Rational general solutions of first order non-autonomous parametric ODEs *

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Abstract

In this paper we study the non-autonomous algebraic ODE $F(x, y, y') = 0$ with a birational parametrization $\mathcal{P}(s, t)$ of the corresponding algebraic surface $F(x, y, z) = 0$. Using this parametrization we drive to a system of ODEs in two parameters s, t of order 1 and of degree 1. We prove the correspondence of a rational general solution of the equation $F(x, y, y') = 0$ and a rational general solution of the new system of ODEs in s, t.

Key words: Rational general solutions, first order non-autonomous ODE, rational surfaces, parametric curves.

1 Introduction

In [Hub96], the general solutions of non-autonomous algebraic ODE $F(x, y, y') = 0$ is studied by giving a method to compute a basis of the general solution of this equation and applied the result to study the local behaviour of the solutions in the neighborhood of a singular solution.

The rational general solutions of first order autonomous algebraic ODEs $F(y, y') = 0$ is well studied by R. Feng and X-S. Gao in current papers [FG04], [FG06]. In fact, $F(y, y')$ is supposed to be a first order non-zero differential polynomial with coefficients in $\mathbb{Q}$ and irreducible over $\overline{\mathbb{Q}}$. One of the key observations in these papers is that a nontrivial rational solution of $F(y, y') = 0$ defines a proper rational parametrization of the corresponding algebraic curve $F(y, z) = 0$. Conversely, if a proper rational parametrization of the algebraic curve $F(y, z) = 0$ satisfies certain conditions, then we can create a rational solution of $F(y, y') = 0$ from this parametrization. Moreover, from a nontrivial rational solution $y(x)$ of $F(y, y') = 0$, we can immediately create a rational general solution by shifting the variable x by an arbitrary constant c, namely $y(x + c)$ is a rational general solution of $F(y, y') = 0$. Therefore, the class of autonomous algebraic ODE $F(y, y') = 0$ with rational general solutions is certainly a

*This work has been supported by the Austrian Science Foundation (FWF) via the project DK11

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subclass of the class of rational algebraic curves. Moreover, the problem of computing a rational general solution is reduced to the problem of computing a nontrivial rational solution. This approach has a great advantage because one can use the theory of rational algebraic curves, which is well-known in [Wal78], [SWD08], to study the nature of rational solutions of first order autonomous algebraic ODEs. For instance, the degree of a nontrivial rational solution is exactly equal to the degree of y' in the differential equation $F(y, y') = 0$ ([FG04], [FG06]).

In this paper we consider a non-autonomous algebraic ODE $F(x, y, y') = 0$ and propose a way to compute a rational general solution of this equation under certain conditions. In fact, one way to consider the non-autonomous differential equation $F(x, y, y') = 0$ is to consider the corresponding surface $\mathcal{S}$ defined by the equation $F(x, y, z) = 0$. Then a nontrivial rational solution $y = f(x)$ of the equation $F(x, y, y') = 0$ defines a parametric curve $(x, f(x), f'(x))$ on the corresponding surface $\mathcal{S}$. Suppose that $\mathcal{S}$ is a rational algebraic surface. Using the birational map $\mathcal{P}(s, t)$, which parametrizes the surface $\mathcal{S}$, we define a new system of differential equations in two indeterminates s, t. Then we prove that a rational general solution $(\bar{s}(x), \bar{t}(x))$ of this system will generate a rational general solution of the original differential equation. Note that this system consists of two differential equations of order 1 in the parameters s, t and degree 1 with respect to s' and t'.

2 Preliminaries

In this section we recall the notion of rational general solutions of algebraic ODEs, in particular for the case of first order (one can find the general setting in [Rit50]). Let $\mathcal{K} = \mathbb{Q}(x)$ be the differential field of rational functions in x with usual differential operator $\frac{d}{dx}$, also written by $'$. Let y be an indeterminate over $\mathcal{K}$. The i-th derivative of y is denoted by y_i. The ring consisting of all polynomials in the y_i with coefficients in $\mathcal{K}$ is called the *ring of differential polynomials over* $\mathcal{K}$, denoted by $\mathcal{K}\{y\}$. Let $\mathcal{U}$ be a universal extension of the differential field $\mathcal{K}$. Let Σ be a set of differential polynomials in $\mathcal{K}\{y\}$. An element $\eta \in \mathcal{U}$ is a *zero of* Σ if it vanishes for all differential polynomials in Σ. Note that the zero set of Σ is the same as the zero set of the differential ideal generated by Σ. The notion of a generic zero of an ideal can be adapted to a differential ideal.

Definition 2.1. Let Σ be a nontrivial prime ideal in $\mathcal{K}\{y\}$. A zero η of Σ is called a *generic zero of* Σ if for any differential polynomial $P \in \mathcal{K}\{y\}$, $P(\eta) = 0$ implies that $P \in \Sigma$.

We are going to define a general solution of a single differential polynomial. Let $F \in \mathcal{K}\{y\}$. The highest derivative of y in F is called the *order of* F, denoted by $\mathrm{ord}(F)$. Suppose that $\mathrm{ord}(F) = p$. Then F has the form

$$F = a_d y_p^d + a_{d-1} y_p^{d-1} + \cdots + a_0,$$

where a_i are differential polynomials in $y, y_1, \ldots, y_{p-1}$ and $a_d \neq 0$. In this case, a_d is called the *initial of* F and $S := \frac{\partial F}{\partial y_p} = a_d d y_p^{d-1} + a_{d-1}(d-1) y_p^{d-2} + \cdots + a_1$ is called the *separant of* F. For any differential polynomial $G \in \mathcal{K}\{y\}$ we have the following representation

$$JG = Q_0F + Q_1F^{(1)} + \cdots + Q_rF^{(r)} + R,$$

where J is a product of certain powers of the initial and separant of F; $F^{(i)}$ are the i-th derivative of F; Q_i and R are differential polynomials in $\mathcal{K}\{y\}$. Moreover, $ord(R) < p$ or $ord(R) = p$ and $\deg_{y_p} R < d$. Then R is called the *differential pseudo remainder of G with respect to F*, denoted by $\mathrm{prem}(G, F)$.

Suppose that F is an irreducible differential polynomial in $\overline{\mathbb{Q}}(x)[y, y_1, \ldots, y_p]$. Let

$$\Sigma_F = \{G \in \mathcal{K}\{y\} | SG \in \{F\}\}$$

where $\{F\}$ is the perfect differential ideal[1] generated by F. Note that $\Sigma_F = \{F\} : S$, $\{F\} \subset \Sigma_F$ and it is well known by [Rit50] that

LEMMA 2.1. *Σ_F is a prime differential ideal and G belongs to Σ_F iff $prem(G, F) = 0$.*

DEFINITION 2.2. Let $F \in \mathcal{K}\{y\}$ be an irreducible differential polynomial. A generic zero of the prime differential ideal Σ_F is called a *general solution of $F = 0$*. A *rational general solution* is defined as a general solution of the form

$$y = \frac{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_mx^m + b_{m-1}x^{m-1} + \cdots + b_0},$$

where a_i, b_j are constants in the constant field of a universal extension of $\mathcal{K}$ and $b_m \neq 0$.

By the above definition and the Lemma 2.1, we have the following.

COROLLARY 2.1. *If η is a general solution of $F = 0$, then for any differential polynomial $G \in \mathcal{K}\{y\}$ we have*

$$G(\eta) = 0 \Leftrightarrow prem(G, F) = 0.$$

3 Main result

In this section we consider a non-autonomous first order ODE

$$F(x, y, y') = 0, \tag{1}$$

where $F \in \mathbb{Q}[x, y, z]$ is an irreducible polynomial over $\overline{\mathbb{Q}}$. A rational solution $y = f(x)$ of (1) is an element of $\overline{\mathbb{Q}}(x)$ such that

$$F(x, f(x), f'(x)) = 0. \tag{2}$$

By viewing x, y and y' as independent variables, whose values are in the field $\overline{\mathbb{Q}}$, the equation $F(x, y, z) = 0$ defines an algebraic surface $\mathcal{S}$ in the space $\mathbb{A}^3(\overline{\mathbb{Q}})$. Then the condition (2) tells us that the parametric space curve $\gamma(x) = (x, f(x), f'(x))$ lies on the surface $\mathcal{S}$.

From now on we assume that the surface $\mathcal{S}$ can be parametrized by rational functions

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)).$$

[1]It is defined as the radical ideal in the ring theory.

Since $\mathcal{P}$ is a birational map $\mathbb{A}^2(\overline{\mathbb{Q}}) \to \mathcal{S} \subset \mathbb{A}^3(\overline{\mathbb{Q}})$, there is a birational inverse map $\mathcal{P}^{-1}$ defining on the surface $\mathcal{S}$ except finitely many curves or points on $\mathcal{S}$.

Definition 3.1. A solution $y = f(x)$ of the equation $F(x, y, y') = 0$ *is parametrizable by* $\mathcal{P}$ if the parametric curve $(x, f(x), f'(x))$ lies in the domain of the image of $\mathcal{P}$.

Proposition 3.1. *Let $F(x, y, z) = 0$ be a rational surface with a parametrization*

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)).$$

The differential equation $F(x, y, y') = 0$ has a rational solution, which is parametrizable by $\mathcal{P}$, if and only if there exist two rational functions $s(x)$ and $t(x)$ such that

$$\begin{cases} \chi_1(s(x), t(x)) = x \\ \dfrac{d\chi_2(s(x), t(x))}{dx} = \chi_3(s(x), t(x)), \end{cases} \tag{3}$$

Proof. Assume that $y = f(x)$ is a rational solution of $F(x, y, y') = 0$, which is parametrizable by $\mathcal{P}$. Then let

$$(s(x), t(x)) = \mathcal{P}^{-1}(x, f(x), f'(x)).$$

This is a plane parametric curve and satisfies the following relations

$$\mathcal{P}(s(x), t(x)) = \mathcal{P}(\mathcal{P}^{-1}(x, f(x), f'(x))) = (x, f(x), f'(x)).$$

In other words, we have

$$\begin{cases} \chi_1(s(x), t(x)) = x \\ \chi_2(s(x), t(x)) = f(x) \\ \chi_3(s(x), t(x)) = f'(x). \end{cases} \tag{4}$$

Moreover, $(s(x), t(x))$ is a rational plane curve in (s, t)-plane because $\mathcal{P}^{-1}$ is a birational map and coordinate functions of $\gamma(x)$ are rational functions in x.

Conversely, if two rational functions $s = s(x)$ and $t = t(x)$ satisfy the system

$$\begin{cases} \chi_1(s(x), t(x)) = x \\ \dfrac{d\chi_2(s(x), t(x))}{dx} = \chi_3(s(x), t(x)), \end{cases}$$

then it is clear that $y = \chi_2(s(x), t(x))$ is a rational solution of the differential equation $F(x, y, y') = 0$. $\square$

Suppose that $s = s(x)$ and $t = t(x)$ are two rational functions such that

$$\begin{cases} \chi_1(s(x), t(x)) = x \\ \dfrac{d\chi_2(s(x), t(x))}{dx} = \chi_3(s(x), t(x)). \end{cases} \tag{5}$$

Differentiate the first equation of (5) and expand the last equation of (5), we get

$$\begin{cases} \dfrac{\partial \chi_1(s(x),t(x))}{\partial s} \cdot s'(x) + \dfrac{\partial \chi_1(s(x),t(x))}{\partial t} \cdot t'(x) = 1 \\ \\ \dfrac{\partial \chi_2(s(x),t(x))}{\partial s} \cdot s'(x) + \dfrac{\partial \chi_2(s(x),t(x))}{\partial t} \cdot t'(x) = \chi_3(s(x),t(x)). \end{cases} \tag{6}$$

If

$$\det \begin{pmatrix} \dfrac{\partial \chi_1(s(x),t(x))}{\partial s} & \dfrac{\partial \chi_1(s(x),t(x))}{\partial t} \\ \\ \dfrac{\partial \chi_2(s(x),t(x))}{\partial s} & \dfrac{\partial \chi_2(s(x),t(x))}{\partial t} \end{pmatrix} \not\equiv 0, \tag{7}$$

then $(s(x), t(x))$ is a solution of the system of differential equations of order 1 in s, t and degree 1 in s', t'

$$\begin{cases} s'(x) = \dfrac{f_1(s,t)}{g(s,t)} \\ \\ t'(x) = -\dfrac{f_2(s,t)}{g(s,t)}, \end{cases} \tag{8}$$

where

$$f_1(s,t) = \frac{\partial \chi_2(s,t)}{\partial t} - \chi_3(s,t) \cdot \frac{\partial \chi_1(s,t)}{\partial t},$$
$$f_2(s,t) = \frac{\partial \chi_2(s,t)}{\partial s} - \chi_3(s,t) \cdot \frac{\partial \chi_1(s,t)}{\partial s}$$

and

$$g(s,t) = \frac{\partial \chi_1(s,t)}{\partial s} \cdot \frac{\partial \chi_2(s,t)}{\partial t} - \frac{\partial \chi_1(s,t)}{\partial t} \cdot \frac{\partial \chi_2(s,t)}{\partial s}.$$

If the determinant (7) is equal to 0, then $(s(x), t(x))$ is a solution of the system

$$\begin{cases} \bar{g}(s,t) = 0 \\ \bar{f}_1(s,t) = 0. \end{cases} \tag{9}$$

where $\bar{g}(s,t)$ and $\bar{f}_1(s,t)$ are numerators of $g(s,t)$ and $f_1(s,t)$ respectively. Thus $(s(x), t(x))$ defines a curve iff $\bar{g}(s,t)|\bar{f}_1(s,t)$ or $\bar{f}_1(s,t)|\bar{g}(s,t)$. Otherwise, $(s(x), t(x))$ is just an intersection point of two algebraic curves $\bar{g}(s,t) = 0$ and $\bar{f}_1(s,t) = 0$, which does not satisfy the relation (5).

We would expect that a rational general solution of the system (8) will define a rational general solution of the equation $F(x, y, y') = 0$. At this point we define what we mean by a rational general solution of the system (8). Let N_i and M_i be the numerator and the denominator of $\dfrac{f_i(s,t)}{g(s,t)}$ for $i = 1, 2$.

Definition 3.2. A rational solution $(\bar{s}(x), \bar{t}(x))$ of the system (8) is called a *rational general solution* if for any differential polynomial $G \in \mathcal{K}\{s, t\}$ we have

$$G(\bar{s}(x), \bar{t}(x)) = 0 \Leftrightarrow prem(G, \{s'M_1(s,t) - N_1(s,t), t'M_2(s,t) + N_2(s,t)\}) = 0,$$

where $prem(G, \{s'M_1(s,t)-N_1(s,t), t'M_2(s,t)+N_2(s,t)\})$ is the pseudo remainder of G with respect to the system of differential polynomials $s'M_1(s,t)-N_1(s,t)$, $t'M_2(s,t)+N_2(s,t)$.

We can see that the $prem(G, \{s'M_1(s,t)-N_1(s,t), t'M_2(s,t)+N_2(s,t)\})$ will be a polynomial in $\mathcal{K}[s,t]$ because the degree of s' and t' are 1. In particular, we have

LEMMA 3.1. *Let $(\bar{s}(x), \bar{t}(x))$ be a rational general solution of the system* (8). *Let G be a bivariate polynomial in $\mathcal{K}[s,t]$. If $G(\bar{s}(x), \bar{t}(x)) = 0$, then $G = 0$ in $\mathcal{K}[s,t]$.*

PROOF. Since $G \in \mathcal{K}[s,t]$, we have

$$prem(G, \{s'M_1(s,t)-N_1(s,t), t'M_2(s,t)+N_2(s,t)\}) = G.$$

Therefore, $G(\bar{s}(x), \bar{t}(x)) = 0$ implies $G = 0$ in $\mathcal{K}[s,t]$. □

THEOREM 3.1. *Let $\bar{y} = f(x)$ be a rational general solution of $F(x,y,y') = 0$. If $\bar{y} = f(x)$ is parametrizable by $\mathcal{P}$, then*

$$(\bar{s}(x), \bar{t}(x)) = \mathcal{P}^{-1}(x, f(x), f'(x))$$

is a rational general solution of the system (8).

PROOF. It turns out that $(\bar{s}(x), \bar{t}(x))$ is a solution of (8). Suppose that $P \in \mathcal{K}\{s,t\}$ is a differential polynomial such that $P(\bar{s}(x), \bar{t}(x)) = 0$. Let

$$R = prem(P, \{s'M_1(s,t)-N_1(s,t), t'M_2(s,t)+N_2(s,t)\}).$$

Then $R \in \mathcal{K}[s,t]$, we have to prove that $R = 0$. We know that

$$R(\bar{s}(x), \bar{t}(x)) = R(\mathcal{P}^{-1}(x, f(x), f'(x))) = 0.$$

Let's consider the rational function $R(\mathcal{P}^{-1}(x,y,z)) = \dfrac{U(x,y,z)}{V(x,y,z)}$. Then $U(x,y,y')$ is a differential polynomial satisfying the condition

$$U(x, f(x), f'(x)) = 0.$$

Since $f(x)$ is a rational general solution of $F = 0$ and both F and U are differential polynomials of order 1, we have

$$U(x,y,y') = Q_0 F,$$

where Q_0 is a differential polynomial of order 1 in $\mathcal{K}\{y\}$. Therefore,

$$R(s,t) = R(\mathcal{P}^{-1}(\mathcal{P}(s,t))) = \frac{U(\mathcal{P}(s,t))}{V(\mathcal{P}(s,t))} = \frac{Q_0(\mathcal{P}(s,t))F(\mathcal{P}(s,t))}{V(\mathcal{P}(s,t))} = 0.$$

Thus $(\bar{s}(x), \bar{t}(x))$ is a rational general solution of (8). □

We are now constructing a rational general solution of $F(x,y,y') = 0$ from a rational general solution of the system (8). Assume that $(\bar{s}(x), \bar{t}(x))$ is a rational general solution of (8). Substituting $\bar{s}(x)$ and $\bar{t}(x)$ into $\chi_1(s,t)$ we get

$$\chi_1(\bar{s}(x), \bar{t}(x)) = x + c$$

for some constant c. Hence

$$\chi_1(\bar{s}(x-c), \bar{t}(x-c)) = x.$$

It follows that $y = \chi_2(\bar{s}(x-c), \bar{t}(x-c))$ is a solution of the differential equation

$$F(x, y, y') = 0.$$

Moreover, we will prove that $y = \chi_2(\bar{s}(x-c), \bar{t}(x-c))$ is a rational general solution of $F(x, y, y') = 0$. The main theorem is the following.

THEOREM 3.2. *Let* $(\bar{s}(x), \bar{t}(x))$ *is a rational general solution of the system* (8). *Then*

$$\bar{y} = \chi_2(\bar{s}(x-c), \bar{t}(x-c))$$

is a rational general solution of $F(x, y, y') = 0$.

PROOF. It is clear that $\bar{y} = \chi_2(\bar{s}(x-c), \bar{t}(x-c))$ is a rational solution of $F(x, y, y') = 0$. Let G be an arbitrary differential polynomial in $\mathcal{K}\{y\}$ such that $G(\bar{y}) = 0$. Let

$$R = prem(G, F)$$

be the differential pseudo-remainder of G with respect to F. It follows that

$$R(\bar{y}) = 0.$$

We have to prove that $R = 0$. Assume that $R \neq 0$. Then

$$R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = \frac{W(s,t)}{Z(s,t)} \in \overline{\mathbb{Q}}(s,t).$$

On the other hand,

$$R(\chi_1(\bar{s}(x), \bar{t}(x)), \chi_2(\bar{s}(x), \bar{t}(x)), \chi_3(\bar{s}(x), \bar{t}(x))) = 0.$$

It follows that $W(\bar{s}(x), \bar{t}(x)) = 0$. By the Lemma 3.1 we have $W(s,t) = 0$. Thus $R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = 0$. Since F is irreducible and $\deg_{y'} R < \deg_{y'} F$, we have $R = 0$ in $\mathbb{Q}[x, y, z]$. Therefore, $\bar{y}$ is a rational general solution of $F(x, y, y') = 0$. $\square$

EXAMPLE 3.1. Consider the differential equation

$$y'^3 - 4xyy' + 8y^2 = 0.$$

The corresponding surface has a proper parametrization

$$\mathcal{P}(s,t) = (t, -4s^2(2s-t), -4s(2s-t)).$$

The inverse map is

$$\mathcal{P}^{-1}(x,y,z) = \left(\frac{y}{z}, x\right).$$

We compute

$$g(s,t) = 8s(3s-t),$$

$$f_1(s,t) = 4s(3s-t), \quad f_2(s,t) = -8s(3s-t).$$

Thus the associated system is

$$\begin{cases} s'(x) = \dfrac{1}{2} \\ t'(x) = 1. \end{cases}$$

Solving this system we obtain a rational general solution $\bar{s}(x) = \dfrac{x}{2} + c_2, \bar{t} = x + c_1$ for arbitrary constants c_1, c_2. It follows that the general solution is

$$\bar{y} = -4\bar{s}(x-c_1)^2(2\bar{s}(x-c_1) - \bar{t}(x-c_1)) = -C(x+C)^2$$

where $C = 2c_2 - c_1$.

Note that in this example $g(s,t) = -8s(t-3s)$. Let $g(s,t) = 0$ we get $s = 0$, or $t = 3s$. This gives us two other solutions $y = 0$, or $y = \dfrac{4}{27}x^3$.

Example 3.2. Consider the differential equation

$$y'x^2 + xy^2 - 2xy - y^2 = 0.$$

It has a birational parametrization

$$\mathcal{P}(s,t) = \left(t, \frac{t^2}{s+1}, \frac{-t(-2s-2+t^2-t)}{(s+1)^2}\right).$$

Its inverse is

$$\mathcal{P}^{-1}(x,y,z) = \left(-\frac{yx - y - 2x + z}{z}, x\right).$$

We compute

$$g(s,t) = \frac{t^2}{(s+1)^2},$$

$$f_1(s,t) = \frac{t^2(t-1)}{(s+1)^2}, \quad f_2(s,t) = \frac{-t^2}{(s+1)^2}.$$

The associated system is

$$\begin{cases} s'(x) = t - 1 \\ t'(x) = 1. \end{cases}$$

Solving this system we obtain

$$\bar{s}(x) = \frac{x^2}{2} + (c_1 - 1)x + c_2, \bar{t}(x) = x + c_1.$$

Therefore,

$$\bar{y} = \frac{2x^2}{x^2 - 2x + 2C}$$

is a rational general solution, where $C = c_2 + c_1 - \frac{c_1^2}{2}$ is an arbitrary constant.

Note that if $g(s,t) = \frac{t^2}{(s+1)^2} = 0$, then it gives us a solution $y = 0$.

Acknowledgments

I would like to express my deep thanks to professor Franz Winkler for his advices and improved comments.

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