

Integration in finite terms for Liouvillian functions

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Introduction

Computing integrals is a common task in many areas of science, antiderivatives are one way to accomplish this. In differential algebra the problem of elementary integration in finite terms can be stated as follows. Given a differential field (F, D) and $f \in F$, compute in finitely many steps g from some elementary extension of (F, D) such that $Dg = f$ if such a g exists.

This problem has been solved for various classes of fields F . For rational functions $(C(x), \frac{d}{dx})$ such a g always exists and algorithms to compute it are known already for a long time. In 1969 Risch [4] published an algorithm that solves this problem when (F, D) is a transcendental elementary extension of $(C(x), \frac{d}{dx})$. Later this has been extended towards integrands being Liouvillian functions by Singer et. al. [5] via the use of regular log-explicit extensions of $(C(x), \frac{d}{dx})$. Also Bronstein [1, 2] and several other authors published related results. Our algorithm extends this to handling transcendental Liouvillian extensions (F, D) of $(C, 0)$ directly without the need to embed them into log-explicit extensions. For example, this means that $\int (z-x)x^{z-1}e^{-x}dx = x^ze^{-x}$ can be computed without including $\log(x)$ in the differential field F .

Before turning to the main results recall the following definitions of the notions used.

Definition: A differential field $(F, D) = (C(t_1, \dots, t_n), D)$ is called a regular Liouvillian extension of its constant field $\text{Const}(F)$, if

1. all t_i are algebraically independent over C ,
2. $\text{Const}(F) = C$, and
3. each t_i is a Liouvillian monomial over $F_i := C(t_1, \dots, t_{i-1})$, i.e. either
 - (a) $Dt_i \in F_i$, in this case t_i is called primitive over F_i , or
 - (b) $\frac{Dt_i}{t_i} \in F_i$, in this case t_i is called hyperexponential over F_i .

Definition: A differential field $(F(t_1, \dots, t_n), D)$ is called an elementary extension of the differential subfield (F, D) if each t_i is elementary over $F_i := F(t_1, \dots, t_{i-1})$, i.e.

1. t_i is algebraic over F_i , or
2. $Dt_i = \frac{Df}{f}$ for some $f \in F_i$ (i.e. t_i is a logarithm of f), or
3. $\frac{Dt_i}{t_i} = Df$ for some $f \in F_i$ (i.e. t_i is an exponential of f).

We say that $f \in F$ has an elementary integral over F if there exists an elementary extension (E, D) of (F, D) and $g \in E$ such that

$$Dg = f.$$

Parametric integration

We present a decision procedure for the following parametric variant of the problem of integration in finite terms.

Problem: parametric elementary integration in finite terms

Given (F, D) a regular Liouvillian extension of its subfield of constants C and $f_0, \dots, f_m \in F$.

Compute in finitely many steps a vector space basis of all $(c_0, \dots, c_m) \in C^{m+1}$ such that the linear combination $c_0f_0 + \dots + c_mf_m$ has an elementary integral over F , together with corresponding g 's from some elementary extension of F such that

$$c_0f_0 + \dots + c_mf_m = Dg.$$

The algorithm follows the general recursive structure of its precursors proceeding through the transcendental extensions one by one. Integrands from $F := K(t_n)$ are reduced to integrands from the differential subfield $K = C(t_1, \dots, t_{n-1})$ and at the same time parts of the integral are computed as follows.

Structure of one step in the recursive reduction algorithm

input integrands from $K(t_n)$

1. Hermite Reduction for reducing the denominators
2. Residue Criterion for computing part of integral in elementary extensions of the form

$$\sum_{i=1}^k \sum_{Q_i(\alpha)=0} \alpha \log(S_i(\alpha, t_n))$$

where $Q_i \in C[z]$ and $S_i \in K[z, t_n]$

3. for integrating reduced integrands from $K\langle t_n \rangle := \{ \frac{a}{b} \in K(t_n) \mid a, b \in K[t_n], b \nmid Db \}$ compute degree bounds and coefficients by solving auxiliary problems in K remaining integrands are from K

Then a refined version of Liouville's theorem has to be used for reducing the question of having an elementary integral over F to having an elementary integral over K . Thereby the original problem is reduced to a problem of the same type but in a smaller field. A special case of the following theorem is already implicitly contained in [5]. When dealing with non-elementary extensions this naturally leads to a parametric version of the problem as above even when we started with just one single integrand.

Theorem:

Assume t is transcendental over (K, D) and $C := \text{Const}(K(t)) = \text{Const}(K)$. Let $f \in K$ such that f has an elementary integral over $K(t)$, then the following statements hold.

1. If t is elementary over K , then f has an elementary integral over K .
2. If t is primitive over K , then there exists a $c \in C$ such that $f - cDt$ has an elementary integral over K .
3. If t is hyperexponential over K , then there exists a $c \in C$ such that $f - c\frac{Dt}{t}$ has an elementary integral over K .

Above refinement is crucial to obtain a decision procedure for Liouvillian extensions. Without it some elementary integrals, e.g. like the first of the examples below, would not be found.

Examples

Some examples of integrals that can be computed by our algorithm (for comparison: the current versions of Maple and Mathematica can compute the first example only, whereas Maxima only succeeds on the second):

- Let $F = \mathbb{Q}(t_1, t_2, t_3)$, where $Dt_1 = 1, Dt_2 = \frac{1}{t_1}, Dt_3 = \frac{1}{t_2}$. Then t_3 represents the logarithmic integral $\text{li}(x) = \int_0^x \frac{1}{\log(t)} dt$. The algorithm detects that $c_0 \frac{(t_1+1)^2}{t_1 t_2} + c_1 t_3$, $c_0, c_1 \in C$, has an elementary integral over F only for $c_0 = c_1$.

$$\int \frac{(x+1)^2}{x \log(x)} + \text{li}(x) dx = (x+2)\text{li}(x) + \log(\log(x))$$

- Let $F = \mathbb{Q}(t_1, t_2, t_3, t_4)$, where $Dt_1 = 1, Dt_2 = \frac{1}{t_1-1}, Dt_3 = -\frac{t_2}{t_1}, Dt_4 = \frac{t_3}{t_1}$. Here the polylogarithms $\text{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t} dt$ and $\text{Li}_3(x) = \int_0^x \frac{\text{Li}_2(t)}{t} dt$ are represented by t_3 and t_4 respectively.

$$\int \frac{\text{Li}_3(x) - x\text{Li}_2(x)}{(1-x)^2} dx = \frac{x}{1-x} (\text{Li}_3(x) - \text{Li}_2(x)) + \frac{\log(1-x)^2}{2}$$

- Let $F = \mathbb{Q}(a, b)(t_1, t_2, t_3, t_4)$, where $Dt_1 = 1, Dt_2 = a\frac{t_2}{t_1}, Dt_3 = b\frac{t_3}{1-t_1}, Dt_4 = \frac{t_2 t_3}{t_1(1-t_1)}$. Then t_4 represents the incomplete Beta function $B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$.

$$\int \frac{(a+b)x-a}{x^{a+1}(1-x)^{b+1}} B_x(a, b) dx = \frac{B_x(a, b)}{x^a(1-x)^b} + \log\left(\frac{1-x}{x}\right)$$

Properties of the algorithm

In some sense our algorithm can be viewed as unification of the algorithms presented in [5, Theorem A1] and [2]: On the one hand it is a full decision procedure for parametric elementary integration over transcendental Liouvillian extensions. On the other hand it also minimizes the computations done in algebraic extensions and tries to avoid factorization into irreducibles as much as possible.

From the algorithmic point of view the main improvement compared to the other algorithms is in how the necessary restrictions for the linear combinations of the integrands are determined during phase 2 of the recursion displayed on the left. To this end [5] relies on irreducible factorization of the denominator in $\overline{CK}[t_n]$ with subsequent partial fraction decomposition. Whereas the algorithm for the single-integrand case given in [2] – a generalization of [3] – avoids computing unnecessary algebraic extensions and complete factorization, but does not carry over to the parametric case. However, reformulating the Rothstein-Trager resultant appropriately we obtained an algorithm which is parametric, eliminates the need for full factorization, and reduces computations in algebraic extensions as can be seen from the theorem below.

Theorem:

Assume t is transcendental over (K, D) , $Dt \in K[t]$, $C := \text{Const}(K(t)) = \text{Const}(K)$ and that we can find a basis for the constant solutions of linear systems with coefficients from K . Let $a_0, \dots, a_m, b \in K[t]$ with $b \neq 0$ and $\text{gcd}(b, Db) = 1$, then using modular inversion in $K[t]$ we can compute a vector space basis of all $(c_0, \dots, c_m) \in C^{m+1}$ such that

$$h + \frac{c_0 a_0 + \dots + c_m a_m}{b} \in K(t) \text{ has an elementary integral over } K(t) \text{ for some } h \in K\langle t \rangle.$$

The last phase of one step in the recursive algorithm requires solving auxiliary problems such as the parametric logarithmic derivative problem and the parametric Risch differential equation. For this the parametric logarithmic derivative heuristic from [2] has been turned into a decision procedure along the idea sketched in [4] for solving the following variant of parametric integration where in addition the integrals are required to be expressible as logarithms of radicals of field elements. This subproblem is the only part of the algorithm where factorization into irreducibles is still required.

Parametric logarithmic derivative problem:

Given (F, D) a regular Liouvillian extension of its subfield of constants C and $f_0, \dots, f_m \in F$.

Compute in finitely many steps a vector space basis of all $(c_0, \dots, c_m) \in \mathbb{Q}^{m+1}$ such that the linear combination $c_0f_0 + \dots + c_mf_m$ is the logarithmic derivative of an F -radical, together with corresponding $g \in F$ and $k \in \mathbb{Z}^*$ such that

$$c_0f_0 + \dots + c_mf_m = \frac{Dg}{kg}.$$

Conclusion

We developed an algorithm suitable for solving indefinite integration problems with Liouvillian integrands. That algorithm has some computationally desirable properties such as avoiding unnecessary algebraic extensions. The extension of this algorithm towards definite integration is currently being explored, especially for the case when additional parameters are involved.

References

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