1/x2 q^(y1-y2)) / (1 - x1/x2) *
          qPochhammer[q^(-N),q,y1+y2] /
          (qPochhammer[q,q,y1] qPochhammer[q,q,y2] *
           qPochhammer[q x1/x2,q,y1] qPochhammer[q x2/x1,q,y2]) *
          q^y2 z^(y1+y2),
          N, {y1,y2}, 2, {0,0}, {2,0}] // qSR // Timing
```

Out[2]=

$$\left\{3.42 \text{ Second},\ (q^{1+N}-z)\,\mathrm{SUM}(N) - q^{1+N}\mathrm{SUM}(1+N) = 0\right\}$$

```
In[3]:= qCR[%[[2]], qPochhammer[z q^(-N),q,N]]

Out[3]= True
```

Next we consider the double-sum case of Milne's (11, Thm. 5.10) first $U(n+1)$ generalization of the $q$-Chu-Vandermonde summation theorem,

$$\sum_{\substack{0\leq y_i\leq N_i\\ i=1,2,\dots,n}} \prod_{1\leq r<s\leq n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \prod_{r,s=1}^{n} \frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\,\frac{x_r}{x_s};q)_{y_r}} \prod_{i=1}^{n} \left(\frac{x_i}{x_n}\,c;q\right)_{y_i}^{-1} (b;q)_{y_1+\dots+y_n}\, q^{y_1+2y_2+\dots+ny_n} = b^{N_1+\dots+N_n} \prod_{i=1}^{n} \frac{\left(\frac{x_i}{x_n}\,c/b;q\right)_{N_i}}{\left(\frac{x_i}{x_n}\,c;q\right)_{N_i}}.$$

For this example it turns out that computing a recurrence in both $N_1$ and $N_2$ is much more efficient than for only one recurrence variable. Also note that, for sake of simplicity, we abbreviated $x_1/x_2$ by $X$.

```
In[4]:= qFR[(1 - X q^(y1-y2)) / (1 - X) qPochhammer[q^(-N1),q,y1] *
          qPochhammer[q^(-N2),q,y2] qPochhammer[X q^(-N2),q,y1] *
          qPochhammer[q^(-N1)/X,q,y2] /
          (qPochhammer[q,q,y1] qPochhammer[q,q,y2] *
           qPochhammer[q X,q,y1] qPochhammer[q/X,q,y2] *
           qPochhammer[X c,q,y1] qPochhammer[c,q,y2]) *
          qPochhammer[b,q,y1+y2] q^(y1+2y2),
          {N1,N2}, {y1,y2}, {0,0}, {1,1}, 1] // qSR[#, 2]& // Timing
```

Out[4]=

$$\{58.25\ \text{Second},$$
$$q^{\text{N2}}\,(-1+q^{1+\text{N1}})\,(q^{1+\text{N2}}-q^{\text{N1}}\,X)\,(-2b+c\,q^{\text{N1}}\,X)\,(-1+q^{1+\text{N1}}\,X)$$
$$(-q^{\text{N2}}+q^{1+\text{N1}}\,X)\,\text{SUM}(\text{N1},1+\text{N2})+$$
$$q^{\text{N1}+\text{N2}}\,(-1+q^{1+\text{N1}})\,(-1+c\,q^{1+\text{N2}})\,X\,(-1+q^{1+\text{N1}}\,X)\,(-q^{\text{N2}}+q^{1+\text{N1}}\,X)$$
$$\text{SUM}(\text{N1},2+\text{N2})-$$
$$q^{\text{N1}}\,(-2b+c\,q^{\text{N2}})\,(-1+q^{1+\text{N2}})\,(q^{1+\text{N2}}-X)\,(q^{1+\text{N2}}-q^{\text{N1}}\,X)\,(-q^{\text{N2}}+q^{1+\text{N1}}\,X)$$
$$\text{SUM}(1+\text{N1},\text{N2})-$$
$$(-q^{\text{N2}}+q^{\text{N1}}\,X)\,(-2q^{1+2\text{N2}}+q^{2+\text{N1}+2\,\text{N2}}+c\,q^{2+\text{N1}+3\,\text{N2}}+q^{3+\text{N1}+3\,\text{N2}}+$$
$$2b\,q^{3+\text{N1}+3\text{N2}}-c\,q^{3+\text{N1}+4\text{N2}}-c\,q^{4+\text{N1}+4\,\text{N2}}+q^{\text{N1}+\text{N2}}\,X-q^{1+\text{N1}+\text{N2}}\,X+$$
$$2q^{2+\text{N1}+\text{N2}}\,X+q^{2+2\text{N1}+\text{N2}}\,X+c\,q^{1+\text{N1}+2\text{N2}}\,X+q^{2+\text{N1}+2\text{N2}}\,X-$$
$$c\,q^{1+2\text{N1}+2\text{N2}}\,X-q^{2+2\text{N1}+2\text{N2}}\,X-2b\,q^{2+2\text{N1}+2\text{N2}}\,X-c\,q^{2+2\text{N1}+2\text{N2}}\,X-$$
$$q^{3+2\text{N1}+2\text{N2}}\,X-2c\,q^{3+2\text{N1}+2\text{N2}}\,X-2q^{4+2\text{N1}+2\text{N2}}\,X-2b\,q^{4+2\text{N1}+2\text{N2}}\,X+$$
$$c\,q^{2+\text{N1}+3\text{N2}}\,X+c\,q^{2+2\text{N1}+3\text{N2}}\,X+c\,q^{4+2\text{N1}+3\text{N2}}\,X+c\,q^{5+2\text{N1}+3\text{N2}}\,X-$$
$$2q^{1+2\text{N1}}\,X^2+c\,q^{1+2\text{N1}+\text{N2}}\,X^2+q^{2+2\text{N1}+\text{N2}}\,X^2+c\,q^{2+3\text{N1}+\text{N2}}\,X^2+$$
$$q^{3+3\text{N1}+\text{N2}}\,X^2+2b\,q^{3+3\text{N1}+\text{N2}}\,X^2-c\,q^{1+2\text{N1}+2\text{N2}}\,X^2-c\,q^{2+2\text{N1}+2\text{N2}}\,X^2-$$
$$2c\,q^{3+2\text{N1}+2\text{N2}}\,X^2+c\,q^{2+3\text{N1}+2\text{N2}}\,X^2+c\,q^{4+3\text{N1}+2\text{N2}}\,X^2+c\,q^{5+3\text{N1}+2\text{N2}}\,X^2+$$
$$c\,q^{2+3\text{N1}+\text{N2}}\,X^3-c\,q^{3+4\text{N1}+\text{N2}}\,X^3-c\,q^{4+4\text{N1}+\text{N2}}\,X^3)\,\text{SUM}(1+\text{N1},1+\text{N2})+$$
$$q^{2+\text{N1}+\text{N2}}\,(-1+c\,q^{1+\text{N2}})\,(q^{1+\text{N2}}-q^{\text{N1}}\,X)\,(-q^{\text{N2}}+q^{\text{N1}}\,X)\,(-q^{\text{N2}}+q^{1+\text{N1}}\,X)$$
$$\text{SUM}(1+\text{N1},2+\text{N2})-$$
$$q^{\text{N1}+\text{N2}}\,(-1+q^{1+\text{N2}})\,(q^{1+\text{N2}}-X)\,(q^{1+\text{N2}}-q^{\text{N1}}\,X)\,(-1+c\,q^{1+\text{N1}}\,X)$$
$$\text{SUM}(2+\text{N1},\text{N2})+$$
$$q^{2+\text{N1}+\text{N2}}\,(q^{1+\text{N2}}-q^{\text{N1}}\,X)\,(-q^{\text{N2}}+q^{\text{N1}}\,X)\,(-q^{\text{N2}}+q^{1+\text{N1}}\,X)\,(-1+c\,q^{1+\text{N1}}\,X)$$
$$\text{SUM}(2+\text{N1},1+\text{N2})=0\}$$

```
In[5]:= qCR[%[[2]], b^(N1+N2) qPochhammer[X c/b,q,N1] *
                    qPochhammer[c/b,q,N2] /
                    (qPochhammer[X c,q,N1] qPochhammer[c,q,N2])]

Out[5]= True
```

## 6.2. A Bailey Pair Identity

Here we consider the case $k = 3$ of a terminating version of Andrews' analytic counterpart of Gordon's partition theorem (see, e.g., Andrews (4; 5), Paule (12), or Warnaar (15))

$$\sum_{L=0}^{n}(-1)^L a^{kL} q^{kL^2+\binom{L}{2}} \frac{(1-aq^{2L})\,(a;q)_L}{(1-a)\,(q;q)_L\,(q;q)_{n-L}\,(aq;q)_{n+L}}$$
$$= \sum_{n\geq n_1\geq\cdots\geq n_{k-1}\geq 0} \frac{a^{n_1+\cdots+n_{k-1}} q^{n_1^2+\cdots+n_{k-1}^2}}{(q;q)_{n-n_1}\,(q;q)_{n_1-n_2}\ldots(q;q)_{n_{k-1}-n_{k-1}}\,(q;q)_{n_k-1}}.$$

Verbaeten completion for the single-sum on the left-hand side leads to a high-order recurrence, which happens quite frequently for sums of this type. Therefore we choose a rectangular structure set.

```
In[2]:= qFR[(-1)^L a^(3L) q^(3L^2+Binomial[L,2]) (1 - a q^(2L)) *
        qPochhammer[a,q,L] /
        ((1 - a) qPochhammer[q,q,L] qPochhammer[q,q,n-L] *
         qPochhammer[a q,q,n+L]),
        n, {L}, 3, {2}, 2, qWZ->True] // qSR // Timing
```

Out[2]=

$$\{390.93\ \text{Second},$$
$$-(q^3\,\mathrm{SUM}(n)) + q\,(1+q+q^2-q^{3+n}+a\,q^{4+2n}+a\,q^{5+2n})\,\mathrm{SUM}(1+n) +$$
$$(-1-q-q^2+q^{3+n}+q^{4+n}-a\,q^{5+2n}-a\,q^{6+2n}+a\,q^{8+3n}-a^2\,q^{10+4n})$$
$$\mathrm{SUM}(2+n) + (1-q^{3+n})\,\mathrm{SUM}(3+n) = 0\}$$

Once again we want to emphasize that this recurrence could be computed much faster (i.e., within a few seconds) by the $q$-Zeilberger algorithm. The recurrence for the double-sum on the right-hand side can be obtained immediately.

```
In[3]:= qFR[a^(i+j) q^(i^2+j^2) / (qPochhammer[q,q,n-i] *
        qPochhammer[q,q,i-j] qPochhammer[q,q,j]),
        n, {i,j}, 3, {0,1}, 1] // qSR // First // Timing
```

Out[3]=

$$\{4.93\ \text{Second},$$
$$-(q^3\,\mathrm{SUM}(n)) + q\,(1+q+q^2-q^{3+n}+a\,q^{4+2n}+a\,q^{5+2n})\,\mathrm{SUM}(1+n) +$$
$$(-1-q-q^2+q^{3+n}+q^{4+n}-a\,q^{5+2n}-a\,q^{6+2n}+a\,q^{8+3n}-a^2\,q^{10+4n})$$
$$\mathrm{SUM}(2+n) + (1-q^{3+n})\,\mathrm{SUM}(3+n) = 0\}$$

### 6.3. Identities Related to Göllnitz's Big Partition Theorem

In their work on colored partitions Alladi, Andrews and Gordon (1, Thm. 2) came up with the identity

$$\sum_{\substack{i=a+ab+ac\\ j=b+ab+bc\\ k=c+ac+bc}} \frac{q^{T_t+T_{ab}+T_{ac}+T_{bc-1}}\left(1-q^a(1-q^{bc})\right)}{(q;q)_a\,(q;q)_b\,(q;q)_c\,(q;q)_{ab}\,(q;q)_{ac}\,(q;q)_{bc}} = \frac{q^{T_i+T_j+T_k}}{(q;q)_i\,(q;q)_j\,(q;q)_k}, \quad (14)$$

where $T_m = m(m+1)/2$ and $t = a+b+c+ab+ac+bc$. Note that, for instance, $ab$ stands for a symbol and not for $a \cdot b$. Clearly the left-hand side denotes a triple-sum. If we choose as summation variables $ab, ac, bc$ then these are our constraints:

```
In[2]:= a = i-ab-ac; b = j-ab-bc; c = k-ac-bc;
        t = a+b+c+ab+ac+bc;
        T[m_] := m(m+1)/2;
```

Our program computes the following:

```
In[3]:= qFR[q^(T[t]+T[ab]+T[ac]+T[bc-1]) (1 - q^a (1-q^bc)) /
            (qPochhammer[q,q,a] qPochhammer[q,q,b] qPochhammer[q,q,c] *
             qPochhammer[q,q,ab] qPochhammer[q,q,ac] *
             qPochhammer[q,q,bc]),
            {i,j,k}, {ab,ac,bc}, {0,0,1}, {1,0,0}, {2,1,1}][[13]] //
         qSR[#, 3]& // Timing
```

Out[3]=

$$\left\{285.25 \text{ Second}, \; -q^{1+j}\,\mathrm{SUM}(i,j,k) + (1-q^{1+j})\,\mathrm{SUM}(i,1+j,k) = 0\right\}$$

```
In[4]:= qCR[%[[2]], q^(T[i]+T[j]+T[k]) /
                    (qPochhammer[q,q,i] qPochhammer[q,q,j] *
                     qPochhammer[q,q,k])]

Out[4]= True
```

The proof is complete after checking the initial case $j = 0$, which could easily be done algorithmically again or by using a result due to Alladi and Gordon (3).

Alladi and Berkovich (2, (1.1)) recently derived a double bounded version of (14),

$$\sum_{\substack{i=a+ab+ac\\ j=b+ab+bc\\ k=c+ac+bc}} q^{T_t+T_{ab}+T_{ac}+T_{bc}} \begin{bmatrix} L-t+a \\ a \end{bmatrix}_q \begin{bmatrix} L-t+b \\ b \end{bmatrix}_q \begin{bmatrix} M-t+c \\ c \end{bmatrix}_q \begin{bmatrix} L-t \\ ab \end{bmatrix}_q \begin{bmatrix} M-t \\ ac \end{bmatrix}_q \begin{bmatrix} M-t \\ bc \end{bmatrix}_q +$$
$$\sum_{\substack{i=a+ab+ac\\ j=b+ab+bc\\ k=c+ac+bc}} q^{T_t+T_{ab}+T_{ac}+T_{bc-1}} \begin{bmatrix} L-t+a-1 \\ a-1 \end{bmatrix}_q \begin{bmatrix} L-t+b \\ b \end{bmatrix}_q \begin{bmatrix} M-t+c \\ c \end{bmatrix}_q \begin{bmatrix} L-t \\ ab \end{bmatrix}_q \begin{bmatrix} M-t \\ ac \end{bmatrix}_q \begin{bmatrix} M-t \\ bc-1 \end{bmatrix}_q$$
$$= \sum_{s\geq 0} q^{s(M+2)-T_s+T_{i-s}+T_{j-s}+T_{k-s}} \begin{bmatrix} L-s \\ s \end{bmatrix}_q \begin{bmatrix} L-2s \\ i-s \end{bmatrix}_q \begin{bmatrix} L-i-s \\ j-s \end{bmatrix}_q \begin{bmatrix} M-i-j \\ k-s \end{bmatrix}_q, \tag{15}$$

employing the slightly modified definition of the $q$-binomial coefficients

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q := \begin{cases} \frac{(q^{m+1};q)_n}{(q;q)_n}, & \text{if } n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Applying our package to this identity reveals several phenomena. First of all, Verbaeten completion for the triple-sum misses the minimal structure set by far. However, Wegschaider's implementation — after setting $q = 1$ — is able to find it, and surprisingly the same set also works in the $q$-case:

```
In[5]:= summand = q^(T[t]+T[ab]+T[ac]+T[bc-1]) *
                   qBinomial[L-t+a,a,q] qBinomial[L-t+b,b,q] *
                   qBinomial[M-t+c,c,q] qBinomial[L-t,ab,q] *
                   qBinomial[M-t,ac,q] qBinomial[M-t,bc,q] *
                   (q^bc + (1-q^a)/(1-q^(L-t+a)) *
                           (1-q^bc)/(1-q^(M-t-bc+1)));

In[6]:= qFR[summand, {L,M,i,j}, {ab,ac,bc}, {0,0,0,0}, {0,0,0},
              StructSet -> {{0,0,0,0,0,0,0}, {1,0,0,0,0,0,0},
                            {1,1,0,1,0,0,0}, {1,1,1,0,0,0,0},
                            {2,1,1,1,1,0,0}, {2,2,1,1,0,0,0}}] //
            qSR[#, 4]& // BackwardShifts // Timing

Out[6]=
```

$$\{5.89\ \text{Second},$$
$$-(q^{-1+2L}\,\mathrm{SUM}(-2+L,-2+M,-1+i,-1+j)) +$$
$$q^L\,\mathrm{SUM}(-2+L,-1+M,-1+i,-1+j) + q^L\,\mathrm{SUM}(-1+L,-1+M,-1+i,j) +$$
$$q^L\,\mathrm{SUM}(-1+L,-1+M,i,-1+j) + \mathrm{SUM}(-1+L,M,i,j) - \mathrm{SUM}(L,M,i,j) = 0\}$$

Note that this recurrence is identical with the one found by Alladi and Berkovich (2, (2.9)). Of course one would expect that the same recurrence could be computed easily by our program also for the single-sum on the right-hand side of (15). However, this is not true, not even for $q = 1$. Nevertheless, with $q$-hypergeometric telescoping (also known as the $q$-Gosper algorithm; see, for instance, Paule and Riese (13)) we can prove algorithmically within a few seconds that the single-sum satisfies this recurrence, too.

Finally we show that both sides of (15) fulfill the same boundary conditions. For this we denote the single-sum by $p_{i,j,k}(L,M)$ and the triple-sum by $g_{i,j,k}(L,M)$. Clearly, if one of the parameters $i, j, k$ is negative it follows that

$$p_{i,j,k}(L,M) = g_{i,j,k}(L,M) = 0.$$

Therefore, following Alladi and Berkovich, it suffices to show that

$$p_{i,j,k}(i+j-1,M) = g_{i,j,k}(i+j-1,M).$$

From the single-sum we immediately read off the relation

$$p_{i,j,k}(i+j-1, M) = \delta_{i,0}\,\delta_{j,0}\,q^{T_k} \begin{bmatrix} \Delta \\ k \end{bmatrix}_q,$$

where $\Delta = M - i - j$. Hence our boundary identity to verify becomes

$$g_{i,j,k}(i+j-1, M) = \delta_{i,0}\,\delta_{j,0}\,q^{T_k} \begin{bmatrix} \Delta \\ k \end{bmatrix}_q. \tag{16}$$

The reason for switching from $M$ to $\Delta$ here is that proving identity (16) for $\Delta = 0$ is easy, whereas for $M = 0$ it is not at all.

Once again we look at the $q = 1$ case to find a structure set of reasonable size:

```
In[7]:= qFR[summand /. {L -> i+j-1, M -> Δ+i+j},
            {Δ,i,j,k}, {ab,ac,bc}, {0,0,0,0}, {0,0,0},
            StructSet -> {{0,0,0,0,0,0,0}, {0,1,1,1,1,0,0},
                          {1,0,0,0,0,0,0}, {1,0,0,1,0,0,0},
                          {1,0,1,1,0,0,0}, {1,0,1,1,0,0,1},
                          {1,1,0,1,0,0,0}, {1,1,0,1,0,1,0},
                          {1,1,1,1,0,0,0}, {1,1,1,2,0,1,1}}] //
          qSR[#, 4]& // Timing

Out[7]=
```

$$\begin{aligned}\{&26.73 \text{ Second},\\ &q^{5+2\Delta+2i+2j}\,\mathrm{SUM}(\Delta, i, j, k) + q^{4+\Delta+3i+3j}\,\mathrm{SUM}(\Delta, i, j, 1+k) -\\ &q^{3+\Delta+i+j}\,(-1+q^{1+i+j})\,\mathrm{SUM}(\Delta, i, 1+j, 1+k) -\\ &q^{3+\Delta+i+j}\,(-1+q^{1+i+j})\,\mathrm{SUM}(\Delta, 1+i, j, 1+k) +\\ &q^{3+\Delta+i+j}\,\mathrm{SUM}(\Delta, 1+i, 1+j, 1+k) + \mathrm{SUM}(\Delta, 1+i, 1+j, 2+k) -\\ &q^{4+\Delta+2i+2j}\,\mathrm{SUM}(1+\Delta, i, j, 1+k) - \mathrm{SUM}(1+\Delta, 1+i, 1+j, 2+k) = 0\}\end{aligned}$$

Obviously $\delta_{i,0}\,\delta_{j,0}\,q^{T_k}\left[\begin{smallmatrix}\Delta\\k\end{smallmatrix}\right]_q$ is a solution of this recurrence. Note that once we have proved the validity of (16) for $\Delta = 0$, which could be done with qMultiSum again or follows immediately from (2, (3.7)), our recurrence implies the validity both for $\Delta \geq 0$ and $\Delta \leq 0$.

We want to remark that in a similar way our package has also successfully proved a triple bounded version of (14); see Berkovich and Riese (7).

Finally, we consider another formula related to Göllnitz's big partition theorem. In (2, (5.6)) Alladi and Berkovich stated the identity

$$\sum_{l=0}^{L} a^{-l}\,\frac{1+a^{2l+1}}{1+a}\,q^{T_l} = \sum_{i,j,k} a^{i-j}(-1)^k q^{T_i+T_j+T_k} \begin{bmatrix} L-k \\ i \end{bmatrix}_q \begin{bmatrix} L-i \\ j \end{bmatrix}_q \begin{bmatrix} L-j \\ k \end{bmatrix}_q, \tag{17}$$

a generalized polynomial version of Jacobi's formula

$$\sum_{l\geq 0}(-1)^l(2l+1)\,q^{T_l}=(q;q)_\infty^3.$$

For the right-hand side of (17) we obtain the following recurrence of order 4:

```
In[8]:= Clear[a]

In[9]:= qFR[a^(i-j) (-1)^k q^(T[i]+T[j]+T[k]) qBinomial[L-k,i,q] *
            qBinomial[L-i,j,q] qBinomial[L-j,k,q],
            L, {i,j,k}, 2, {0,0,0}] // qSR // Timing

Out[9]=
```

$$\begin{aligned}&\{17.34\ \text{Second},\\ &-\left(a\,q^{9+3L}\,\mathrm{SUM}(L)\right)+q^{7+2L}\,(1-a+a^2+a\,q^{2+L})\,\mathrm{SUM}(1+L)-\\ &(1-a+a^2)\,q^{4+L}\,(-1+q^{3+L})\,\mathrm{SUM}(2+L)+\\ &(-a-q^{4+L}+a\,q^{4+L}-a^2\,q^{4+L})\,\mathrm{SUM}(3+L)+a\,\mathrm{SUM}(4+L)=0\}\end{aligned}$$

Now we plug in the left-hand side of (17):

```
In[10]:= Simplify[%[[2]] /. SUM[L + m_] :> SUM[L] +
                   Sum[a^(-l) (1+a^(2l+1))/(1+a) q^T[l], {l,L+1,L+m}]]

Out[10]= True
```

### 6.4. A Generalization of the Pentagonal Number Theorem

Recently, Andrews (6) came up with a generalization of Euler's Pentagonal Number Theorem whose proof relies on verifying the triple sum identity

$$\sum_{i,j,k}(-1)^{i+j+k}q^{\binom{i+j+k}{2}}\begin{bmatrix}2m\\ m+i\end{bmatrix}_q\begin{bmatrix}2n\\ n+j\end{bmatrix}_q\begin{bmatrix}2p\\ p+k\end{bmatrix}_q$$
$$=\frac{(q;q)_{2m}\,(q;q)_{2n}\,(q;q)_{2p}}{(q;q)_{m+n-p}\,(q;q)_{m+p-n}\,(q;q)_{p+n-m}}.$$

Also here the computation of the structure set causes problems mainly due to the summand's symmetry. But, as P. Paule observed, in situations like this it is often of advantage to uncouple parameters. For instance, if we substitute $l$ for $i+j+k$ in order to destroy the symmetry manually, we succeed quite fast:

```
In[2]:= qFR[(-1)^l q^Binomial[l,2] qBinomial[2m,m+i,q] *
            qBinomial[2n,n+j,q] qBinomial[2p,p+l-i-j,q],
            {m,n}, {i,j,l}, {2,2}, {0,0,0}, {0,0,1}] //
          qSR[#, 2]& // Timing
```

Out[2]=

$$\{21.98\ \text{Second},\ -\big((-1+q^{1+m})\,(1+q^{1+m})\,(-1+q^{1+2m})\,\text{SUM}(m,1+n)\big) + (-1+q^{1+n})\,(1+q^{1+n})\,(-1+q^{1+2n})\,\text{SUM}(1+m,n) + q^2\,(1+q)\,(q^m-q^n)\,(q^m+q^n)\,\text{SUM}(1+m,1+n) + \text{SUM}(1+m,2+n) - \text{SUM}(2+m,1+n) = 0\}$$

```
In[3]:= qCR[%[[2]], qPochhammer[q,q,2m] qPochhammer[q,q,2n] *
                    qPochhammer[q,q,2p] / (qPochhammer[q,q,m+n-p] *
                    qPochhammer[q,q,m+p-n] qPochhammer[q,q,p+n-m])]

Out[3]= True
```

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