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Abstract – In this letter we show how the nonlinear evolution of a resonant triad depends on the special combination of the modes' phases chosen according to the resonance conditions. This phase combination is called dynamical phase. Its evolution is studied for two integrable cases: a triad and a cluster formed by two connected triads, using a numerical method which is fully validated by monitoring the conserved quantities known analytically. We show that dynamical phases, usually regarded as equal to zero or constants, play a substantial role in the dynamics of the clusters. Indeed, some effects are i) to diminish the period of energy exchange τ within a cluster by 20% and more; ii) to diminish, at time scale τ, the variability of wave energies by 25% and more; iii) to generate a new time scale, $T \gg \tau$, in which we observe considerable energy exchange within a cluster, as well as a periodic behaviour (with period T) in the variability of the modes' energies. These findings can be applied, for example, to the control of energy input, exchange and output in tokamaks; for the explanation of some experimental results; to guide and improve the performance of experiments; to interpret the results of numerical simulations, etc.

Introduction. – In this letter we will regard nonlinear resonant systems corresponding to the 3-wave resonance conditions. Examples of these nonlinear resonant systems are, in order of simplicity, triads (which are integrable), and small groups of connected triads which are known to be important for various physical applications (large-scale motions in the Earth's atmosphere [1], laboratory experiments with gravity-capillary waves [2], etc.). The dynamical system for a triad has the standard Manley-Rowe form:

$$\dot{B}_1 = Z B_2^* B_3, \qquad \dot{B}_2 = Z B_1^* B_3, \qquad \dot{B}_3 = -Z B_1 B_2, \quad (1)$$

where (B_1, B_2, B_3) are complex amplitudes of the resonantly interacting modes $B_j \exp i(\mathbf{k}_j \cdot \mathbf{x} - \omega(\mathbf{k}_j)t)$. The corresponding resonance conditions are

$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) - \omega(\mathbf{k}_3) = 0, \qquad \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 = 0, \quad (2)$$

with $\omega(\mathbf{k})$ the dispersion relation and $\mathbf{k}$ the wave vector. System (1) has been studied both in its real and complex

form (*e.g.*, [3,4], etc.). In the amplitude-phase representation $B_j = |B_j| \exp i\theta_j$, (1) is equivalent to a system for the 3 real amplitudes $|B_j|$ and the phase combination $\varphi = \theta_1 + \theta_2 - \theta_3$, the individual phases θ_j being slave variables and they can be obtained by quadratures [5]. The dynamical equation for φ is also known ([4], p. 43, eq. (28)). Still, a sort of general misunderstanding persists, concerning the relevance of φ for the general dynamics of the system. It is a common belief that for an exact resonance to occur, it is necessary that φ is either zero ([6], p. 132, eq. (6.7); [7], p. 156, eq. (3.26.19), etc.) or constant (*e.g.* [2]).

In this letter we to show that for generic initial conditions, φ, which we call *dynamical phase*, affects the evolution of the amplitudes and, therefore, has a direct impact on the behaviour of any physical system governed by a triad as well as by small clusters of resonant triads. As was shown in [8], some of these clusters are described by integrable systems and for them a complete set of conservation laws (CLs) was given explicitly, showing that dynamical phases are relevant in the determination of these CLs. In this paper we present differential equations for the two independent dynamical phases appearing in the *butterfly*,

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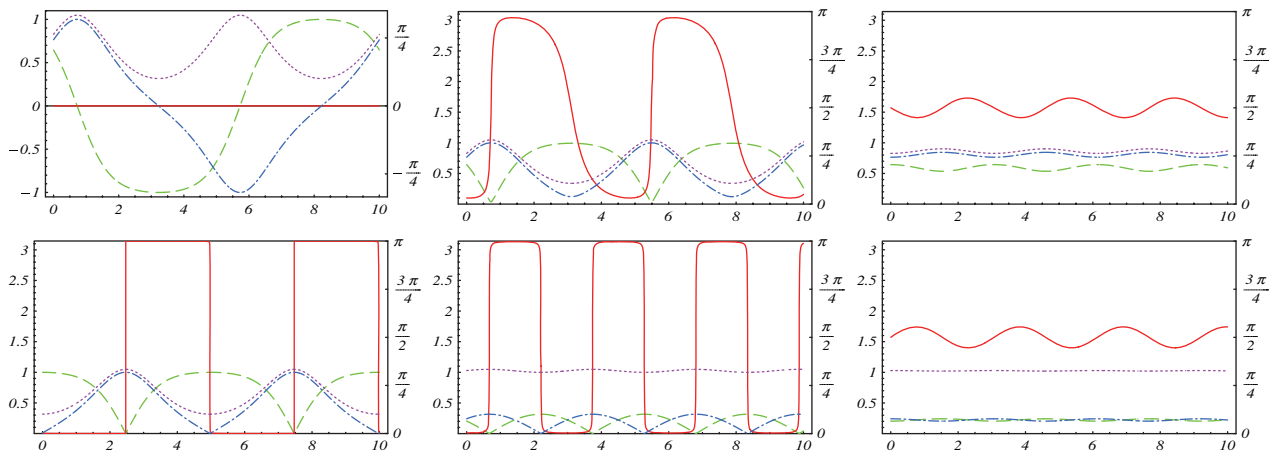(b)E-mail: lena@risc.uni-linz.ac.at

Fig. 1: (Colour on-line) Plots of the modes' amplitudes and dynamical phase as functions of time, for a triad with $Z = 1$. For each frame, $\varphi(t)$ is the (red) solid line, $C_1(t)$ is the (purple) dotted line, $C_2(t)$ is the (blue) dash-dotted line, $C_3(t)$ is the (green) dashed line. Upper panel: initial condition $\alpha = 0.7$ and conserved quantities $I_{23} = 1$, $I_{13} = 1.1$ for all frames. Initial conditions for the phase and corresponding cubic conserved quantity: left: $\varphi = 0$, $I_T = 0$; middle: $\varphi = 0.1$, $I_T = 0.041$; right: $\varphi = \pi/2$, $I_T = 0.408$. Lower panel: conserved quantity $I_{13} = 1.1$ for all frames. Initial conditions and corresponding conserved quantities: left: $\alpha = 1.56$, $\varphi = 0.01$ and $I_{23} = 1$, $I_T = 3.2 \times 10^{-5}$; middle: $\alpha = 0.7$, $\varphi = 0.01$ and $I_{23} = 0.1$, $I_T = 5.1 \times 10^{-4}$; right: $\alpha = 0.7$, $\varphi = \pi/2$ and $I_{23} = 0.1$, $I_T = 5.1 \times 10^{-2}$. Here, the horizontal axis denotes the non-dimensional time; the vertical left and right axes denote amplitude and phase, respectively.

a resonance cluster formed by two triads connected *via* one mode. We investigate the integrable case of butterfly by solving numerically the reduced evolutionary differential equations shown in [8] for the phases and amplitudes. The numerical integration is fully validated by monitoring the conservation laws, known analytically from our previous work [8]. We study the effects of the phases on the modes' amplitudes, and some physical implications are briefly discussed.

Triad with complex amplitudes. – The CLs for (1) have the form

$$
\begin{cases}
I_{13} = |B_1|^2 + |B_3|^2, \quad I_{23} = |B_2|^2 + |B_3|^2, \\
I_T = \text{Im}(B_1 B_2 B_3^*),
\end{cases}
\tag{3}
$$

which is enough for the explicit solution to be constructed. The conserved quantities I_{13} and I_{23} are linear combinations of energy and enstrophy [1]. The analytical solution of (1) can be found in [4], as well as the equations on two of the three phases θ_j (the standard amplitude-phase representation $B_j = C_j \exp(i\theta_j)$ is used). Below we use slightly different notations, introduced in [8], because they are more convenient for further studies of bigger groups of connected triads. The equation for the dynamical phase can be easily deduced and reads as

$$
\dot{\varphi} = -I_T(C_1^{-2} + C_2^{-2} - C_3^{-2}).
\tag{4}
$$

Combining (3) and (4), the constraint $I_T = 0$ implies that $\varphi = 0$.

If we put $I_T = 0$, the solution for the amplitudes takes the familiar form

$$
\begin{cases}
C_1(t) = \text{dn}((-t + t_0)\ z\sqrt{I_{13}}, \tfrac{I_{23}}{I_{13}})\sqrt{I_{13}}, \\
C_2(t) = \text{cn}((-t + t_0)\ z\sqrt{I_{13}}, \tfrac{I_{23}}{I_{13}})\sqrt{I_{23}}, \\
C_3(t) = \text{sn}((-t + t_0)\ z\sqrt{I_{13}}, \tfrac{I_{23}}{I_{13}})\sqrt{I_{23}},
\end{cases}
\tag{5}
$$

where t_0 is defined by the initial conditions and can also be written out explicitly.

Triad, case $I_T \neq 0$. – Now we present some results for the cases when $I_T \neq 0$, otherwise the resonance conditions in the standard form (2) are satisfied. Figure 1 shows the evolution of the characteristic amplitudes, depending on the value of I_T. To characterize the initial conditions, the variable $\alpha = \arctan(C_3/C_2)$ has been chosen for a triad and variables $\alpha_a = \arctan(C_{3a}/C_{2a})$ and $\alpha_b = \arctan(C_{3b}/C_{2b})$ have been chosen for a butterfly. These variables appear naturally from the explicit form of the corresponding Hamiltonians, and simplify the form of the dynamical systems and conservation laws (see [8] for more details).

As is shown in fig. 1, when the initial dynamical phase is zero, it will remain zero at all times, but the amplitudes will change sign periodically (upper-left panel). When the dynamical phase is initially very small but non-zero, the amplitudes become purely positive and the dynamical phase will have abrupt jumps, at those times when the amplitudes used to change sign (lower-left panel). Physically, in terms of squares of amplitudes, the dynamics in both cases is quite the same (figure not shown). However, the phase's dynamics, with its periodic motion, is revealed in the non-zero case. As is shown in

the upper panel (left, middle and right), the non-zero dynamical phase influences the evolution of amplitudes, so that as the initial φ increases from 0 to $\pi/2$, the range of amplitude variations decreases from 1 to 0.1 and the period of the motions decreases from 5 to 3.

In the lower panel, we show that the notion of A-mode (active) and P-mode (passive) introduced in [9] (compare to the stability criterion [10]) is useful also in the case of the non-zero dynamical phase φ. The A-mode is the mode with the highest frequency, and two other modes are called P-modes. In the pictures, C_1 is a P-mode and C_3 is an A-mode. Lower-middle picture: when the initial value of the amplitude $C_1 \gg C_3$, C_2 (4 times in the figure), the P-mode C_1 keeps its energy and the A-mode C_3 interacts strongly with the remaining P-mode C_2. If $C_2 \gg C_3$, C_1 then the situation will be qualitatively the same, with the P-mode C_2 keeping the energy. Lower-left figure: on the other hand, if $C_3 \gg C_1$, C_2, then a completely different time evolution is observed and all modes interact.

One more important general feature of the dynamical phase is shown in the upper- and lower-right panels. Indeed, independently of the details of the initial values of C_1, C_2, C_3, the variation range of the amplitudes is minimized when the initial condition for the dynamical phase φ is equal to $\pi/2$. This can be used in real physical systems in order to control the exchange of energy between resonant modes, *at no energy cost*: the choice of the initial dynamical phase does not change the energy of the system, which is a sum of squares of amplitudes, obviously independent of the dynamical phase (and of any phase, for that matter).

Butterfly with complex amplitudes. – As was shown in [9], clusters formed by two triads a and b connected *via* one mode can have one of the three types accordingly to the types of connecting mode in each triad: PP-, AP- and AA-butterfly. In this letter, a PP-butterfly is taken as a representative example. The dynamical system describing the evolution of a PP-butterfly has the form

$$\begin{cases} \dot{B}_1 = Z_a B_{2a}^* B_{3a} + Z_b B_{2b}^* B_{3b}, \\ \dot{B}_{2a} = Z_a B_1^* B_{3a}, \quad \dot{B}_{2b} = Z_b B_1^* B_{3b}, \\ \dot{B}_{3a} = -Z_a B_1 B_{2a}, \quad \dot{B}_{3b} = -Z_b B_1 B_{2b}, \end{cases} \quad (6)$$

where the notation $B_1 = B_{1a} = B_{1b}$ is chosen for the amplitude of the mode common for both triads, while B_{2a}, B_{3a}, B_{2b}, B_{3b} are other four modes of the butterfly cluster. The set of constructed conservation laws reads

$$\begin{cases} I_{23a} = |B_{2a}|^2 + |B_{3a}|^2, \quad I_{23b} = |B_{2b}|^2 + |B_{3b}|^2, \\ I_{ab} = |B_1|^2 + |B_{3a}|^2 + |B_{3b}|^2, \\ I_B = \mathrm{Im}(Z_a B_1 B_{2a} B_{3a}^* + Z_b B_1 B_{2b} B_{3b}^*). \end{cases} \quad (7)$$

Similar to the triad, the use of the standard representation $B_j = C_j \exp(i\theta_j)$ shows that the syst. (6) has effectively 3 degrees of freedom and two dynamical phases are

important: $\varphi_a = \theta_{1a} + \theta_{2a} - \theta_{3a}$, $\varphi_b = \theta_{1b} + \theta_{2b} - \theta_{3b}$, with the requirement $\theta_{1a} = \theta_{1b}$ which corresponds to the choice of connecting mode. Accordingly, equations on the dynamical phases take the form

$$\begin{cases} \dot{\varphi}_a = -I_B(C_1^{-2} + C_{2a}^{-2} - C_{3a}^{-2}), \\ \dot{\varphi}_b = -I_B(C_1^{-2} + C_{2b}^{-2} - C_{3b}^{-2}). \end{cases} \quad (8)$$

In order to study the effect of non-zero dynamical phases for the butterfly, we performed numerical simulations for the integrable case $Z_a = Z_b$. This allows us to compare the results with the triad, which is just a particular case of the integrable butterfly.

In fig. 2, left column: we show phases, amplitudes and amplitudes squared (energies) for initial conditions $\varphi_a = \varphi_b = 0$. The dynamics is quite similar to the triad with initial condition $\varphi = 0$ shown in fig. 1 upper left. In fig. 2, middle column: phases, amplitudes and energies are shown for initial conditions $\varphi_a = \pi/2$, $\varphi_b = 0$, while in the right column the same data are presented, for initial conditions $\varphi_a = \varphi_b = \pi/2$. We observe from figs. 1 and 2 some effects from the dynamical phases $\varphi_a(t)$ and $\varphi_b(t)$ of a butterfly (respectively, the phase $\varphi(t)$ of a triad): to diminish the period of energy exchange τ within a cluster by 20% and more; to reduce the variability of wave energies by 25% and more; to generate a new time scale, $T \gg \tau$, in which there is considerable energy exchange within a cluster, as well as a periodic behaviour (with period T) in the variability of the modes' energies.

All computations have been done using Mathematica and we have validated the code by checking the corresponding conservation laws, particularly those of cubic and quartic dependence on the amplitudes (introduced in [8]). These conservation laws are stably conserved during the whole numerical simulation, within a relative error of 10^{-12}.

A comment regarding the ergodicity of the integrable butterfly. We observe in a parametric plot of $\cos(\varphi_a)$ *vs.* $\cos(\varphi_b)$ as functions of time (figure not shown), that the seemingly periodic motions are indeed precessing with precession speed depending on the initial conditions. This is a generic feature of integrable systems which are not superintegrable.

Conclusions. – Effects of non-zero dynamical phases should be taken into account in the following situations.

– To control energy input, exchange and output in laboratory experiments, *e.g.* in tokamaks. Indeed, a possibility of concentrating energy in a small set of drift waves via some instability mechanisms has been conjectured by Petviashvili some 15 years ago [11]. As soon as energy is concentrated in a resonant cluster, mode amplitudes can become dangerously large. In [12] it was shown that the appearance of resonances can be completely avoided by a special choice of the form of the laboratory facilities which, of course, is too costly a game with tokamaks. On
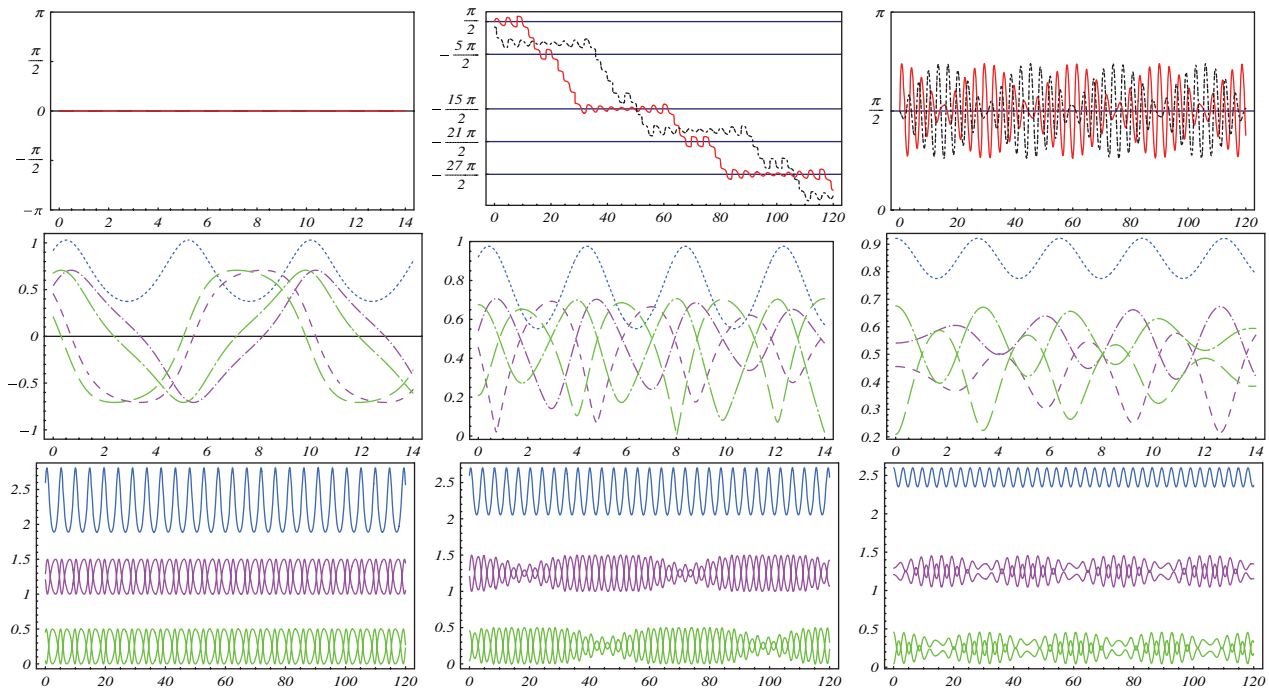
Fig. 2: (Colour on-line) Upper panel: plots of dynamical phases as functions of time, for a butterfly with $Z_a = Z_b = 1$. For each frame, $\varphi_a(t)$ is the (red) solid line and $\varphi_b(t)$ is the (black) dashed line. Initial conditions $\alpha_a = 0.3$, $\alpha_b = 0.7$; and conserved quantities $I_{ab} = 1.1$, $I_{23a} = I_{23b} = 0.5$ for all frames. Initial conditions for the phases and corresponding cubic conserved quantities: left: $\varphi_a = \varphi_b = 0$, $I_B = 0$; middle: $\varphi_a = 0$, $\varphi_b = \pi/2$, $I_B = 0.13$; right: $\varphi_a = \varphi_b = \pi/2$, $I_B = 0.36$. Middle panel: plots of the modes' amplitudes of a butterfly as functions of time, same initial conditions and parameters as in the corresponding left, middle and right frames of the upper panel. Connecting mode C_1 is the (blue) dotted line, C_{2a} is the (green) long-dash-dotted line, C_{3a} is the (green) long-dashed line, C_{2b} is the (purple) short-dash-dotted line, C_{3b} is the (purple) short-dashed line. Lower panel: plot of amplitudes squared, as functions of time, same initial conditions and parameters as in the corresponding left, middle and right frames of upper and middle panels. Colours as above. To facilitate the view, C_{2b}^2, C_{3b}^2 are shifted upwards by the value 1, and C_1^2 is shifted upwards by the value 1.75. Here, in all three panels, the horizontal axis denotes the non-dimensional time; the vertical axis denotes the phase in the upper panel and the amplitudes in the middle and lower panels.

the other hand, adjustment of dynamical phases can diminish the amplitudes of resonantly interacting drift waves 10 times and more for the same technical equipment.

– To gain more insight into the phenomenon of zonal flows in plasmas which are now regarded as the main component in all regimes of drift wave turbulence. "The progress of plasma physics induced a paradigm shift from the previous 'linear, local and deterministic' view of turbulent transport to the new 'nonlinear, nonlocal (both in real and wave number space), statistical' view of turbulent transport. Physics of the drift wave-zonal flow system is a prototypical example of this evolution in understanding the turbulence and structure formation in plasmas" [13]. In [14], a modulational instability of Rossby/plasma drift waves leads to the generation of zonal jets through a process in which the wave amplitudes are initially well approximated by a *kite*, a cluster consisting of two triads connected *via* two modes. The study of the behaviour of associated dynamical phases could lead to a deeper understanding of zonal jet

formation. Structure formation (for 3-wave resonance processes) is presented in [15], examples of non-local interactions are given in [16] as well as the cases of "weak" locality (waves with wave numbers of order n and n^2 can form a resonance cluster, while waves with wave numbers of order n and n^3 cannot); effects of initial energy distribution among the modes of a cluster are studied in [9]. In this letter we identify the dynamical phase as an additional important parameter for any theoretical study of nonlinear wave systems. We would like to point out that some equations for the dynamical phase of a triad have been known for more than 40 years in plasma physics and even earlier in nonlinear optics (*e.g.* eq. (3.31), [17]). We consider as our material impact in this *métier* the detailed study of the effect of dynamical phase on the nonlinear evolution of a triad and a butterfly.

– To guide and improve the performance and analysis of laboratory experiments. We showed that non-zero phases can dramatically reduce the variability of the oscillations (figs. 1 and 2, left columns). It would

then be possible to tune initial conditions and/or forcing (in rotating water tanks, for example) in order that the measurement of the resonant modes' oscillations be less subject to errors. In [2] results of laboratory experiments with gravity-capillary waves are presented. The corresponding dynamical system for three connected triads is written out and solved explicitly. The authors report qualitative agreement of the observation with the solutions of the dynamical system though the magnitudes of observed amplitudes are higher than those theoretically predicted. For all calculations the dynamical phase of the initially excited triad was set to π which might be the source of this discrepancy.

– To interpret the results of numerical simulations. For instance, in [1] a generic model of intra-seasonal oscillations in the Earth's atmosphere has been presented which describes the processes with periods of the order of 30–90 days. In this model, a dynamical phase has not been taken into account yet. A new time scale, $T \gg \tau$, which is clearly observable in fig. 2, corresponds, for the resonant triads of atmospheric planetary waves, to periods of the order of 2–5 years. This is in the time range of climate variability which might imply that in numerical modeling of climate variability the control of the dynamical phase is a matter of prodigious importance.

$* * *$

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