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Zusammenfassung

Diese Dissertation befasst sich mit der Untersuchung der symbolisch-algebraischen Faktorisierung, der Klassifikation und von Methoden zur Integration Linearer Partieller Differentialoperatoren (LPDOs).

Ein neuer theoretischer Begriff, ein Hindernis (obstacle) zur Faktorisierung von LPDOs in allgemeiner Form, wird eingeführt. Damit kann die Untersuchung von Faktorisierungsalgorithmen vereinfacht werden. Ein volles System von Invarianten für bivariate hyperbolische LPDOs dritter Ordnung wird gefunden. Die Faktorisierungen von LPDOs der Ordnung zwei, drei und vier mit vollständig faktorisierbarem Symbol und ohne jede andere Bedingung werden untersucht. Wir zeigen, dass es "irreduzible" parametrische Faktorisierungen nur geben kann für einige wenige Typen von Faktorisierungen. Explizite Beispiele für diese Fälle werden angeführt. Für Operatoren der Ordnung zwei und drei wird nachgewiesen, dass eine Familie durch höchstens eine Funktion in einer Variablen parametrisiert werden kann. Neue Transformationen (verallgemeinerte Laplace Transformationen, generalized Laplace transforms) bivariater hyperbolischer LPDO zweiter Ordnung werden eingeführt. Als wichtige Anwendung ergibt sich die Möglichkeit der Ausweitung der Klasse der analytisch lösbaren partiellen Differentialgleichungen. Beispiele werden angegeben.

Die Resultate wurden erzielt mithilfe eines zu diesem Zweck erstellten Programmpakets in MAPLE. Auch die Prozeduren zur Berechnung der Hindernisse (obstacles) zur Faktorisierung und der Invarianten sind in dem Programmpaket implementiert.

Schlüsselwörter: linearer partieller Differentialoperator, Faktorisierung, Invarianten, Transformationsmethode, symbolisches Rechnen.

Abstract

This thesis is devoted to the study of symbolic-algebraic factorization, classification, and integration methods for Linear Partial Differential Operators (LPDOs).

A new theoretical notion, an obstacle to factorizations of LPDOs of general form, that simplifies the considerations of factorization algorithms is introduced. A full system of invariants for third-order bivariate hyperbolic LPDOs is found. The factorizations of LPDOs of orders two, three, and four with completely factorable symbols and without any additional requirement are studied. We prove that "irreducible" parametric factorizations can exist only for a few certain types of factorizations. For these cases explicit examples are given. For operators of orders two and three, it is shown that a family may be parameterized by at most one function in one variable. New transformations (Generalized Laplace Transformations) of bivariate hyperbolic second order LPDOs are introduced. The important application is the possibility to extend the class of analytically solvable partial differential equations. Examples are given.

The results have been obtained with the help of a specially created Maple-package. Also the procedures for computing the obstacles to factorizations and invariants are implemented in the package.

Keywords: linear partial differential operator, factorization, invariants, transformation method, symbolic computation.

Contents

1 Introduction

1.1 Place in Science

The solution of Partial Differential Equations (PDEs) is one of the most important problem of mathematics, and has an enormous area of applications. The study of PDEs started in the 18th century in the works of Euler, d'Alembert, Lagrange and Laplace as a central tool in the description of the mechanics of continua and, more generally, as the principal mode of analytical study of models in the physical sciences. The analysis of physical models has remained to the present day one of the fundamental concerns of the development of PDEs. However, beginning in the middle of the 19th century, particularly with the work of Riemann, PDEs also became an essential tool in other branches of mathematics.

The most famous contributions to algebraical methods for the solution of PDEs were made by Galois (Galois theory uses the technique of the continuous transformation group), by Cartan (his theory makes use of the equivalence method of differential geometry, which determines whether two geometrical structures are the same up to a diffeomorphism), by Ritt (who studied the integrals of algebraic differential system of equations), and by Weyl (the famous Weyl algebra of differential operators with polynomial coefficients).

1.2 Symbolic Methods for Analytical Solutions of PDEs

In common with many other types of mathematical problems (for example integration), the solution of PDEs can be attacked either symbolically or numerically. (In addition, the so-called symbolic-numeric approaches to algebraic problems might be promising.) Of course, an analytical solution is to be preferred. Indeed, using an analytical solution, one can compute a numerical solution to any precision and on any segment of the domain, or analyze the solution's behavior at infinity and at extremal points; on can explore dependence on parameters, etc. Some simple Ordinary Differential Equations (ODEs) can often be solved analytically, but as they are more complicated, it happens much more rarely. Only a few separate PDEs can be solved analytically. Such solutions are often expressions in quadratures, but it is a matter of great luck to have even such a solution.

It is difficult even to list analytically solvable PDEs. Indeed, every method is usually an algorithm for a certain class of equations, which is used in combination with change of variables, conjugations, and other transformations. Thus, for a successful solution of a certain PDE, one has to transform it to some known canonical form. This, however, is a very difficult problem, as even a simple change of variables is not easy for PDEs. Moreover, the notion of an analytical solution is not rigorously defined. Can one consider implicit functions? Suppose, say, as one example, we made an analytical change of variables: $(x, y) \to (u, v)$, where

$$\begin{aligned} u &= x + e^{x+y^2}, \\ v &= y + e^{xy}/(y + \sin(xy)). \end{aligned}$$

Can we consider the inverse functions $x = x(u, v), y = y(u, v)$ analytically defined?

One of the methods for extending the quantity of analytically solvable PDEs consists in transformations of PDEs and the corresponding transformations of their solutions. Thus, based on the fact that a second-order equation

$$z_{xy} + az_x + bz_y + c = 0 \ , \tag{1}$$

where $a = a(x, y), b = b(x, y), c = c(x, y)$, can be solved if one of its factorizations is known, the famous method of Laplace Transformations suggests a certain sequence of transformations of a given equation of the form (1). Then, if at a certain step of the transformations an equation becomes factorable, an analytical solution of this transformed equation — and then of the initial one — can be found.

Another approach to the same problem lays in the factorization of a given PDEs, possibly taken in some transformed form.

1.3 Computer Algebra Resources

Symbolic methods for the finding of analytical solutions of PDEs have been extensively investigated by mathematicians, since the first appearance of PDEs in the 18th century. It seems that almost all the things which can be done by hand have already been done. Formally speaking, the transformations that have been considered are not very complicated, and any of them in principle could be computed by hand. However, computation of a sequence of transformations, even only a little bit advanced, is an almost impossible task if is being done by hand.

The main distinctive feature of the modern stage in research is the possibility of using computer algebra systems, which may help to solve problems that neither Laplace nor Newton, nor 20th century mathematicians, could hope to approach. Of course, this is not to say that all the problems that seemed complicated before can be solved now easily. Computer algebra systems are not intelligent, and serve as advanced symbolic calculators. For instance, the expanded form $(x^{100} + \dots)$ of the expression $(x + a)^{100}$ is very large, and yet every computer algebra system can easily factor the expanded form back to the original. On the other hand, no computer algebra system can simplify the expanded form of $(x+a)^{100}+2$ automatically to the compact form shown. Because of this, the modern computer algebra systems are not a substitute for the theoretical investigations of mathematicians, but they do hold out the promise, or at least the hope, of moving a step or more further than Laplace, Euler, etc.

This possibility first appeared not long ago, and not all scientists use it to the full: mathematicians developing symbolic and algebraic methods for PDEs are often not accustomed to use computer algebra systems, while computer scientists do not serve usually as specialists in PDEs. Thus, now there is a good chance to use computer algebra tools for essential progress in the solution of PDEs.

1.4 Definitions and Notations

Consider a field K with a set

$$\Delta = \{\partial_1, \ldots, \partial_n\}$$

of commuting derivations acting on it. We work with the ring of linear differential operators

$$K[D] = K[D_1, \ldots, D_n] \ ,$$

where $D_1, \ldots, D_n$ correspond to the derivations $\partial_1, \ldots, \partial_n$, respectively.

The totality of all linear differential operators of orders $\leqslant i$ with defined left and right multiplication is a K-bimodule, which we denote by $K_{\leqslant i}$. Thus we have the filtration

$$\cdots \supset K_{\leqslant i} \supset K_{\leqslant i-1} \supset \cdots \supset K_{\leqslant 0} \ .$$

Consider the associate algebra

$$Smbl_* = \sum_{i \geq 0} Smbl_i, \quad Smbl_i = K_{\leqslant i} \diagup K_{\leqslant i-1}.$$

K-module $Smbl_*$ is a commutative K-algebra, which is isomorphic to the ring of polynomials $K[X] = K[X_1, \ldots, X_n]$ in n variables. The image of the operator $L \in K[D]$ by the natural projection is some element Sym_L of $K[X]$. Actually, the is a homogeneous polynomial corresponding to the sum of the highest terms. We use the notation

$$D^{(i_1, \ldots, i_n)} := D_1^{i_1} \ldots D_n^{i_n} \ ,$$

and define the order as follows:

$$|D^{(i_1, \ldots, i_n)}| = \mathrm{ord}(D^{(i_1, \ldots, i_n)}) := i_1 + \cdots + i_n \ ,$$

and in addition the order of the zero operator is $-\infty$.

For a homogeneous polynomial $S \in K[X]$ we define the operator $\widehat{S} \in K[D]$, which is the result the substitution of the operator D_i for each variable X_i, that is for

$$S = \sum_{|J|=k} s_J X^J$$

the corresponding operator is

$$\widehat{S} = \sum_{|J|=k} s_J D^J \ ,$$

where $J \in \mathbf{N}^n$. If there is no danger of misunderstanding we use just S to denote the operator $\widehat{S}$.

By $K_i[D]$ we denote the set of all operators in $K[D]$ of order i.

Thus, any operator $L \in K[D]$ is of the form

$$L = \sum_{|J|\leq d} a_J D^J \ , \tag{2}$$

where $a_J \in K,\ J \in \mathbf{N}^n$ and $|J|$ is the sum of the components of J. Then the polynomial

$$\mathrm{Sym}_L = \sum_{|J|=d} a_J X^J$$

is *the symbol* of L.

One can recollect the components of the sum (2) so that the components of one order are in the same group. Denote the sum of all components of order $i \leq d$ by L_i. We say that L_i is *the i-th component* of L. Now the operator $L \in K[D]$ can be written as

$$L = \sum_{i=0}^{d} L_i \ . \tag{3}$$

For an operator $L \in K[D]$ *the operation of* is

$$L \to L^t(f) = \sum_{|J|\leq d} (-1)^{|J|} D^J(a_J f) \ ,$$

where $f \in K$.

Let K^* denotes the set of invertible elements in K. Then for $L \in K[D]$ and every $g \in K^*$ we consider the gauge transformation

$$L \to g^{-1} L g \ .$$

We also can say that this is *the operation of* . Then an algebraic differential expression I in coefficients of L is *invariant* under the gauge transformations if it is unaltered under the these transformations. Trivial examples of an invariant are coefficients of the symbol of the operator.

An operator $L \in K[D]$ is said to be *hyperbolic* (*separable*) if its symbol is completely factorable (all factors are of first order) and each factor has multiplicity one.

We use the usual abbreviations:

- LPDO for Linear Partial Differential Operator,
- LODO for Linear Ordinary Differential Operator,
- LPDE for Linear Partial Differential Equation,
- LODE for Linear Ordinary Differential Equation.

1.5 Laplace Transformations Method of Integration

The classical *Laplace cascade method* [7] (or as in some literature it is called *the*) has been known since the end of the 18th century. It is the oldest known algebraic method of integration of partial differential equations. The method does for second-order linear hyperbolic equations on the plane, which has the normalized form

$$z_{xy} + az_x + bz_y + c = 0, \tag{4}$$

where $a = a(x, y), b = b(x, y), c = c(x, y)$.

Now we highlight the main points, while the exhaustive exposition may be found in [7, 14, 12]. For a given second-order partial differential equation of the form (4), consider the corresponding differential operator

$$L = D_x \circ D_y + aD_x + bD_y + c. \tag{5}$$

This operator can be rewritten in the following ways:

$$L = (D_x + b) \circ (D_y + a) + h = (D_y + a) \circ (D_x + b) + k, \tag{6}$$

where

$$h = c - a_x - ab, \quad k = c - b_y - ab \tag{7}$$

are known as *the* , while two representations (6) are said to be incomplete factorizations of the operator L. Note, that the operator L is factorable if and only if h or k is zero.

I. Suppose h or k is equal zero. Then the operator L is factorable, and whence the equation (4) is integrable. Indeed, if, for example, $h = 0$, we have $L = (D_x + b)(D_y + a)$, and the problem of determination of all the integrals of the equation (4) is reduced to the problem of the integration of the two first order equations:

$$\begin{cases} (D_x + b)(z_1) = 0, \\ (D_y + a)(z) = z_1. \end{cases}$$

Accordingly one gets the general solution of the initial equation (4):

$$z = \Big(A(x) + \int B(y) e^{\int a dy - b dx} dy\Big) e^{-\int a dy} \tag{8}$$

with two arbitrary functions $A(x)$ and $B(y)$.

II. Suppose now that h and k do not vanish. Then one can apply two $L \to L_1$ and $L \to L_{-1}$, which are defined by the substitutions

$$z_1 = (D_y + a)(z), \quad z_{-1} = (D_x + b)(z) \ . \tag{9}$$

Such the transformations preserve the equations' form (4). Indeed, we have

$$L \to L_1 = D_x \circ D_y + a_1 D_x + b_1 D_y + c_1 \ , \tag{10}$$

$$\begin{array}{rcl} a_1 & = & a - \partial_y(ln|h|) \ , \\ b_1 & = & b \ , \\ c_1 & = & c + b_y - a_x - b\partial_y(ln|h|) \ , \end{array}$$

and

$$L \to L_{-1} = D_x \circ D_y + a_{-1} D_x + b_{-1} D_y + c_{-1} \ , \tag{11}$$

$$\begin{array}{rcl} a_{-1} & = & a \ , \\ b_{-1} & = & b - \partial_x(ln|k|) \ , \\ c_{-1} & = & c - b_y + a_x - a\partial_x(ln|k|) \ . \end{array}$$

The Laplace invariants of the new operators can be expressed in terms of the invariants of the initial operator, thus, for the operators L_1 and L_{-1}, we have

$$\begin{array}{ll} h_1 = 2h - k - \partial_{xy}(ln|h|), & k_1 = h, \\ h_{-1} = k, & k_{-1} = 2k - h - \partial_{xy}(ln|k|). \end{array}$$

The invariants k_1, and h_{-1} are certainly non-zero. So we have to check whether the invariants h_1 and k_{-1} are zero, that is whether the new operators L_1 and L_{-1} are factorable. If for example $h_1 = 0$, we solve the new equation $L_1(z_1) = 0$ in quadratures as described above. Then using the inverse substitution

$$z = \frac{1}{h}(z_1)_{-1}, \tag{12}$$

we obtain the complete solution of the original equation $L(z) = 0$. One considers the case $k_{-1} = 0$ analogously. If neither h_1, nor k_{-1} equal zero, we can apply the Laplace transformations again.

Thus, in the generic case, we obtain two sequences:

$$\begin{array}{rl} \dots \to L_{-2} \to L_{-1} \to & L, \\ & L \quad \to L_1 \to L_2 \to \dots \end{array}$$

The inverse substation (12) implies $L = h^{-1}(L_1)_{-1}h$, and one can prove that the Laplace invariants do not change under such substitution. This means that essentially we have one chain

$$\cdots \leftrightarrow L_{-2} \leftrightarrow L_{-1} \leftrightarrow L \leftrightarrow L_1 \leftrightarrow L_2 \leftrightarrow \ldots, \tag{13}$$

and the corresponding chain of invariants

$$\cdots \leftrightarrow k_{-2} \leftrightarrow k_{-1} \leftrightarrow k \leftrightarrow h \leftrightarrow h_1 \leftrightarrow h_2 \leftrightarrow \ldots \tag{14}$$

In that way one iterates the Laplace transformations until one of the Laplace invariants in the sequence (14) vanishes. In this case, one can solve the corresponding transformed equation in quadratures and using the same differential substitution (9) obtains the complete solution of the original equation.

What is more, one may prove (see for ex. [12]) that if the chain (14) if finite in both directions, then one may obtain a quadrature free expression of the general solution of the original equation.

Example 1.1. [34] As a straightforward computation shows, for the equation

$$z_{xy} - \frac{n(n+1)}{(x+y)^2} z = 0$$

the chain (13) has the length n in either direction.

For example, for $n = 1$ we have the short chain $L_{-1} \leftrightarrow L \leftrightarrow L_1$, where

$$\begin{aligned} L &= D_{xy} - \frac{2}{(x+y)^2}\ , \\ L_1 &= D_{xy} + \frac{2}{x+y} D_x - \frac{2}{(x+y)^2}\ , \\ L_{-1} &= D_{xy} + \frac{2}{x+y} D_y - \frac{2}{(x+y)^2}\ . \end{aligned}$$

And the corresponding chain (14) of the Laplace invariants is

$$0 \leftrightarrow k \leftrightarrow h \leftrightarrow 0\ .$$

Since the Laplace invariant h is zero for L_1, then L_1 is factorable, and the equation $L_1(z_1) = 0$, $z \in K$ can be analytically solved (see (8)):

$$z_1 = \frac{1}{(x+y)^2} \Big(\int B(y)(x+y)^2 dy + A(x) \Big)\ .$$

Using the substitution (12), we compute the solution of the initial equation:

$$z = \frac{1}{h} D_x(z_1) = \frac{1}{2} A(x) + \frac{1}{(x+y)} \Big((x+y) \int (x+y) B(y) dy - \int (x+y)^2 B(y) dy - A(x) \Big)\ .$$

Though the Laplace transformations method is one of the important methods of symbolic integration of the PDEs of the form $z_{xy} + az_x + bz_y + c = 0$, it leads to solutions not very often. In the chapter 5 a generalization of this method is suggested.

1.6 Variations of the Laplace Method

A hundred years later after Laplace, Darboux suggested [7] an explicit integration method of non-linear second-order scalar equations of the form

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0. \tag{15}$$

The idea is to consider a linerization of the equation (15). For example, for a second order hyperbolic non-linear equation

$$z_{xy} = f(x, y, z, z_x, z_y) \tag{16}$$

one considers the substitution $z(x, y) \to z(x, y) + \epsilon u(x, y)$ and cancels the terms with $\epsilon^n,\ n > 1$. Thus, one has the linearized equation $P(u) = 0$, where P is the linearized operator

$$P = D_x \circ D_y + \frac{\partial f}{\partial z_x} D_x + \frac{\partial f}{\partial z_y} D_y + \frac{\partial f}{\partial z}.$$

Then one apply the Laplace method. The relationship between the Laplace invariants of the linearized operator P and Darboux integrability of the initial equation (16) was established by Sokolov, Ziber, Startsev [30, 31], who proved that a second order hyperbolic non-linear equation is Darboux integrable if and only if the both Laplace sequences are finite. Later Anderson, Juras, and Kamran [1, 2, 15] generalized this to the case of the equations of the general form (15) as a consequence of their analysis of higher degree conservation laws for different types of partial differential equations.

In [8, 9] Dini suggested a generalization of the Laplace transformations for certain class of second-order operators in the space of arbitrary dimension. But no general statement was given on the range of applicability of his trick. Recently, Tsarev proved [35] that for a generic second-order linear partial differential operator in three-dimensional space,

$$L = \sum_{i+j+k \le 2} a_{ijk}(x, y, z) D_x D_y D_z$$

there exist two Dini transformations $L \to L_1$ and $L \to L_{-1}$ under the assumption that its principal symbol factors.

There were also several attempts to generalize the Laplace method to some systems of equations. Thus, Athorne and Yilmaz [3, 4] proposed a special transformation, which is applicable to systems whose order coincides with the number of independent variables. A serious effort to generalize the classical theory to operators of higher order (in two independent variables) was undertaken in [23]. Recently Tsarev described another procedure [35], which generalizes the Cascade Method to the case of arbitrary order hyperbolic operators.

1.7 Grigoriev-Schwarz Factorization Algorithm

In this section a recent result of Grigoriev and Schwarz [10] is presented in the form of a theorem, the proof of which is the algorithm of factorization of LPDOs. This algorithm extends the factorization of the symbol of an operator to a factorization of the operator, or concludes that there is no such factorization. The authors call it Hensel descent algorithm, since it is close in nature to the Hensel lifting algorithm of factorizations of polynomials.

Theorem 1.2. *Let $L \in K[D]$, and its symbol be factored into two coprime factors:*

$$Sym_L = S_1 \cdot S_2 \ . \tag{17}$$

Then there exists at most one factorization of the form

$$L = \left(\widehat{S}_1 + G\right) \circ \left(\widehat{S}_2 + H\right) \ ,$$

where $G, H \in K[D]$ and $\mathrm{ord}(G) < \mathrm{ord}(\widehat{S}_1)$*,* $\mathrm{ord}(H) < \mathrm{ord}(\widehat{S}_2)$*.*

Proof. Consider L, G, H as the sums of their components:

$$L = \sum_{i=0}^{d} L_i \ , \quad G = \sum_{i=0}^{k_1-1} G_i \ , \quad L = \sum_{i=0}^{k_2-1} H_i$$

where $d = \mathrm{ord}(L)$, $k_1 = \mathrm{ord}(\widehat{S}_1)$, $k_2 = \mathrm{ord}(\widehat{S}_2)$. Then the considered factorizations has the form

$$\sum_{i=0}^{d} L_i = \left(\widehat{S}_1 + G_{k_1-1} + \cdots + G_0\right) \circ \left(\widehat{S}_2 + H_{k_2-1} + \cdots + H_0\right) .$$

When equates the components of the both sides of equality, one gets the following system in the corresponding to the operators H_i and G_j homogeneous polynomials, which we denote by the same letters:

$$\begin{cases} L_{d-1} & = & S_1 \cdot H_{k_2-1} & + & G_{k_1-1} \cdot S_2 \ , & & \\ L_{d-2} & = & S_1 \cdot H_{k_2-2} & + & G_{k_1-2} \cdot S_2 & + & P_{d-2} \ , \\ \dots & & & & & & \\ L_i & = & S_1 \cdot H_{i-k_1} & + & G_{i-k_2} \cdot S_2 & + & P_i \ , \\ \dots & & & & & & \end{cases}$$

where P_i are some expressions of derivatives of H_{k_2-j}, G_{k_1-j} with $j < i$. Thus, if one solves the system in descent order, the polynomials P_i can be considered as known. Also here L_i stands for the homogeneous polynomial corresponding to the component L_i of L.

Consider one equation of the system:

$$L_i = S_1 \cdot H_{i-k_1} + G_{i-k_2} \cdot S_2 + P_i.$$

It is equivalent to a linear algebraical system in coefficients of the polynomials H_{i-k_1} and G_{i-k_2}. Since S_1 and S_2 are coprime, then there is at most one solution of the system, and so is for the equation. Thus, at every step one either gets the next components of H and G, or (in the case the linear algebraic system is unfeasible) concludes that there is no factorization of the operator L that extends the polynomial factorization of the symbol (17). $\square$

By induction on the number of factors one proves the following theorem:

Theorem 1.3. *Let $L \in K[D]$, and*

$$\mathrm{Sym}_L = S_1 \cdot S_2 \dots S_k \ ,$$

where $S_1, \dots, S_k$ be coprime. Then there exists at most one factorization

$$L = F_1 \circ \dots \circ F_k \ ,$$

such that

$$\mathrm{Sym}_{F_i} = S_i \ , \ i = 1, \dots k \ .$$

Example 1.4. Consider operator

$$L = D_{xyy} + D_{xx} + D_{xy} + D_{yy} + xD_x + D_y + x \ .$$

It is not hyperbolic, therefore, not every factorization of its symbol has coprime factors. However, there is, for example, factorization of the symbol

$$Sym_L = (X) \cdot (Y^2) \ ,$$

which has coprime factors. The corresponding factorizations of L has the form

$$L = (D_x + G_0) \circ (D_{yy} + H_1 + H_0) \ ,$$

where $G_0 = r$, $H_1 = aD_x + bD_y$, and $H_0 = c$, where $r, a, b, c \in K$. Equates the components on the both sides of the equality:

$$\begin{cases} L_2 &= (aX + bY)X + rY^2 \ , \\ L_1 &= (c + ra + a_x)X + (b_x + rb)Y \ , \\ L_0 &= rc + c_x \ , \end{cases} \tag{18}$$

where L_i stands for the homogeneous polynomial corresponding to the component L_i of L, that is

$$L_2 = X^2 + XY + Y^2, \quad L_1 = xX + Y, \quad L_0 = x \ .$$

Solve the system (18) in descent order. Consider the first equation. This equation in polynomials is equivalent to a linear algebraical system in their coefficients:

$$\begin{cases} 1 &= a \ , \\ 1 &= b \ , \\ 1 &= r \ , \end{cases}$$

which gives us

$$a = b = r = 1 \ .$$

Thus, solving of the first equation of the system (18) we determine all the coefficients of H_1 and G_0.

After substitution of 1 for a, b, r, the second equation of the system (18) has the form:

$$xX + Y = (c+1)X + Y \ ,$$

that gives us

$$c = x - 1 \ ,$$

that makes the last (third) equation of the system (18) become an identity. Therefore operator L can be factored as follows:

$$L = (D_x + 1) \circ (D_{yy} + D_x + D_y + x - 1) \ .$$

1.8 Contributions of this Thesis

The results expounded in this thesis extend and generalize existing theorems and algorithms for Linear Partial Differential Operators (LPDOs). Specifically, improvements are made in the symbolic-algebraic factorization of LPDOs, in the classification of LPDOs, and in integration methods for them. Each chapter of the thesis except this one — the introduction — and the last one — conclusion — is devoted to one topic in the list above.

1.8.1 Obstacles to Factorizations of LPDOs

Chapter 2 is devoted to the factorization properties of LPDOs. The key idea is a shift in emphasis from *complete* factorization to *partial* factorization. Using the Grigoriev-Schwarz factorization algorithm (section 1.7) I introduce the notions of *partial factorizations* of LPDOs and *common obstacles to their factorization.* The first notion allows me to establish Theorem 2.9, a generalization of the Grigoriev-Schwarz Theorem (given above as theorem 1.3) to the case of non-coprime factors of the initial factorization of the operator's symbol.

Although for operators of order two the common obstacles are the Laplace invariants of the same operator (see section 1.5), for operators of orders higher than two the obstacles do not enjoy good properties. Therefore, I introduce a certain algebraic structure, which I call the *ring of obstacles* (section 2.5). For types of factorizations in which the symbols of the factors are pairwise coprime, the symbols of all common obstacles to factorization belong to the same class in the ring of obstacles. Then I define *the obstacles to the factorization* of a certain factorization type as the class of the symbols of the common obstacles to factorizations of the same type.

I establish some properties of obstacles to factorizations in section 2.6. For example, obstacles to the factorization of an operator L are invariant with respect to the gauge transformations $L \to g^{-1}Lg$. In addition, the obstacles for operators of order two and three have been computed explicitly in the last sections of chapter 2. The computation of the obstacles for operators of orders two and three (for order two they are the Laplace invariants) have been implemented in my MAPLE-package, described in Chapter 6.

The results of this chapter have been published [26, 27].

1.8.2 A Full System of Invariants for Third-Order Hyperbolic LPDOs

Chapter 3 addresses the problem of finding a full system of invariants for LPDOs with respect to gauge transformations $L \to g^{-1}Lg, g \in K^*$. A full system is successfully found for operators of the form

$$L = D_x D_y (pD_x + qD_y) + a_{20} D_x^2 + a_{11} D_{xy} + a_{02} D_y^2 + a_{10} D_x + a_{01} D_y + a_{00} \ , \quad (19)$$

where all the coefficients belong to K. Recall that by the classical definition, an invariant of an operator is a function of the operator's coefficients which is not altered by transformations of the operator in a certain way.

I used the invariance of obstacles to find twelve invariants of LPDOs of the form above. Eight of these invariants can be expressed in terms of the other four, and therefore I found four independent invariants. Although these four invariants are already useful, it is highly preferable to have a full system of invariants. Such a system uniquely defines the class of an operator with respect to the considered transformations. Thus, the invariant properties of operators can be described in terms of invariants, and moreover some normalized form of an operator can be easily found.

The four invariants obtained from the theory of obstacles do not form a full system. However, I found a fifth invariant by a separate method, and proved that these five invariants together form a full system of invariants. Thus, I showed that the theory of obstacles provides us with an easy way to obtain some invariants for hyperbolic LPDOs of arbitrary orders, but the problem of finding a full system of invariants for the general case remains open.

These results have been published. The case $q = p = 1$, in the notation of equation (19) is considered in [24], and in the newly submitted paper [25] the general case was considered.

The computation of the five invariants has been implemented in my MAPLE-package (chapter 6).

1.8.3 Parametric Factorizations of Non-Separable LPDOs

Up to now, most of the activity in factorization has concentrated on the separable case (e.g. hyperbolic operators) and there is as yet a lack of knowledge about the non-separable case, meaning that the symbol is non-separable, e.g. parabolic operators. There is a distinction in kind between these two cases. For a separable LPDO on the plane a factorization is determined uniquely by a factorization of the operator's symbol (principal symbol) [10]. In contrast, for the non-separable case, not only is uniqueness lost, but even parametric factorizations (families of factorizations) may appear. Therefore, in chapter 4, I study parametric factorizations of LPDOs. I investigate the case of non-separable LPDOs of orders two, three and four on the plane. I prove that families of factorizations can exist only for a few certain types of factorizations. For these cases I give explicit examples. For the operators of orders two and three it is shown that a family may be parameterized by just one function in one variable (the function may be the constant function).

Chapter 4 gives the first non-trivial example of a parametric factorization of an LPDO of high order. Specifically, I have found the following fourth-order irreducible family of factorizations:

$$D_{xxyy} = \Big(D_x + \frac{\alpha}{y+\alpha x+\beta}\Big)\Big(D_y + \frac{1}{y+\alpha x+\beta}\Big)\Big(D_{xy} - \frac{1}{y+\alpha x+\beta}(D_x + \alpha D_y)\Big),$$

where $\alpha, \beta \in K\backslash\{0\}$ (see Example 4.12).

As with the previous chapters, the results have been obtained with the help of my MAPLE-package (chapter 6).

The results have been accepted for publishing [28].

1.8.4 Generalized Laplace Transformations

Generally speaking, the final goal of all the investigations of LPDOs is the solution of the corresponding LPDEs. Factorization is one of the solution methods for LPDEs. Another important method is one based on transformations. Transformations can be used in two different ways. The first way applies certain transformations to a given LDPO until some LPDO with a known analytical solution is obtained. Then one computes the solution of the original LPDO. The second way consists in applying certain transformations to LPDOs which already have a known analytical solution. This leads to new analytically solvable LPDOs, and therefore one can extend the class of LPDOs which have a known analytical solution. In chapter 5, I suggest new transformations of LPDOs that help to find new classes of analytically solvable LPDEs.

In section 5.2 I suggest some new transformations (I call them GL-transformations) for LPDOs of arbitrary orders and in multidimensional space. The idea appeared through considering a definition that is not commonly used of Laplace transformations. Thus, GL-transformations generalize the Laplace ones.

I prove some properties of GL-transformations for LPDOs of general form, and then concentrate on LPDOs of the form

$$L = D_x \circ D_y + aD_x + bD_y + c \ ,$$

where all the coefficients belong to K. For this case, all possible GL-transformations are described. Finally, I show several examples of applications of GL-transformations.

The results of the chapter have been submitted for publication [29].

1.8.5 A Maple Package for LPDOs with Parametric Coefficients

In chapter 6 I introduce my package, implemented inMaple, which has been an essential computational tool, used to obtain the results of this thesis. The package is useful for the consideration of LPDOs with symbolic coefficients. Basic arithmetic operations, Laplace invariants and transformations, invariants for third-order bivariate LPDOs, obstacles to factorizations and factorizations of LPDOs of orders two and three (Grigoriev-Schwarz algorithm) are implemented in the package.

Among the distinctive features of the package, I would like to mention that LPDOs are kept as a set of coefficients, and, therefore, one has easy access to them. This is in contrast to previous representations of LPDOs in Maple, from which coefficients could be extracted only with great difficulty. Another feature of the package is the absence of restrictions on the number of variables, on the orders of considered LPDOs, and on the parameters occurring in the operators. The only thing one has to declare at the beginning of every worksheet is a list of variables. However, this information is usually known.

2 Obstacles to Factorizations of Linear Partial Differential Operators

2.1 Introduction

The Laplace Cascade method and all its generalizations and variations essentially use factoring of a linear partial differential operator of a certain form. They work on the assumption that there is a constructive factorization algorithm to factor an LPDO of a certain form. In the classical Laplace Cascade method the factorization problem is completely investigated and relatively easy. Indeed, an LPDO of the form

$$L = D_x \circ D_y + aD_x + bD_y + c$$

can possibly have only the following factorizations:

$$L = (D_x + b) \circ (D_y + a), \ L = (D_y + a) \circ (D_x + b).$$

The first factorization is realized if and only if the Laplace invariant h vanishes, while the second one if and only if the Laplace invariant k equals zero.

For an ordinary linear differential operator L the Loewy uniqueness theorem [18] states that if

$$L = P_1 \circ \cdots \circ P_k = \widetilde{P}_1 \circ \cdots \circ \widetilde{P}_t$$

are two different irreducible factorizations, then they have the same number of factors (that is $k = t$) and the factors are pairwise "similar" in some transposed order. In the scope of the Loewy-Ore theory there is a factorization algorithm for LODOs over the field $\mathbb{Q}(x, y)$ of rational functions.

Unfortunately not much is known about factorization properties of LPDOs. One of the main problems of the LPDOs case lies in non-uniqueness of factorizations. There is an interesting example given by Landau [5]: the operator

$$L = D_x^3 + xD_x^2 D_y + 2D_x^2 + (2x + 2)D_x D_y + D_x + (2 + x)D_y$$

has two factorizations into different numbers of irreducible factors:

$$L = Q \circ Q \circ P = R \circ Q \ ,$$

for the operators

$$P = D_x + xD_y, \quad Q = D_x + 1, \quad R = D_{xx} + xD_{xy} + D_x + (2 + x)D_y \ .$$

Note that the second-order operator R is absolutely irreducible, that is one cannot factor it into product of first-order operators with coefficients in any extension of $\mathbb{Q}(x, y)$.

Due to the fast development of the integration algorithms for PDEs involving its factorization, the problem of construction of an efficient factorization algorithm has been quite popular over the last decades.

One important direction of the development has been attacking of the non-uniqueness of factorizations, and inventing new definitions of factorizations [33, 17, 11]. Then the conventional factorization becomes the special case of the generalized factorization, and some analogues of the Loewy-Ore uniqueness theorem can be proved. In one of the earliest attempts [17] the factoring of a linear homogeneous partial differential system is treated as finding superideals of a left ideal in the ring of LPDOs rather than factoring a single LPDO, and a generalization of the Beke-Schlesinger algorithm for factoring LODOs, whose coefficients belong to $\overline{\mathbb{Q}}(x, y)$ has been given. The algorithm is based on an algorithm for finding hyperexponental solutions of such ideals. In [11] a given LPDO is considered as a generator of a left D-module over an appropriate ring of differential operators. In this algebraic approach decomposing a D-module means finding overmodules which describe various parts of the solution of the original problem.

Another direction was founded by Miller [21], who was first to think about some analogue of well-known Hensel lifting of polynomial theory for LPDOs. The scientist has been considering LPDOs of order two and three only. In [10] Grigoriev and Schwarz generalize the idea to the LPDOs of arbitrary order (see in detail in the section 1.7).

Despite of all these results, and many others (for ex. [3, 36, 34]), the general factorization problem remained open. All the theories are either pure theoretical, and do not provide an algorithmic way of establishing factorizability of a given LPDO, or treat some special class of LPDOs.

In this chapter I study properties of factorizations of LPDOs over a field. The starting point is the algorithm of Grigoriev–Schwarz, which extends a factorization (into coprime factors) of the operator's symbol to a factorization of the whole operator. At the first step of the algorithm, only the highest terms of the factors of a factorization are known. At every succeeding step, either we determine the next component in each factor, or we conclude that there is no such factorization. In the latter case we lose all of the information about the operator that we obtained implicitly during the execution of the algorithm. Here, I suggest that the information can be used, and introduce the notions of partial factorizations and common obstacles.

The partial factorizations help to prove Theorem 2.9 below, which states that a factorization is uniquely defined from a certain moment on (here we do not require that the initial factorization of the operator's symbol has coprime factors). Theorem 1.3 of Grigoriev-Schwarz is a particular case of this theorem.

For the operators of order two the common obstacles are the invariants of Laplace, however, for operators or higher order the common obstacle is not so good notion, and does not enjoy useful properties like invariance under gauge transformations $L \mapsto g(x,y)^{-1}Lg(x,y))$. Some examples can be seen in [16]. On the other hand we would like to describe all factorable (or unfactorable) LPDOs in some algebraic terms, and understand what actually prevents an LPDO to be factorable. Thus we suggest to consider some special algebraic structure, which we call the ring of obstacles. Then one can prove that the symbols of all common obstacles belong to the same class of this factor ring. We call this class the obstacle to factorizations. Obstacles to factorization enjoy some important properties, for example uniqueness, invariability w.r.t. the Gauge transformations.

The last property leads us to an application in the invariants finding area. Thus, a full system of invariants w.r.t. the gauge transformations $L \to g^{-1}Lg$ is found in the chapter 3.

The results of this chapter have been published [26, 27].

2.2 Partial Factorizations

Definition 2.1. Let $L \in K[D]$ and suppose that its symbol has a decomposition $\mathrm{Sym}_L = S_1 \dots S_k$. Then we say that the factorization

$$L = F_1 \circ \dots \circ F_k, \quad \text{where} \quad \mathrm{Sym}_{F_i} = S_i \ , \ \forall i \in \{1, \dots, k\},$$

is *of the* $(S_1)(S_2)\dots(S_k)$.

Definition 2.2. Let for some operators L, for $F_i \in K[D]$, $i = 1, \dots, k$ and for some $t \in \{0, \dots, \mathrm{ord}(L)\}$

$$\mathrm{ord}(L - F_1 \circ \dots \circ F_k) < t \tag{20}$$

holds. Then we say that $F_1 \circ \dots \circ F_k$ is *a partial factorization* of order t of the operator L. If in addition $S_i = \mathrm{Sym}_{F_i}$, $i = 1, \dots, k$ (so $\mathrm{Sym}_L = S_1 \dots S_k$), then this partial factorization is of the factorization type $(S_1)\dots(S_k)$.

Remark 2.3. Every usual factorization of $L \in K[D]$ is a partial factorization of order 0.

Remark 2.4. Let $L \in K[D]$, $\mathrm{ord}(L) = d$. Then for every factorization of the symbol $\mathrm{Sym}_L = S_1 \dots S_k$ the corresponding composition of operators $\widehat{S_1} \circ \dots \circ \widehat{S_k}$ is a partial factorization of order d.

Let $L \in K[D]$ and $F_1 \circ \dots \circ F_k$ be a partial factorization of order t. Note that the condition (20) still holds if we change any term whose order is less than or equal to $t - (d - d_j)$ in any factor F_j, $j \in \{1, \dots, k\}$. Thus we obtain new partial factorizations of order less than or equal t. Then we naturally introduce the following definition.

Definition 2.5. Let $L \in K[D]$, $\mathrm{Sym}_L = S_1 \dots S_k$, $\mathrm{ord}(S_i) = d_i,\ i = 1, \dots, k$ and

$$F_1 \circ \dots \circ F_k, \quad F_1' \circ \dots \circ F_k'$$

be partial factorizations of orders t and t' respectively. Let $t' < t$, then $F_1' \circ \dots \circ F_k'$ is an extension of $F_1 \circ \dots \circ F_k$ if

$$\mathrm{ord}(F_i - F_i') < t - (d - d_i),\ \forall i \in \{1, \dots, k\}\ .$$

Example 2.6. Consider the fifth-order operator

$$L = (D_x^2 + D_y + 1) \circ (D_x^2 D_y + D_x D_y + D_x + 1)\ .$$

Compositions of the type

$$(D_x^2 + \dots) \circ (D_x^2 D_y + \dots)\ ,$$

where ellipses mean arbitrarily chosen terms of lower orders, are partial factorizations of order 5. Their extensions are the following fourth-order partial factorizations of the type

$$(D_x^2 + D_y + \dots) \circ (D_x^2 D_y + D_x D_y + \dots).$$

Remark 2.7. Let $L \in K[D]$. Then $F_1 \circ \dots \circ F_k$ is a partial factorization of L of the type $(S_1) \dots (S_k)$ if and only if $F_1 \circ \dots \circ F_k$ is an extension of a partial factorization $S_1 \circ \dots \circ S_2$.

2.3 Generalization of Grigoriev-Schwarz Theorem

Consider an operator $L \in K[D]$ of some order d, and some factorization of its symbol

$$Sym_L = S_1 \cdot S_2\ .$$

Then the corresponding composition of operators $\widehat{S}_1 \circ \widehat{S}_2$ is a partial factorization of the operator L, and

$$L = \widehat{S}_1 \circ \widehat{S}_2 + R\ ,$$

for some $R \in K[D]$ of order less than d.

Suppose S_1 and S_2 are coprime, then by the Grigoriev-Schwarz Theorem 1.3, there exists at most one extension of this partial factorization to a factorization of the whole operator L .

Suppose now that there exists a nontrivial common divisor of S_1 and S_2. Then the uniqueness of the extension is not necessarily the case.

Example 2.8. Consider the operator of Landau [5]

$$L = D_x^3 + xD_x^2 D_y + 2D_x^2 + (2x+2)D_x D_y + D_x + (2+x)D_y,$$

which is a frequently cited instance of an operator that has two factorizations into different numbers of irreducible factors (that is, factors that cannot be factored into factors of smaller orders):

$$L = (D_x+1)\circ(D_x+1)\circ(D_x+xD_y) \ = \ (D_x^2+xD_xD_y+D_x+(2+x)D_y)\circ(D_x+1).$$

The same operator L (the symbol of L is $X^3 + xX^2Y$) has a whole family of factorizations into two factors with the symbols $S_1 = X$ and $S_2 = X(X + XY)$ respectively:

$$L = \Big(D_x+1+\frac{1}{x+f_1(y)}\Big)\circ\Big(D_x^2+xD_xD_y+(1-\frac{1}{x+f_1(y)})D_x+(x+1-\frac{x}{x+f_1(y)})D_y\Big),$$

where $f_1(y) \in K$ is a functional parameter.

Though there is no uniqueness of factorization in this case, we may formulate the following theorem:

Theorem 2.9. *Let $L \in K[D]$ be of some order d, and a factorization of its symbol be known:*

$$\mathrm{Sym}_L = S_1 \cdot S_2 \ ,$$

where the greatest common divisor of S_1 and S_2 be some homogeneous polynomial S_0 of order s.

Then for every $(d-d_0)$th order partial factorization of the type $(S_1)(S_2)$, there is at most one extension to a complete factorization of L of the same type.

To prove the theorem we will use the following two propositions.

Proposition 2.10. *Let S_1, S_2, p be homogeneous polynomials, in an arbitrary number of variables, of orders d_1, d_2, s $(0 < s < d_1 + d_2)$ respectively. Let, in addition, S_1 and S_2 be coprime. Then there exists at most one pair (u, v) of homogeneous polynomials u and v of orders $s - d_1$ and $s - d_2$ respectively, such that*

$$S_1 \cdot u + S_2 \cdot v = p \ . \tag{21}$$

Proof. Suppose there are two pairs (u_1, v_1) and (u_2, v_2) that satisfy the equation (21). Then

$$S_1 \cdot (u_1 - u_2) + S_2 \cdot (v_1 - v_2) = 0 \ ,$$

that implies

$$S_1 \cdot (u_1 - u_2) = S_2 \cdot (v_2 - v_1) \ .$$

Since the polynomials S_1 and S_2 are coprime, S_1 divides $(v_2 - v_1)$. However,

$$\mathrm{ord}(v_2 - v_1) \leq s - d_2 < d_1 + d_2 - d_2 = d_1 .$$

Thus, $\mathrm{ord}(v_2 - v_1) < \mathrm{ord}(S_1)$. Contradiction. □

The second fact is the generalization of Proposition 2.10 to the case of non-coprime polynomials.

Proposition 2.11. *Let S_1, S_2, p be homogeneous polynomials of orders d_1, d_2, s respectively and in an arbitrary number of variables. Let a polynomial S_0 of order d_0 be the greatest common divisor of S_1 and S_2, and $0 < s < d_1 + d_2 - d_0$. Then there exist at most one pair (u, v) of homogeneous polynomials u and v of orders $s - d_1$ and $s - d_2$ respectively, such that*

$$S_1 \cdot u + S_2 \cdot v = p. \tag{22}$$

Proof. Divide the both sides of the equation (22) by the polynomial S_0. Then use the proposition 2.10. □

To prove Theorem 2.9 is enough to prove the following lemma.

Lemma 2.12. *Let $L \in K[D]$, $\mathrm{Sym}_L = S_1 \cdot S_2$, $\mathrm{ord}(L) = d$, and let the greatest common divisor of S_1 and S_2 be a homogeneous polynomial S_0 of order s. Then for every t-th ($t \leq (d - d_0)$) order partial factorization of the type $(S_1)(S_2)$, there is at most one (up to lower order terms) extension to the partial factorization of order $t - 1$ of the same type.*

Proof. If $d_0 = 0$, then the statement of the lemma is implies from Theorem 1.3. If $d_0 > 0$, consider the general form of a complete factorization of L, that extends the given t-th order partial factorization:

$$L = \left(\widehat{S}_1 + \sum_{j=0}^{k_1 - 1} G_j \right) \circ \left(\widehat{S}_2 + \sum_{j=0}^{k_2 - 1} H_j \right) , \tag{23}$$

where $k_1 = \mathrm{ord}(S_1)$, $k_2 = \mathrm{ord}(S_2)$ and $G_j \in K_j[D], \quad H_i \in K_i[D], \quad j = 0, \ldots, (k_1 - 1),\ i = 0, \ldots, (k_2 - 1)$. By comparing components of order $t - 1$ on the both sides of the equality (23), we get

$$L_{t-1} = H_{t-k_1-1} \cdot S_1 + G_{t-k_2-1} \cdot S_2 + P_{t-1}, \tag{24}$$

where P_{t-1} is a homogeneous polynomial of order t, which is determined uniquely by the polynomials G_i, H_j, $i > t - k_1 - 1$, $j > t - k_2 - 1$, which are components of order t of the given partial factorization. We assume that polynomials G_i, H_i are 0 for $i < 0$.

Now, since $t - 1 < d - d_0$, we may apply proposition 2.11: there is at most one solution of equation (24). Thus there exists at most one extension to a partial factorization of order $t - 1$. □

Corollary 2.13. *Let* $L \in K[X]$, $\mathrm{Sym}_L = S_1 \cdot S_2$, *and* S_1 *and* S_2 *be coprime. Then there is at most one factorization of* L *of the type* $(S_1)(S_2)$. *Thus the theorem is a generalization of the Grigoriev-Schwarz Theorem 1.3.*

Corollary 2.14. *In the case of ordinary differential operators, the greatest common divisor of* S_1 *and* S_2 *is*

$$\gcd(S_1, S_2) = X^{d_0}, \quad \textit{where} \quad d_0 = min(\mathrm{ord}(S_1), \mathrm{ord}(S_2)).$$

Then for every partial factorization of order

$$max\big(\mathrm{ord}(S_1), \mathrm{ord}(S_2)\big) - 1$$

there is at most one extension to a complete factorization.

Corollary 2.15. *Let* $L \in K[D]$, $\mathrm{Sym}_L = S_1 \cdot S_2$, *and let* S_1 *be coprime with* S_2. *Then for every* t, $t < \mathrm{ord}(L)$ *there is at most one (up to lower order terms) partial factorization of order* t.

2.4 Factorable with Certain Type LPDOs

Reformulate Theorem 1.3:

Theorem 2.16. *Let* $L \in K[D]$, $\mathrm{Sym}_L = S_1 \cdot S_2 \dots S_k$ *and let* $S_1, \dots, S_k$ *be coprime. Then there exists at most one factorization of the type* $(S_1)(S_2)\dots(S_k)$.

Consider LPDOs with the same symbol and describe the quantity of those of them that are factorable with a certain factorization type.

Theorem 2.17. *Consider the variety of all the operators in* $K[D]$ *that have the symbol* $\mathrm{Sym} = S_1 \dots S_k$, $\mathrm{ord}(S_i) = d_i$, $i = 1, \dots, k$. *Then the codimension of the subvariety of the operators that have a factorization of the type* $(S_1)(S_2)\dots(S_k)$ *equals*

$$\binom{n+d-1}{n} - \sum_{i=1}^{k} \binom{n+d_i-1}{n} .$$

Proof. Consider the problem of the factorization of L of the type $(S_1)(S_2)\dots(S_k)$ in the general form:

$$L = \Big(S_1 + \sum_{i=0}^{d_1-1} G_i^1\Big) \circ \dots \circ \Big(S_k + \sum_{i=0}^{d_k-1} G_i^k\Big), \tag{25}$$

where G_i^j denotes the i-th component in the j-th factor. Compare components of orders t, $0 \le t \le \mathrm{ord}(L) - 1$ on both sides of (25), then we have

$$P_t = (\mathrm{Sym}/S_1) \cdot G^1_{t-d+d_1} + \dots + (\mathrm{Sym}/S_k) \cdot G^k_{t-d+d_k}, \tag{26}$$

where P_t is a homogeneous polynomial of order t, which is determined uniquely by the polynomials $G_i,\ H_j,\ \ i > t-k_1,\ \ j > t-k_2$, and so it is known if we solve equations (26) in "descent" order, that is if we start with $t = \mathrm{ord}(L)-1$, and reduce t by one at each succeeding step.

Polynomials $G_i, H_j,\ \ i > t-k_1,\ \ j > t-k_2$, and so P_t are determined uniquely: it is an immediate consequence of the following lemma:

Lemma 2.18. *Let $S_1, \ldots, S_k$ are pairwise coprime homogeneous polynomials of orders $d_1, \ldots, d_k$ respectively. Denote $S = S_1 \ldots S_k$. Then there is at most one tuple $(A_1, \ldots, A_k)$ such that*

$$P_t = (S/S_1) \cdot A_1 + \cdots + (S/S_k) \cdot A_k, \tag{27}$$

where $\mathrm{ord}(P_t) = t,\ t < \mathrm{ord}(S)$, *and* $\mathrm{ord}(A_i) + \mathrm{ord}(S/S_i) = t$.

Proof. Assume we have two such tuples: $(A'_1, \ldots, A'_k)$ and $(A''_1, \ldots, A''_k)$. Consider the difference of the equations corresponding to them, so we have

$$0 = (S/S_1) \cdot B_1 + \cdots + (S/S_k) \cdot B_k, \tag{28}$$

where $B_i = A'_i - A''_i$, $i = 1, \ldots, k$. Without loss of generality we may assume $B_1 \neq 0$ and rewrite equation (28) in the form

$$-(S/S_1) \cdot B_1 = (S/S_2) \cdot B_2 + \cdots + (S/S_k) \cdot B_k.$$

Every component on the right is divisible by S_1, while (S/S_1) is not so. Thus, B_1 is divisible by S_1, and so $\mathrm{ord}(B_1) \geq \mathrm{ord}(S_1)$.

On the other hand, we have $\mathrm{ord}(A_i)+\mathrm{ord}(S/S_i) = t$ and $t < \mathrm{ord}(S)$, that is $\mathrm{ord}(A_i) < \mathrm{ord}(S_i)$, and so $\mathrm{ord}(B_1) < \mathrm{ord}(S_i)$. This is a contradiction with the results of the previous paragraph. □

The factorization exists if the system of all the equations (26), $t = d - 1, \ldots, 0$ is compatible. The codimension equals the number of independent equations in the coefficients of the operator.

For every t we have the linear equation (26) in the polynomials $G^1_{t-d+d_1}, \ldots, G^k_{t-d+d_k}$, which is equivalent to the system of linear equations in their coefficients. Let the system be $A \cdot \vec{g} = \vec{c}$, where A is the matrix of the system. The system has a unique solution, and so the rank of the matrix A equals the number v of variables. That is the columns of the matrix A are linearly independent.

The system $A \cdot \vec{g} = \vec{c}$ is compatible when vector $\vec{c}$ belongs to a v-dimensional affine space, generated by the columns of A. The length of vector $\vec{c}$ equals the number of equations in the system. Thus the codimension of the solution space is the difference between the number of equations and the number of variables.

Now the codimension of the variety of all the operators that have factorizations of the type $(S_1)(S_2)\ldots(S_k)$ equals the difference between the number of equations and the number of variables at all the steps together. This can be computed using the following combinatorial fact:

Lemma 2.19. *The cardinality of the set*

$$\{M = x_1^{d_1} \dots x_n^{d_n} \mid d_1 + \dots + d_n = t\}$$

of monomials in n *independent variables* $x_1, \dots, x_n$ *is* $\binom{n+t-1}{t} = \binom{n+t-1}{n-1}$.

The theorem about codimension is proved. $\square$

Example 2.20. Consider all second-order operators in $K[D_x, D_y]$ with the symbol $S_1 \cdot S_2$, where S_1, S_2 are certain coprime homogeneous operators of the first order. By Theorem 2.17, the codimension of the variety of all the operators that have a factorization of the type $(S_1)(S_2)$, is 1.

One may find explicit formulae for the equation which defines this variety. Let, for example,

$$\widehat{S}_1 = D_x, \; \widehat{S}_2 = D_y \; .$$

Consider all the operators of the form

$$L = D_x D_y + a_{10} D_x + a_{01} D_y + a_{00} \; .$$

Such an operator has a factorization of the type $(S_1)(S_2)$ if and only if coefficients a_{10}, a_{01}, a_{00} satisfy the condition

$$a_{00} - a_{10} a_{01} - \partial_x(a_{10}) = 0 \; .$$

Example 2.21. Consider all third-order operators in $K[D_x, D_y]$ with the symbol $S_1 \cdot S_2$, where S_1, S_2 are certain coprime homogeneous operators of first and second orders respectively. By Theorem 2.17, the codimension of the variety of all the operators that have a factorization of the type $(S_1)(S_2)$, is 2.

If we consider a factorization of the type $(S_1)(S_2)(S_3)$, where S_1, S_2, S_3 are coprime homogeneous operators of the first order, then, by Theorem 2.17, the codimension is 3.

2.5 Obstacles to Factorizations

Even the LPDOs with a factorable symbol happen to be factorable quite rarely. On the other hand one can always introduce the following notion.

Definition 2.22. Let $L \in K[D]$, $\mathrm{Sym}_L = S_1 \dots S_k$. An operator $R \in K[D]$ is called a to factorization of the type $(S_1)(S_2) \dots (S_k)$ if there exists a factorization of this type for the operator $L - R$ and R has minimal possible order.

Example 2.23. Consider the operator

$$L = D_{xy} + a D_x + b_y + c \; ,$$

where all the coefficients belong to K. Then regardless whether it is factorable or not, one can present L in the form

$$L = (D_x + b) \circ (D_y + a) + h = (D_y + a) \circ (D_x + b) + k,$$

where

$$h = c - a_x - ab, \quad k = c - b_y - ab$$

are the Laplace invariants.

Naturally, common obstacles are closely related to partial factorizations:

Proposition 2.24. *Let $L \in K[D]$, $\mathrm{Sym}_L = S_1 \dots S_k$. A common obstacle to a factorization of the type $(S_1) \dots (S_k)$ is of order t if and only if the minimal order of a partial factorization of this type is $t + 1$.*

Though common obstacles are the natural generalization of the Laplace invariants, they do not preserve the important properties of those. Neither common obstacles nor their symbols are unique in general, or invariant (w.r.t. the gauge transformations $L \to g^{-1}Lg$). On the other hand we would like to describe all factorable (or unfactorable) LPDOs in some algebraic terms, and understand what actually prevents an LPDO to be factorable. Thus, we suggest to consider the following algebraic structure:

Definition 2.25. Let $L \in K[D]$ and $\mathrm{Sym}_L = S_1 \cdot S_2 \cdot \dots \cdot S_k$. Then we say that the to factorizations of the type $(S_1) \dots (S_k)$ is the factor ring

$$K(S_1, \dots, S_k) = K[X]/I,$$

where

$$I = \left(\frac{\mathrm{Sym}_L}{S_1}, \dots, \frac{\mathrm{Sym}_L}{S_k} \right)$$

is a homogeneous ideal.

Example 2.26. In the case of two factors ($k = 2$), the ring of obstacles is

$$K(S_1, S_2) = K[X]/(S_1, S_2).$$

Now we prove an important property of rings of obstacles.

Theorem 2.27. *Let $L \in K[D]$ and $\mathrm{Sym}_L = S_1 \cdot S_2 \dots S_k$, where S_i, $i \in \{1, \dots, k\}$ are pairwise coprime. Then the symbols of all common obstacles to factorization of the type $(S_1) \dots (S_k)$ belong to the same class in the factor-ring $K(S_1, \dots, S_k)$.*

Proof. Denote $d_i = \text{ord}(S_i),\ \ i \in \{1, \dots, k\}$ and let t be the order of common obstacles. In the same way as in the proof of Theorem 2.17, we obtain the equation (26), that is the symbol of every common obstacle can be written in the form

$$P_t - ((\text{Sym}_L/S_1) \cdot G^1_{t-d+d_1} + \dots + (\text{Sym}_L/S_k) \cdot G^k_{t-d+d_k}),$$

where P_t is known, uniquely determined and the same for all common obstacles polynomial. Thus all common obstacles belong to the class $[P_t]$ of the factor-ring $K(S_1, \dots, S_k)$. □

Definition 2.28. We say that the class of common obstacles in the ring of obstacles is the *obstacle to factorization.*

Remark 2.29. Every element of this class is again a common obstacle.

2.6 Properties of Obstacles to Factorizations

Definition 2.30. We say that two types of factorizations $(S_1)\dots(S_k)$ and $(b_1S_1)\dots(b_kS_k)$ are *similar*, if $b_1, \dots, b_k \in K$ and $b_1 \dots b_k = 1$.

Theorem 2.31. *For an operator in $K[X]$ the rings of obstacles and the obstacles of similar types are the same.*

Proof. Consider an operator $L \in K[D]$ and two similar types of factorizations of L: $(S_1)\dots(S_k)$ and $(b_1S_1)\dots(b_kS_k)$, where $b_i \in K,\ i = 1, \dots, k$. Then the homogeneous ideals $(S_1, \dots, S_k)$ and $(b_1S_1, \dots, b_kS_k)$ are the same, thus the rings of obstacles are also.

Every common obstacle of the type $(S_1)\dots(S_k)$ and of order d_0 may be written as

$$P = L - (\widehat{S}_1 + T_1) \circ \dots \circ (\widehat{S}_k + T_k), \tag{29}$$

where T_i is the sum of components of orders $d_i - 1, \dots, d - d_i - d_0 + 1$, and $\text{ord}(P) = d_0$.

There exist $T'_1, \dots, T'_k$ such that T'_i is the sum of components of orders $d_i - 1, \dots, d - d_i - d_0 + 1$ and

$$(S_1 + T_1) \circ \dots \circ (S_k + T_k) = (b_1S_1 + T'_1) \circ \dots \circ (b_kS_k + T'_k).$$

Thus P is a common obstacle of order d_0 of the type $(b_1S_1)\dots(b_kS_k)$. On the other hand, we know that the rings of obstacles $K(S_1, \dots, S_k)$ and $(b_1S_1, \dots, b_kS_k)$ are the same. Thus obstacles are the same also. □

Theorem 2.32. *Let $L \in K[D]$, and P be a common obstacle to factorizations of L. Then $g^{-1}Pg$ will be a common obstacle for the gauge transformed operator $g^{-1}Lg$, where $g \in K^*$.*

Proof. Consider a common obstacle (29) for L of order d_0. Then we have

$$g^{-1}Pg = g^{-1}Lg - g^{-1} \circ (S_1 + T_1) \circ \Pi_{j=2}^{k-1}(S_i + T_i) \circ (S_k + T_k) \circ g.$$

There exist $T_1', \dots, T_k'$ such that T_i' is the sum of components of orders $d_i - 1, \dots, d - d_i - d_0 + 1$ and

$$g^{-1}Pg = g^{-1}Lg - (g^{-1}S_1 + T_1') \circ \Pi_{j=2}^{k-1}(S_i + T_i') \circ (gS_k + T_k').$$

□

Corollary 2.33. *Obstacles are invariant under the gauge transformations $L \to g^{-1}Lg$.*

Proof. Under the gauge transformations common obstacles are conjugated, and so symbols of common obstacles are the same. □

Theorem 2.34. *Let $n = 2$, $L \in K[D]$, $\mathrm{ord}(L) = d$, and let $\mathrm{Sym}_L = S_1 \dots S_k$, where S_i, $i \in \{1, \dots, k\}$ are pairwise coprime. Thus the ring of obstacles $K(S_1, \dots, S_k)$ is 0 to order $d-1$. (That is, non-zero obstacles may be only less than or equal to $d-2$.)*

Proof. Denote $d_i = \mathrm{ord}(S_i)$, $i \in \{1, \dots, k\}$ and repeat the reasoning of the proof of Theorem 2.17. Thus we write equation (26) for $t = d - 1$:

$$P_{d-1} = (\mathrm{Sym}_L/S_1) \cdot G^1_{d_1-1} + \dots + (\mathrm{Sym}_L/S_k) \cdot G^k_{d_k-1}.$$

It has at most one solution w.r.t. $G^1_{d_1-1}, \dots, G^k_{d_k-1}$. Consider the corresponding system of equations in their coefficients. By Lemma 2.19 the number of equations in this system is d, the number of variables is d also. Thus the system has a unique solution, and so we have a partial factorization of order $d-1$. □

Theorem 2.35. *Let $L \in K[D]$ be a bivariate hyperbolic of order d. Then for each type of factorization, a common obstacle is unique.*

Proof. Let the type of the factorizations be $(S_1)\dots(S_d)$, and let P be a common obstacle for this type. Let the order of common obstacles be p. Assume there is another common obstacle for this type, then it is of the form

$$P + (\mathrm{Sym}_L/S_1) \cdot A_1 + ... + (\mathrm{Sym}_L/S_d) \cdot A_d,$$

where A_i are some homogeneous polynomials of orders $p_i = p - \mathrm{ord}(\mathrm{Sym}_L/S_i) = p - (d-1)$. That is $p \geq d - 1$.

On the other hand, by Theorem 2.34, the ring of obstacles is 0 to order $d-1$, and so $p \leq d - 2$. □

2.7 Bivariate Operators of Order Two

Consider a second-order hyperbolic operator $L \in K[D_x, D_y]$ in such a system of coordinate that the symbol of L is XY. By Theorems 2.34 and 2.35, both common obstacles to factorizations of L have order 0 and are uniquely defined. We compute explicit formulas.

Theorem 2.36. *Let*

$$L = D_x \cdot D_y + aD_x + bD_y + c,$$

where $a_{10}, a_{01}, a_{00} \in K$*. Then obstacles of types* $(X)(Y)$, $(Y)(X)$ *are*

$$\begin{array}{l} c - ab - \partial_x(a), \\ c - ab - \partial_y(b) \end{array}$$

respectively.

Proof. A factorization of L of type $(X)(Y)$ has the form

$$L = (D_x + g_{00}) \circ (D_y + h_{00}) \, ,$$

where g_{00}, h_{00} are some elements of K. Comparing components of order 1 on the right and on the left, we have

$$(a - h_{00})D_x + (b - g_{00})D_y = 0 \, , \tag{30}$$

that is $a = h_{00}$, $b = g_{00}$. Now we compute the obstacle as

$$L - (D_x + b) \circ (D_y + a) = c - ab - \partial_x(a).$$

One may find the obstacle for type $(Y)(X)$ analogously. □

Remark 2.37. The obtained obstacles are the invariants of Laplace [34].

2.8 Bivariate Operators of Order Three

Consider some operator $L \in K[D_x, D_y]$ of order three. Let the symbol of L be

$$Sym_L = S_1 \cdot S_2 \cdot S_3 \, ,$$

then the following types of factorizations are possible: six types of factorization into three factors:

$$(S_1)(S_2)(S_3),\ (S_1)(S_3)(S_3), (S_2)(S_1)(S_3),\ (S_2)(S_3)(S_1),\ (S_3)(S_1)(S_2),\ (S_3)(S_2)(S_1),$$

and six types of factorization into two factors:

$$(S_1)(S_2S_3),\ (S_2)(S_1S_3),\ (S_3)(S_1S_2),\ (S_1S_2)(S_3),\ (S_1S_3)(S_2),\ (S_2S_3)(S_1).$$

2.8.1 Factorizations into Two Factors

The theory introduced above applies for the case of pairwise coprime symbols of factors. That is, if the considered type is $(S_1)(S_2S_3)$, then S_1 and S_2S_3 should be coprime. Taking this and the symmetry into account, we restrict ourselves to considering two important special cases: factorization of the type $(X)(X^2+XY)$ for an operator with symbol X^2Y+XY^2 and of the type $(X)(Y^2)$ for an operator with symbol XY^2.

Note that by Theorem 2.34 common obstacles of these types may be of orders one and zero only, in the first case a common obstacle is not unique.

Theorem 2.38. *Let*

$$L = \widehat{\mathrm{Sym}}_L + a_{20}D_{xx} + a_{11}D_{xy} + a_{02}D_{yy} + a_{10}D_x + a_{01}D_y + a_{00},$$

where all $a_{ij} \in K$.

Let $\mathrm{Sym}_L = XY(X+Y)$, *then*

$$\begin{aligned}\mathrm{Obst}_{(X)(YX+YY)} &= \Big(a_{02}^2 - a_{11}a_{02} + a_{01} + \partial_x(a_{02}-a_{11})\Big)D_y + \\ &a_{00} - a_{02}a_{10} + a_{02}^2a_{20} + 2a_{02}\partial_x(a_{20}) - \partial_x(a_{10}) + a_{20}\partial_x(a_{02}) + \partial_{xx}(a_{20}),\end{aligned}$$

is a common obstacle to factorizations of L *of type* $(X)(YX+YY)$.

Let $\mathrm{Sym}_L = X^2Y$, *then*

$$\begin{aligned}\mathrm{Obst}_{(Y)(XX+XY)} &= \Big(a_{10} - a_{20}a_{11} - \partial_y(a_{11})\Big)D_x + \\ &a_{00} - a_{20}a_{01} + a_{20}^2a_{02} + 2a_{20}\partial_y(a_{02}) - \partial_y(a_{01}) + a_{02}\partial_y(a_{20}) + \partial_{yy}(a_{02}),\end{aligned}$$

is a common obstacle to factorizations of L *of type* $(Y)(XX)$.

Proof. All factorizations of type $(X)(YX+YY)$ have the form

$$L = (D_x + G_0) \circ (D_{xy} + D_{yy} + H_1 + H_0), \tag{31}$$

where $G_0 = g_{00} \in K$, $H_1 = h_{10}D_x + h_{01}D_y \in K[D_x, D_y]$, $H_0 = h_{00} \in K$. Compare components of order 2 on both sides of equality (31), then we get a system of linear equations in coefficients h_{10}, h_{01}, g_{00}:

$$\begin{cases} a_{20} = h_{10}, \\ a_{11} = h_{01} + g_{00}, \\ a_{02} = g_{00}. \end{cases}$$

We find the unique solution of the system. Then, we compare coefficients in D_x on the both sides of (31), and so we get

$$h_{00} = a_{10} - a_{20}a_{02} - \partial_x(a_{20}).$$

Now we may compute a common obstacle as $P = L - (D_x + G_0) \circ (D_{xy} + D_{yy} + H_1 + H_0)$.

One may find the obstacle for type $(Y)(XX)$ analogously. □

Example 2.39. Consider operator

$$L = D_x D_y (D_x + D_y) + aD_x + bD_y + c \ ,$$

where $a = a(x, y), b = b(x, y), c = c(x, y)$ - parameters. Below we demonstrate common obstacles to factorizations into two factors and the corresponding partial factorizations. There are three factorizations' types, where the first factor is of order one, and the second is of order two:

$$\begin{aligned} L &= D_x \circ (D_{xy} + D_{yy} + D_y + a) + bD_y + c - a_x \ , \\ L &= D_y \circ (D_{xx} + D_{xy} + D_x + b) + aD_x + c - b_y \ , \\ L &= (D_x + D_y + 1) \circ (D_{xy} + a) + (b - a)D_y + c - a_x - a_y \ . \end{aligned}$$

Also there are three factorizations' types, where the first factor is of order two, and the second is of order one:

$$\begin{aligned} L &= (D_{xy} + D_{yy} + D_y + a) \circ D_x + bD_y + c \ , \\ L &= (D_{xx} + D_{xy} + D_x + b) \circ D_y + aD_x + c \ , \\ L &= (D_{xy} + a) \circ (D_x + D_y + 1) + (b - a)D_y + c - a \ . \end{aligned}$$

There are no other types of factorizations into two factors. I computed this example using my package for LPDOs with symbolic coefficients (see chapter 6).

2.8.2 Factorizations into Three Factors

Here we restrict ourselves in considering of an important special case of the hyperbolic third-order operators in $K[D_x, D_y]$: the operators with the symbol

$$XY(X + Y) \ .$$

The common obstacles for the general case are found analogously, but they are too large to present them on the paper. An easy way to have them computed is to use my package (see the chapter 6 for details). For the same reason, it is enough to consider here the case of the factorization type $(X)(Y)(X + Y)$.

Note that in this case a common obstacle may be of orders 1 and 0 only (Theorem 2.34) and it is unique (Theorem 2.35).

Theorem 2.40. *Let*

$$L = D_x D_y (D_x + D_y) + a_{20} D_{xx} + a_{11} D_{xy} + a_{02} D_{yy} + a_{10} D_x + a_{01} D_y + a_{00},$$

where all $a_{ij} \in K$. The common obstacle of type $(X)(Y)(X + Y)$ is

$$\begin{aligned} \mathrm{Obst}_{(X)(Y)(X+Y)} = \ & (a_{10} - a_{20}a_{11} + a_{20}^2 - \partial_x(a_{20}) + \partial_y(s_2))D_x + \\ & (a_{01} - a_{02}a_{11} + a_{02}^2 + \partial_x(-a_{11} + a_{02}))D_y + \\ & a_{00} + a_{20}a_{02}s_2 + s_2\partial_x(a_{20}) + (a_{20}\partial_x + \partial_{xy} + a_{02}\partial_y)(s_2), \end{aligned}$$

where $s_2 = a_{20} - a_{11} + a_{02}$.

Proof. Every factorization of type $(X)(Y)(X+Y)$ has the form:

$$L = (D_x + g_0) \circ (D_y + h_0) \circ (D_x + D_y + f_0)\,. \qquad (32)$$

Compare components of order 2 and get the only solution

$$h_0 = a_{20},\ g_0 = a_{02},\ f_0 = a_{11} - a_{02} - a_{20}.$$

Now, we may compute the common obstacle as the difference of the left and the right sides of the equation (32). □

Example 2.41. Consider the operator

$$L = D_x D_y (D_x + D_y) + aD_x + bD_y + c$$

of the example 2.39. The common obstacles to factorizations and the corresponding partial factorizations are

$$\begin{array}{rcl} L &=& D_x \circ D_y \circ (D_x + D_y + 1) + R\ , \\ L &=& D_x \circ (D_x + D_y + 1) \circ D_y + R\ , \\ L &=& D_y \circ D_x \circ (D_x + D_y + 1) + R\ , \\ L &=& D_y \circ (D_x + D_y + 1) \circ D_x + R\ , \\ L &=& (D_x + D_y + 1) \circ D_x \circ D_y + R\ , \\ L &=& (D_x + D_y + 1) \circ D_y \circ D_x + R\ , \end{array}$$

where

$$R = aD_x + bD_y + c\ .$$

Thus, all the common obstacles to factorizations into three factors of this operator L are the same.

Also, since the common obstacle is unique, we conclude that the factorization into three factors exists for L if only if the parameters a, b, c are zero simultaneously.

I computed this example using my package for LPDOs with symbolic coefficients (see chapter 6).

2.9 Conclusion

The concept of obstacles to factorization provides us with a new tool for the general purpose for studying and describing all factorable LPDOs. The foundations of the theory have been laid, and now some further investigations, such as the description of functorial properties, the computation of obstacles and obstacle rings, and the generalization to the case of non-coprime factors of the symbol are planned.

The common obstacles have been implemented in MAPLE (see in more detail in the chapter 6).

The results of the following chapter serve as an example of the application of obstacle theory.

3 Invariants of Linear Partial Differential Operators

3.1 Introduction

The Laplace Cascade method essentially uses quantities h, k known as the Laplace invariants. They are indeed invariants (that is unaltered) under the gauge transformations

$$L \mapsto g(x_1, x_2)^{-1} \circ L \circ g(x_1, x_2),$$

which corresponds to the linear transformation of the dependent variable

$$z = \lambda(x, y)z, \quad \lambda(x, y) \neq 0 \tag{33}$$

in the corresponding equation $L(z) = 0$.

The search of invariants is a classical problem in the classification of PDEs. Indeed, whenever we know a full system of invariants for a certain class of equations under the certain transformations, we may easily solve the equivalence problem, as well as uniquely classify a number of simple equations in terms of their invariants, or describe some invariant properties of the considered class of equations. For example, the equation of the form

$$z_{xy} + az_x + bz_y + c = 0, \tag{34}$$

where $a = a(x, y), b = b(x, y), c = c(x, y)$, the Laplace invariants h and k together form a full system of invariants w.r.t. Gauge transformations of a dependent variable, i.e. any other invariant can be expressed in terms of these two [7]. So there is an easy way to judge whether two equations of the form (34) are Gauge equivalent. Thus, it was proved that the equation of the form (34) is Gauge equivalent to the wave equation

$$z_{xy} = 0$$

whenever $h = k = 0$.

Whereas the operators of order two are very actively investigated (classical Laplace's hyperbolic second-order LPDOs, scalar hyperbolic non-linear LPDOs (for ex. [1, 2]), etc), for the hyperbolic operators of high orders not much is known.

A method to obtain some invariants for an hyperbolic operator of arbitrary order has been mentioned in [35]. As well as in the paper [16] a method to compute some invariants for operators of order three is suggested.

In this chapter I present a complete set of invariants for bivariate third-order hyperbolic operators.

The five invariants of the obtained complete system of invariants have been implemented in Maple (see in more detail in the chapter 6).

The results have been partially published [24]. The rest has been submitted [25].

3.2 Connection of Obstacles to Factorizations with Invariants

Consider in $K[D_x, D_y]$ a hyperbolic LPDO of order three. The symbol of such the operator has the form

$$(\alpha_1 X + \beta_1 Y)(\alpha_2 X + \beta_2 Y)(\alpha_3 X + \beta_3 Y) \ ,$$

where all the coefficients belong to K, and the factors are pairwise coprime. In an appropriate system of coordinates, the symbol has the form $XY(pX + qY)$, where $p, q \in K$ and neither p, nor q is equal to zero, and the operator has the normalized form

$$L = D_x D_y (pD_x + qD_y) + a_{20} D_x^2 + a_{11} D_{xy} + a_{02} D_y^2 + a_{10} D_x + a_{01} D_y + a_{00} \ , \quad (35)$$

where the coefficients belong to K. Accumulate all the knowledge of the chapter 2 about the (common) obstacles to factorizations of such operators in the following theorem:

Theorem 3.1. *[24] For an LPDO of the form* (35) *consider its factorizations into first-order factors. Then*

1. *the order of common obstacles is zero or one;*
2. *a common obstacle is unique for each factorization type, and therefore, the corresponding obstacles consist of just one element;*
3. *there are* 6 *common obstacles to factorizations into exactly three factors;*
4. *the symbol of a common obstacle is an invariant w.r.t. the gauge transformations $L \to g^{-1} L g$.*

3.3 Computing of Invariants

First of all, since the symbol of the LPDO L does not altered under the gauge transformations $L \to g^{-1} L g$, then the symbol, and therefore, the coefficients of the symbol are invariants w.r.t. these transformations. Thus, p and q are invariants.

Now we use Theorem 3.1 to compute a number of invariants for the operator L. Suppose for a while that

$$p = 1 \ .$$

Denote the factors of the symbol $\mathrm{Sym}_L = XY(X + qY)$ of L:

$$S_1 = X, \ S_2 = Y, \ S_3 = X + qY \ .$$

Denote the common obstacle to factorizations of the type $(S_i)(S_j)(S_k)$ by Obst_{ijk}.

Then the coefficient at Y of the symbol of the common obstacle Obst_{123} is

$$(a_{01}q^2 + a_{02}^2 - (3q_x + a_{11}q)a_{02} + q_xqa_{11} - \partial_x(a_{11})q^2 + q\partial_x(a_{02}) + 2q_x^2 - q_{xx})/q^2 \ .$$

By Theorem 3.7 this expression is invariant w.r.t. gauge transformations $L \to g^{-1}Lg$. Since the term $(2q_x^2 - q_{xx})/q^2$ and multiplication by q^2 does not influence on being an invariant (because q is an invariant), the following expression is invariant also:

$$I_4 = a_{01}q^2 + a_{02}^2 - (3q_x + a_{11}q)a_{02} + q_xqa_{11} - \partial_x(a_{11})q^2 + q\partial_x(a_{02}) \ .$$

The coefficient at Y of the symbol of the common obstacle Obst_{213} is

$$(I_4 - (\partial_x(a_{20})q^2 - \partial_y(a_{02})q + a_{02}q_y)q + a_{02}q_y)q - q_xq_yq + q_{xy}q^2 + 2q_x^2 - q_{xx}q)/q^2 \ .$$

Again the expressions in q can be omitted, while I_4 is an invariant itself. Therefore,

$$I_2 = \partial_x(a_{20})q^2 - \partial_y(a_{02})q + a_{02}q_y$$

is an invariant.

Similarly, we obtain the invariants

$$\begin{aligned} I_1 &= 2a_{20}q^2 - a_{11}q + 2a_{02} \ , \\ I_3 &= a_{10} + a_{20}(qa_{20} - a_{11}) + \partial_y(a_{20})q - \partial_y(a_{11}) + 2a_{20}q_y \ . \end{aligned}$$

Generally speaking, by Theorem 3.1, there are six different obstacles to factorizations into exactly three factors. In fact, all the coefficients of the symbols of the common obstacles can be expressed in terms of four invariants

$$I_1, I_2, I_3, I_4 \ .$$

Denote the symbol of the common obstacle Obst_{ijk} by Sym_{ijk}. The direct computations justify the following theorem:

Theorem 3.2.

$$\begin{aligned} q^2\mathrm{Sym}_{123} &= (q^2I_3 + I_2 - q_{xy}q + q_{yy}q^2 + q_xq_y)D_x &&+ (I_4 + 2q_x^2 - q_{xx})D_y \ , \\ q^2\mathrm{Sym}_{132} &= (i_2 + I_2)D_x &&+ (I_4 + 2q_x^2 - q_{xx})D_y \ , \\ q^2\mathrm{Sym}_{213} &= (q^2I_3 + q^2q_{yy})D_x &&+ i_3D_y \ , \\ q^2\mathrm{Sym}_{231} &= (q^2I_3 + q^2q_{yy})D_x &&+ i_1D_y \ , \\ q^2\mathrm{Sym}_{312} &= (i_2 + I_2)D_x &&+ (i_1 + I_2q)D_y \ , \\ q^2\mathrm{Sym}_{321} &= i_2D_x &&+ i_1D_y \ , \end{aligned}$$

where

$$\begin{aligned} i_1 &= I_4 - 2\partial_x(I_1)q + 4q_xI_1 - 2I_2q \ , \\ i_2 &= q^2I_3 - 2\partial_y(I_1)q + 2I_1q_y + I_2 \ , \\ i_3 &= I_4 - I_2q - q_xq_yq + q_{xy}q^2 + 2q_x^2 - q_{xx}q \ . \end{aligned}$$

Note that neither of the obtained invariants I_1, I_2, I_3, I_4 depends on the "free" coefficient a_{00} of the operator L, and, therefore, we need at least one another.

3.4 A Full System of Invariants for Third Order LPDOs

Theorem 3.3. *For some non-zero $q \in K$ consider the operators of the form*

$$L = D_x D_y (D_x + q D_y) + a_{20} D_x^2 + a_{11} D_{xy} + a_{02} D_y^2 + a_{10} D_x + a_{01} D_y + a_{00} \ , \quad (36)$$

where the coefficients belong to K. Then the following is a full system of invariants of such an operator w.r.t. the gauge transformations $L \to g^{-1} L g$:

$$\begin{aligned}
I_1 &= 2a_{20}q^2 - a_{11}q + 2a_{02} \ , \\
I_2 &= \partial_x(a_{20})q^2 - \partial_y(a_{02})q + a_{02}q_y \ , \\
I_3 &= a_{10} + a_{20}(qa_{20} - a_{11}) + \partial_y(a_{20})q - \partial_y(a_{11}) + 2a_{20}q_y \ , \\
I_4 &= a_{01}q^2 + a_{02}^2 - (3q_x + a_{11}q)a_{02} + q_x q a_{11} - \partial_x(a_{11})q^2 + q\partial_x(a_{02}) \ , \\
I_5 &= a_{00} - \frac{1}{2}\partial_{xy}(a_{11}) + q_x\partial_y(a_{20}) + q_{xy}a_{20} + \\
& \left(2qa_{20} + \frac{2}{q}a_{02} - a_{11} + q_y\right)\partial_x(a_{20}) - \frac{1}{q}a_{02}a_{10} - a_{01}a_{20} + \frac{1}{q}a_{20}a_{11}a_{02} \ .
\end{aligned}$$

Thus, an operator $L' \in K[D]$

$$L' = D_x D_y (D_x + q D_y) + b_{20} D_x^2 + b_{11} D_x D_y + b_{02} D_y^2 + b_{10} D_x + b_{01} D_y + b_{00} \quad (37)$$

is equivalent to L (w.r.t. the gauge transformations $L \to g^{-1} L g$) if and only if their corresponding invariants I_1, I_2, I_3, I_4, I_5 are equal.

Remark 3.4. Since the symbol of an LPDO L does not alter under the gauge transformations $L \to g^{-1} L g$, we consider the operators with the same symbol.

Proof. 1. The direct computations show that the five expressions from the statement of the theorem are invariants w.r.t. the gauge transformations $L \to g^{-1} L g$. One just has to check that these expressions do not depend on g, when calculate them for the operator $g^{-1} L g$. Basically, we have to check the fifth expression I_5 only, since the others are invariants by construction.

2. Prove that these five invariants form a complete set of invariants, in other words, the operators L and L' are equivalent (w.r.t. the gauge transformations $L \to g^{-1} L g$) if and only if their corresponding invariants are equal.

The direction "$\Rightarrow$" is implied from 1. Prove the direction "$\Leftarrow$". Let

$$I_1', I_2', I_3', I_4', I_5'$$

be the invariants computed from the coefficients of the operator L' by the formulas from the statement of the theorem, and

$$I_i = I'_i, \; i = 1, 2, 3, 4, 5 \; . \tag{38}$$

Look for a function $g = e^f, \; f, g \in K$, such that

$$g^{-1} L g = L' \; . \tag{39}$$

Equate the coefficients of D_{xx}, D_{yy} on both sides of (39), and get

$$\begin{aligned} \partial_y(f) &= b_{20} - a_{20} \; , \qquad (40) \\ \partial_x(f) &= (b_{02} - a_{02})/q \; . \qquad (41) \end{aligned}$$

In addition, the assumption $I_2 = I'_2$ implies

$$(b_{20} - a_{20})_x = ((b_{02} - a_{02})/q)_y .$$

Therefore, there is only one (up to a multiplicative constant) function f, which satisfies the conditions (40) and (41).

Consider such the function f. Then substitute the expressions

$$\begin{aligned} b_{20} &= a_{20} + f_y \; , \qquad (42) \\ b_{02} &= a_{02} + q f_x \; . \qquad (43) \end{aligned}$$

for b_{20}, b_{02} in (39), and prove that it holds for $g = e^f$.

Subtracting the coefficients of D_{xy} in $g^{-1}Lg$ from that in L' we get

$$b_{11} - a_{11} - 2f_x - 2q f_y \; ,$$

which equals

$$2q(I_1 - I'_1) \; ,$$

which is zero by the assumption (38). Now we can substitute

$$b_{11} = a_{11} + 2f_x + 2q f_y \; .$$

Analogously, subtracting the coefficients of D_x,D_y in $g^{-1}Lg$ from those in L', correspondingly, we get

$$\begin{aligned} b_{10} - a_{10} - 2a_{20} f_x - a_{11} f_y - 2f_{xy} - 2f_x f_y - q f_{yy} - q f_y^2 &= \\ I'_3 - I_3 &= 0 \; , \\ b_{01} - a_{01} - 2a_{02} f_y - a_{11} f_x - 2q f_{xy} - 2q f_x f_y - f_{xx} - f_x^2 &= \\ I'_4 - I_4 &= 0 \; . \end{aligned}$$

Now we can express b_{10} and b_{01}. Now, subtracting the "free" coefficient of $g^{-1}Lg$ from that of L', we get

$$\begin{aligned} b_{00} - a_{00} - a_{10}f_x - a_{01}f_y - a_{20}(f_{xx} + f_x^2) - a_{11}(f_{xy} + f_xf_y) - a_{02}(f_{yy} + f_y^2) - \\ f_{xxy} - 2f_{xy}f_x - f_yf_{xx} - f_yf_x^2 - qf_xf_{yy} - qf_xf_y^2 - qf_{xyy} - 2qf_yf_{xy} &= \\ I_5' - I_5 = 0 \ . \end{aligned}$$

Thus, we proved that for the chosen function f, the equality (39) holds, and therefore, the operators L and L' are equivalent. □

Thus, a full system of invariants for the case $p = 1$ has been found. Now we give the formulas for the general case.

Theorem 3.5. *For some non-zero $p, q \in K$ consider the operators of the form*

$$L = D_xD_y(pD_x+qD_y)+a_{20}D_x^2+a_{11}D_{xy}+a_{02}D_y^2+a_{10}D_x+a_{01}D_y+a_{00} \ , \quad (44)$$

where the coefficients belong to K. Then the following is a full system of invariants of such an operator w.r.t. the gauge transformations $L \to g^{-1}Lg$:

$$\begin{aligned} I_1 &= 2a_{20}q^2 - a_{11}pq + 2a_{02}p^2 \ , \\ I_2 &= \partial_x(a_{20})pq^2 - \partial_y(a_{02})p^2q + a_{02}p^2q_y - a_{20}q^2p_x \ , \\ I_3 &= a_{10}p^2 - a_{11}a_{20}p + 2a_{20}q_yp - 3a_{20}qp_y + a_{20}^2q - \partial_y(a_{11})p^2 + a_{11}p_yp + \partial_y(a_{20})pq \ , \\ I_4 &= a_{01}q^2 - a_{11}a_{02}q + 2a_{02}qp_x - 3a_{02}pq_x + a_{02}^2p - \partial_x(a_{11})q^2 + a_{11}q_xq + \partial_x(a_{02})pq \ , \\ I_5 &= a_{00}p^3q - p^3a_{02}a_{10} - p^2qa_{20}a_{01} + \\ &\quad (pI_1 - pq^2p_y + qp^2q_y)a_{20x} + (qq_xp^2 - q^2p_xp)a_{20y} \\ &\quad +(4q^2p_xp_y - 2qp_xq_yp + qq_{xy}p^2 - q^2p_{xy}p - 2qq_xpp_y)a_{20} \\ &\quad +(\frac{1}{2}p_{xy}p^2q - p_xp_ypq)a_{11} - \frac{1}{2}p^3qa_{11xy} + \frac{1}{2}a_{11x}p_yp^2q + \frac{1}{2}a_{11y}p_xp^2q \\ &\quad +p^2a_{02}a_{20}a_{11} + pqp_xa_{20}a_{11} - 2p_xq^2a_{20}^2 - 2p^2p_xa_{20}a_{02} \ . \end{aligned}$$

Proof. Since $p \neq 0$ we can multiply (44) by p^{-1} on the right, and get some new operator

$$L_1 = D_xD_y(D_x + \frac{q}{p}D_y) + \frac{a_{20}}{p}D_x^2 + \frac{a_{11}}{p}D_{xy} + \frac{a_{02}}{p}D_y^2 + \frac{a_{10}}{p}D_x + \frac{a_{01}}{p}D_y + \frac{a_{00}}{p} \ .$$

The invariants of the operator L and L_1 are the same. We compute the invariants of the operator L_1 by the formulas of the Theorem 3.3, and get the invariants of the statement of the current theorem up to multiplication by integers and p, q. □

Example 3.6. For some $p, q, c \in K$ consider the simple operator

$$L = D_x D_y (pD_x + qD_y) + c \ . \tag{45}$$

Compute the full system of invariants of Theorem 3.5 for L:

$$\begin{aligned} I_1 &= 0 \ , \\ I_2 &= 0 \ , \\ I_3 &= 0 \ , \\ I_4 &= 0 \ , \\ I_5 &= p^3 qc \ . \end{aligned}$$

Thus, every LPDOs in $K[D_x, D_y]$ with the symbol $XY(pX+qY)$ that has the same set of values for these five invariants is equivalent to the simple operator (45). The fact is useful, since the operators equivalent to the operator (45) are not trivially looking. Such the operators has the form

$$\begin{aligned} L \ = \ & D_x D_y (pD_x + qD_y) + pf_y D_x^2 + (2pf_x + 2qf_y) D_{xy} + qf_x D_y^2 + \\ & (2pf_{xy} + 2pf_x f_y + qf_{yy} + qf_y f_y) D_x + (pf_{xx} + pf_x f_x + 2qf_{xy} + 2qf_x f_y) D_y + \\ & c + pf_{xxy} + 2pf_{xy} f_x + pf_y f_{xx} + pf_y f_x^2 + qf_x f_{yy} + qf_x f_y^2 + qf_{xyy} + 2qf_y f_{xy} \ , \end{aligned}$$

for some $f \in K$.

3.5 Invariants for LPDOs of Arbitrary Orders

Consider hyperbolic LPDOs in $K[D_x, D_y]$ (of arbitrary order). Accumulate all the knowledge of the chapter 2 about the (common) obstacles to factorizations of such operators in the following theorem:

Theorem 3.7. *[24] Consider a hyperbolic operator $L \in K[D_x, D_y]$ of order d, and the factorizations of L into first-order factors. Then*

1. *the order of common obstacles less than or equal to $d - 2$;*
2. *a common obstacle is unique for each factorization type, and, therefore, the corresponding obstacles consist of just one element;*
3. *there are $d!$ common obstacles;*
4. *if $d = 2$, then the common obstacles of order 0 are the Laplace invariants;*
5. *the symbol of a common obstacle is an invariant.*

Thus, in much the same way it was done for the case $d = 3$, one can find a number invariants w.r.t. the gauge transformations $L \to g^{-1} L g$ for operators of arbitrary order. One of the difficulty lays in the large expressions, which appear already for third-order operators, when consider them in general form. The outlet can be in the guessing of the forms of invariants by analyzing the structure of the obtained invariants for third-order operators.

3.6 Conclusion

Consider operators of the form

$$L = D_x D_y (pD_x + qD_y) + a_{20} D_x^2 + a_{11} D_{xy} + a_{02} D_y^2 + a_{10} D_x + a_{01} D_y + a_{00} \ ,$$

where all the coefficients belong to K. This is the normalized form for hyperbolic bivariate LPDOs of order three. A full system of invariants with respect to the gauge transformations $L \to g^{-1} L g$ for such operators has been found. A way to find a number of invariants for operators of higher order has been suggested.

Whenever we have a full system of invariants for a given class of Linear Partial Differential Operators (LPDOs), we have an easy way to judge whether two operators of the class are equivalent. This means that it is possible to classify some of the corresponding partial differential equations in terms of their invariants. Naturally, classification has an immediate application to the integration of PDEs. Indeed, to solve a given PDE, we consider the corresponding LPDO and compute its basic invariants; the invariants then allow us to determine the normal form of the operator.

Furthermore, a full system of invariants for a certain class of operators can be used for the description of all the invariant properties of the operators, by expression the properties in terms of the invariants of the full system. For instance, it can be interesting to describe the properties "to be factorable with a certain type of factorization" and "to have an obstacle ring of a certain form" using the full system.

4 Parameterized Factorizations of Linear (Partial/Ordinary) Differential Operators

4.1 Introduction

The factorization of the Linear Partial Differential Operator (LPDO)

$$L = \sum_{i_1+\dots+i_n \le d} a_{i_1 \dots i_n} D_1^{i_1} \dots D_n^{i_n} \ ,$$

where the coefficients belong to some differential ring, is an important technique, used by modern algorithms for the integration of the corresponding Linear Partial Differential Equation (LPDE) $L(f) = 0$ (Mostly these integration algorithms are advanced modifications and generalizations of the Laplace transformation method). Over the last decade, a number of new modifications of the classical algorithms for the factorization of LPDOs (for example, [10, 11, 26, 27, 33, 34]) have been given. However, so far most of the activity has addressed the hyperbolic case, and there is as yet a lack of knowledge concerning the non-hyperbolic case.

There is a distinction in kind between the two cases. A factorization of a hyperbolic LPDO on the plane is determined uniquely by a factorization of the operator's symbol (principal symbol) (see Theorem 1.3 and [10]). Thus, the operator (5) may have at most one factorization of each of the forms $(D_x + \dots) \circ (D_y + \dots)$ and $(D_y + \dots) \circ (D_x + \dots)$. On the other hand for the non-hyperbolic operator D_{xx} there is the stereotypical example

$$D_{xx} = D_x \circ D_x = \big(D_x + \frac{1}{x+c}\big) \circ \big(D_x - \frac{1}{x+c}\big) \ ,$$

where c is an arbitrary parameter. A more significant example is provided by the Landau operator $L = D_{xx}(D_x + xD_y) + 2D_{xx} + 2(x+1)D_{xy} + D_x + (x+1)D_y$, which factors as

$$L = \Big(D_x + 1 + \frac{1}{x+c(y)}\Big) \circ \Big(D_x + 1 - \frac{1}{x+c(y)}\Big) \circ \Big(D_x + xD_y\Big) \ , \qquad (46)$$

where the function $c(y)$ is arbitrary. This shows that some LPDOs may have essentially different factorizations, and, further, that the factors may contain arbitrary parameters or even functions. Thus we may have *families of factorizations.*

An LPDO is hyperbolic if its symbol is completely factorable (all factors are of first order) and each factor has multiplicity one. In the present chapter we consider the case of LPDOs of orders two, three, and four, that have completely factorable symbols, without any additional requirement. We prove that "irreducible" (see Definition 4.3) families of factorizations can exist only for a few certain types of factorizations. For these cases explicit examples are given. For operators of orders two and three, it is shown that a family may be parameterized by at most one function in one variable. The investigations cover the case of ordinary differential operators as well. Some related remarks about parametric factorizations for ordinary differential operators may be found in [32].

The results of this chapter has been accepted for publishing [28].

4.2 Definitions

Definition 4.1. Let $L, F_1, \dots, F_k \in K[D]$. A factorization $L = F_1 \circ \dots \circ F_k$ is said to be of the *factorization type* $(S_1) \dots (S_k)$, where $S_i = \mathrm{Sym}_{F_i}$ for all i.

Definition 4.2. Let $L \in K[D]$. We say that

$$L = F_1(T) \circ \dots \circ F_k(T) \tag{47}$$

is a *family of factorizations* of L *parameterized by the parameter* T if, for any value $T = T_0$, we have that $F_1(T_0), \dots, F_k(T_0)$ are in $K[D]$ and $L = F_1(T_0) \circ \dots \circ F_k(T_0)$ holds. Here T is an element from the space of parameters $\mathbb{T}$. Usually $\mathbb{T}$ is the Cartesian product of some (functions') fields, in which the number of variables is less than that in K.

We often consider families without mentioning or designating the corresponding operator; we define the symbol and the order of the family to be equal to symbol and order of the operator.

Definition 4.3. We say that a family of factorizations (47) is *reducible*, if there is i, $1 \le i \le k$, such that the product

$$F_1(T) \circ \dots \circ F_i(T)$$

does not depend on the parameter T (in this case the product $F_{i+1}(T) \circ \dots \circ F_k(T)$ does not depend on the parameters as well). Otherwise the family is *reducible.*

Thus, the family (46) is reducible. However, the product of the first two factors does not depend on the parameter, while the factors themselves do. So we have an example of a second-order irreducible family of factorizations.

Remark 4.4. Note that any irreducible family of the type $(S_1)(S_2)(S_3)$ serves as an irreducible family of the types $(S_1S_2)(S_3)$ and $(S_1)(S_2S_3)$ as well. Indeed, the irreducibility of the family of the type $(S_1)(S_2)(S_3)$ means that the product of the first and the second factors, as well as that of the second one and the third one, depends on the parameter.

Analogous property enjoys the families of arbitrary orders.

Theorem 1.3 can be reformulated as

Theorem 4.5. *[10] Let $L \in K[D]$, and $\mathrm{Sym}_L = S_1 \dots S_k$, and let the S_i be pairwise coprime. There is at most one factorization of L of the type $(S_1)\dots(S_k)$.*

The theorem implies that, for instance, there are no irreducible families of the types $(X)(Y^3)$ or $(X^2)(Y^2)$.

Remark 4.6. The properties of factorizations, such as the existence of the factorizations, or the number of parameters, or again the number of variables in parametric functions, are invariant under a change of variables and the gauge transformations $L \mapsto g^{-1}Lg$, $g \in K$, of the initial operator.

Definition 4.7. We say that a partial differential operator $L \in K[D]$ is *almost ordinary* if it is an ordinary differential operator in some system of coordinates (transformation's functions belong to K).

4.3 The Linearized Problem

The basic tool in our study of families of factorizations will be their . Let an operator $L \in K[D]$ have a family of factorizations

$$L = M_1(T) \circ M_2(T) \ ,$$

parameterized by some parameters $T = (t_1, \dots, t_k)$, with $M_1(T), M_2(T) \in K[D]$. By means of a multiplication by a function from K, one can make the symbols of $M_1(T)$ and $M_2(T)$ independent of the parameters. Take some point T_0 as an initial point, make the substitution $T \to T_0 + \varepsilon R$, and equates the coefficient at the power ε. This implies

$$F_1 \circ L_2 + L_1 \circ F_2 = 0. \tag{48}$$

where we have denoted the initial factorization factors by $L_i = M_i(T_0)$, and $F_i = F_i(R)$ for $i = 1, 2$.

Below in this chapter we apply the linearization to obtain some important information about families of factorizations.

4.4 Second-Order Operators

Theorem 4.8. *A second-order operator in $K[D_x, D_y]$ has a family of factorizations (in some extension of the field K) if and only if it is almost ordinary. Any such family is unique for a given operator. Further, in appropriate variables it has the form*

$$\left(D_x + a + \frac{Q}{W + f_1(y)} \right) \circ \left(D_x + b - \frac{Q}{W + f_1(y)} \right) \ ,$$

where $Q = e^{\int (b-a)dx}$, $W = \int Q dx$, $a, b \in K$, and $f_1(y) \in K$ is a parameter.

Proof. Consider a second-order operator $L \in K[D_x, D_y]$. By a change of variables we can make the symbol of L equal to either X^2 or XY. In the latter case, L has no family of factorizations because of Theorem 1.3.

Consider the case $\mathrm{Sym}_L = X^2$. Then operator L has a factorization only if it is ordinary. Suppose we know one factorization: $L = L_1 \circ L_2 = (D_x + a) \circ (D_x + b)$, where $a, b \in K$, and we are interested in deciding whether there exists a family. Consider the linearized problem, that is the equation (48) w.r.t. $F_1, F_2 \in K$: $F_1 \circ L_2 + L_1 \circ F_2 = 0$. The equation always has a solution

$$\begin{cases} F_1 = f_1(y)e^{(b-a)x}, \\ F_2 = -F_1, \end{cases}$$

where $f_1(y) \in K$ is a parameter function. Thus, any family can be parameterized by only one function of one variable.

In fact, such a family always exists, and it is given explicitly in the statement of the theorem. Moreover, one can prove straightforwardly that such a family is unique for a given operator L. □

4.5 Third-Order Operators

Theorem 4.9. *Let a third-order operator in $K[D_x, D_y]$ with the completely factorable symbol has an irreducible family of factorizations. Then it is almost ordinary.*

Any such family depends by at most three (two) parameters if the number of factors in factorizations is three (two). Each of these parameters is a function of one variable.

Proof. Consider a third-order operator L in $K[D_x, D_y]$. For the symbol Sym_L only the following three are possible: it has exactly three, two, or no coprime factors. In the first case no family is possible because of Theorem 4.5.

Suppose exactly two factors of the symbol are coprime. Thus, in some variables the symbol of L is X^2Y. Consider factorization into two factors. Then the following types of factorizations are possible: $(X)(XY), (Y)(X^2), (XY)(X), (X^2)(Y)$. By Theorem 4.5, there is no family of factorizations of the types $(Y)(X^2), (X^2)(Y)$. Because of the symmetry, it is enough to consider just the case $(X)(XY)$. Indeed, if there exists a family of the type $(XY)(X)$ for some operator L of the general form

$$L = \sum_{|J| \le d} a_J D^J \ ,$$

$a_J \in K$, then the adjoint operator

$$L^t(f) = \sum_{|J| \le d} (-1)^{|J|} D^J (a_J f).$$

has a family of the type $(X)(XY)$, and the number of parameters in the family is the same.

Thus, we consider a factorization of the factorization type $(X)(XY)$:

$$L = L_1 \circ L_2 = (D_x + r) \circ (D_{xy} + aD_x + bD_y + c),$$

where $r, a, b, c \in K$ as the initial factorization for some family of factorizations of the factorization type $(X)(XY)$. By means of the gauge transformations, we make the coefficient a equal zero in this initial factorization (of course, the coefficient at D_x in the second factor of other factorizations of the family may be still non-zero). To study possible families in this case, we consider the linearized problem: the equation $F_1 \circ L_2 + L_1 \circ F_2 = 0$ w.r.t. $F_1 = r_1$, $F_2 = a_{10}D_x + a_{01}D_y + a_{00}$, where $r_1, a_{10}, a_{01}, a_{00} \in K$. The only non-trivial solution is

$$a_{10} = a_{00} = 0, \ r_1 = -a_{01}, \ a_{01} = f_1(y) \cdot Q,$$

where $Q = e^{\int (b-r)dx}$ and $f_1(y) \in K$ is a parameter, while

$$c = 0$$

is a necessary condition of the solution's existence. Therefore, every family of the type $(X)(XY)$ is parameterized by one function of one parameter (can be a constant function). Secondly, the initial factorization has the form

$$L = (D_x + r) \circ (D_x + b) \circ D_y \ , \tag{49}$$

that is the operator L itself has very special form.

Now, if we consider a factorization of the family in general form, namely

$$\widetilde{L}_1 \circ \widetilde{L}_2 = (D_x + \widetilde{r}) \circ (D_{xy} + \widetilde{a}D_x + \widetilde{b}D_y + \widetilde{c}) \ ,$$

where all the coefficients belong to K, and equates the corresponding product to the expression (49), we obtain

$$\widetilde{a} = \widetilde{c} = 0,$$

and so any factorization of the family has the form

$$L = (D_x + \widetilde{r}) \circ (D_x + \widetilde{b}) \circ D_y \ .$$

Therefore, only reducible families of factorizations into two factors may exist in this case. Then, by Remark 4.4, there is no any irreducible family of factorizations into any number of factors in this case.

Consider the case in which all the factors of the symbol Sym_L are the same (up to a multiplicative function from K). Then one can find variables in which the symbol is X^3. Note that any irreducible factorization of the factorization type $(X)(X)(X)$ is an irreducible factorization of the types $(X)(X^2)$ and $(X^2)(X)$ also. Then because of the symmetry only one of two types $(X)(X^2)$ and $(X^2)(X)$ has to be considered. Therefore, it is sufficient to consider the factorization type $(X)(X^2)$. Thus, consider an initial factorization of the form

$$L = L_1 \circ L_2 = (D_x + r) \circ (D_{xx} + aD_x + bD_y + c)$$

where $r, a, b, c \in K$. Under the gauge transformations we may assume $a = 0$ (while the coefficient at D_x in the second factor of other factorizations of the family may be still non-zero). Consider the linearized problem (48) for such the initial factorization: the equation $F_1 \circ L_2 + L_1 \circ F_2 = 0$ w.r.t. $F_1 = r_1$, $F_2 = a_{10}D_x + a_{01}D_y + a_{00}$, where $r_1, a_{10}, a_{01}, a_{00} \in K$. The only non-trivial solution is $a_{01} = 0$, $r_1 = -a_{10}$, $a_{00} = -ra_{10} - \partial_x(a_{10})$, provided both

$$b = 0$$

and $ca_{10}+r^2a_{10}+2r\partial_x(a_{10})+a_{10}\partial_x(r)+\partial_{xx}(a_{10}) = 0$. The solution of the latter equation depends on two arbitrary function in the variable y. Therefore, any family of the type $(X)(XX)$ is parameterized by two functions of one variable (can be constant functions), and such a family may exist only for an *almost ordinary* operator L. This implies that a family of the factorization type $(X)(X)(X)$ may exist only for an *almost ordinary* operator L.

Any irreducible family of the type $(X)(X)(X)$ serves as an irreducible family of the type $(X)(XX)$. Therefore, a family of the type $(X)(X)(X)$ can have two parameters (functions in one variables), that appear in the corresponding family of the type $(X)(XX)$, and additional parameters, that can appear when we consider two last factors separately. By the theorem 4.8, there is at most one additional parameter (a function in one variable). Thus, for the family of the type $(X)(X)(X)$ the maximal number of parameters is three, and these parameters are functions in one variable (may be constant functions). This agrees with [33]. □

The theorem implies that for an operator (with the completely factorable symbol), that is not almost ordinary, only reducible families may exist. Any such family is obtained by the multiplication (on the left or on the right) of a second-order family by some non-parametric first order operator. Note that this second-order family should be almost ordinary, by Theorem 4.8.

Example 4.10. The family of the Landau operator (46) is reducible, which is obtained from a second-order family.

4.6 Fourth-Order Operators

Here we start with an example of a fourth-order irreducible family for an almost ordinary operator.

Example 4.11. The following is a fourth-order irreducible family of factorizations:

$$D_{xxxx} = \Big(D_{xx} + \frac{2}{x+2f_1(y)} + y\Big)\Big(D_{xx} - \frac{2}{x+2f_1(y)} + y\Big),$$

where $f_1(y) \in K$ is a parameter.

Unlike the irreducible families of orders two and three, an irreducible fourth-order family need not be almost ordinary.

Example 4.12. The following is a fourth-order irreducible family of factorizations:

$$D_{xxyy} = \Big(D_x + \frac{\alpha}{y+\alpha x+\beta}\Big)\Big(D_y + \frac{1}{y+\alpha x+\beta}\Big)\Big(D_{xy} - \frac{1}{y+\alpha x+\beta}(D_x + \alpha D_y)\Big),$$

where $\alpha, \beta \in K\backslash\{0\}$. Note that the first two factors commute.

Again, we actually have several examples here. Namely, for the same operator D_{xxyy}, we have families of the types $(X)(XY^2)$, $(XY)(XY)$ and $(X)(Y)(XY)$.

Theorem 4.13. *In $K[D_x, D_y]$, irreducible fourth order families of factorizations with a completely factorable symbol exist only for factorizations of the types $(XY)(XY)$ and $(X^2)(X^2)$.*

Proof. For the symbol Sym of the family, there are exactly four possibilities: to have exactly four, three, two or no different factors.

I. Case of four different factors. No family is possible because of Theorem 4.5.

II. Case of three different factors. In this case, in appropriate variables, the symbol has the form $\mathrm{Sym}_L = X^2Y(\alpha X + Y)$, where $\alpha \in K\backslash\{0\}$. At first consider factorizations into two different factors. Then by Theorem 4.5 and because of the symmetry (in the same sense as it is in the proof of Theorem 4.9), it is enough to consider the types of factorizations $(XY)(X(\alpha X + Y))$ and $(X)(XY(\alpha X + Y))$.

1) Case of the type $(XY)(X(\alpha X + Y))$. We prove that there is no irreducible family of this type. Let

$$L_1 \circ L_2 = (D_{xy} + a_1 D_x + b_1 D_y + c_1) \circ (\alpha D_{xx} + D_{xy} + a_2 D_x + b_2 D_y + c_2)\ ,$$

where all coefficients are in K, be the initial factorization of such a family. Then under the gauge transformation we may assume $b_2 = 0$ (while the coefficient at D_y in the second factor of any other factorization of the family may be still non-zero). Consider the linearized problem: the equation $F_1 \circ L_2 + L_1 \circ F_2 = 0$ w.r.t. $F_1, F_2 \in K[D]$ and $\mathrm{ord}(F_1) = \mathrm{ord}(F_2) = 1$. The only non-trivial solution is parameterized by a function $f_1(y) \in K$ and exists only under two conditions on the coefficients of L_1 and L_2:

$$c_1 = \partial_y(b_1) + b_1 a_1 \quad \text{and} \quad c_2 = \partial_{xx}(\alpha) - \partial_x(a_2). \tag{50}$$

Thus, if a family exists, then the coefficients of the initial factorization $L_1 \circ L_2$ satisfy the two conditions (50).

Now we come back to the initial problem and look for a family of factorizations in the general form:

$$L_1 \circ L_2 = (L_1 + S_1) \circ (L_2 + S_2) \ ,$$

where S_1, S_2 are arbitrary first-order operators in $K[D]$. This gives us a system of equations in the coefficients of S_1 and S_2. The system together with the conditions (50) has a unique non-trivial solution. The corresponding (to this solution) family of factorizations is complete, that is, both factors are factorable themselves:

$$L_1 \circ L_2 =$$
$$(D_y + a_1) \circ \Big(D_x + b_1 + \tfrac{Q}{W + f_1(y)}\Big) \circ \Big(D_x - \tfrac{Q}{W + f_1(y)}\Big) \circ \Big(\alpha D_x + D_y + a_2 - \partial_x(\alpha)\Big),$$

where $Q = e^{-\int b_1 dx}$ and $W = \int Q dx$ and $f_1(y)$ is the only parameter function. Now it is clear that the first and the last factors do not depend on a parameter, and so any factorization of any family of the type $(XY)(X(\alpha X + Y))$ is reducible.

2) Case of the type $(X)(XY(\alpha X + Y))$. There is no irreducible family of this type either. To prove this we consider a factorization of this type,

$$L_1 \circ L_2 = (D_x + c_1) \circ (D_{xxy} + D_{xyy} + a D_{xx} + b D_{xy} + c D_{yy} + d D_x + e D_y + f),$$

where all the coefficients belong to K, as the initial factorization of a family of factorizations. Under the gauge transformations we may assume $c = 0$ (note that the analogous coefficients in the other factorizations of the family do not necessary become zero). Proceeding as in the previous case, we also get that there is only one non-trivial solution, which is parameterized by a function $f_1(y) \in K$. Also we have two conditions which provide the existence of such an equation:

$$\begin{aligned} e_2 &= b_2 c_2 + \partial_x(b_2) - \alpha \partial_x(c_2) - \alpha c_2^2 - 2\partial_x(\alpha) c_2 - \partial_{xx}(\alpha) \\ f_2 &= d_2 c_2 + \partial_x(d_2) - a_2 \partial_x(c_2) - a_2 c_2^2 - 2\partial_x(a_2) c_2 - \partial_{xx}(a_2) \ . \end{aligned}$$

Now, we use the obtained conditions for the initial problem, where a family of factorizations is considered in general form. Thus, we get that if such a family exists, then the second factor of the family can be always factored into first and second-order operators, and the second-order operator does not depend on the parameter:

$$L_1 \circ L_2 = \Big(D_x + c_1 + m_{00}\Big) \circ \Big(D_x + c_2 - m_{00}\Big) \circ$$
$$\Big(\alpha D_x + D_y + a_2 D_x + (b_2 - \alpha c_2 - \partial_x(\alpha))D_y + d_2 - a_2 c_2 - \partial_x(a_2)\Big),$$

where only $m_{00} \in K$ may depend on a parameter. Thus any family (if it exists) of the type $(X)(XY(\alpha X + Y))$ is reducible.

Finally, by Remark 4.4, we conclude that there is no an irreducible family of factorizations into any number of factors.

III. Case of two different factors of the symbol. Then there exist variables in which the symbol has either the form X^2Y^2 or XY^3. At first consider factorizations into two factors. By Theorem 4.5 and because of the symmetry, it is enough to consider types of factorizations $(X)(XY^2)$ and $(XY)(XY)$ in the case of the symbol X^2Y^2, and the types $(Y)(XY^2)$ and $(XY)(Y^2)$ in the case XY^3.

1) Case of the type $(X)(XY^2)$. Let us prove that there is no irreducible family of this type. As we did in the cases above, we consider an initial factorization of the family in general form:

$$L_1 \circ L_2 = (D_x + c_1) \circ (D_{xyy} + aD_{xx} + bD_{xy} + cD_{yy} + dD_x + e_1 D_y + f),$$

with all coefficients in K, and by means of gauge transformations we assume $c = 0$ (the coefficients at D_{yy} in the second factor of other factorizations of the family may be still non-zero). Then solve the linearized problem: the equation $F_1 \circ L_2 + L_1 \circ F_2 = 0$ w.r.t. $F_1, F_2 \in K[D]$ and $\mathrm{ord}(F_1) = 0$, $\mathrm{ord}(F_2) = 2$. This leads us to two necessary conditions of existence of such a family: $e_1 = \partial_x(b)$, $f = \partial_x(d) - \partial_{xx}(a)$. Then we apply this to the initial problem. That is consider a family of factorizations in general form (in much the same way as in the case I.1)), and obtain that every such family, if it exists, can be factored further, that is the second factor of such a family (which has the symbol XY^2) can be always factored itself into two factors, and the right factor does not depend on any parameter. More precisely, we have

$$L_1 \circ L_2 = \Big(D_x + c_1 + \frac{Q}{W - f_1(y)}\Big)\Big(D_x - \frac{Q}{W - f_1(y)}\Big)\Big(D_{yy} + aD_x + bD_y + d - \partial_x(a)\Big),$$

where $Q = e^{-\int c_1 dx}$, $W = \int Q dx$. Thus, there is no irreducible family of this type.

2) Case of the type $(XY)(XY)$. In this case a family can exist: see the Example 4.12.

3) Case of the type $(Y)(XY^2)$. We prove that there is no irreducible family of this type. Indeed, consider a factorization of this type:

$$L_1 \circ L_2 = (D_y + c_1) \circ (D_{xyy} + aD_{xx} + bD_{xy} + cD_{yy} + dD_x + e_1 D_y + f),$$

with all coefficients in K, as the initial factorization of a family of factorizations. By gauge transformations we may assume the coefficient at D_{yy} in this initial factorization is zero, that is $c = 0$. The linearized problem is the equation $F_1 \circ L_2 + L_1 \circ F_2 = 0$ w.r.t. $F_1, F_2 \in K[D]$ and $\mathrm{ord}(F_1) = 0,\ \mathrm{ord}(F_2) = 2$. This equation has a non-trivial solution, provided $a = 0$ and

$$d = e_1^{-2}(bfe_1 - be_1\partial_y(e_1) + e_1^2\partial_y(b) + 3f\partial_y(e_1) - e_1\partial_y(f) - 2(\partial_y(e_1))^2 + e_1\partial_{yy}(e_1) - f^2)$$

Then, when we consider the corresponding family of factorizations in general form, we may apply these conditions, and easily get that such a family cannot exist.

4) Case of the type $(XY)(Y^2)$. We prove that there is no irreducible family of this type. Indeed, consider a factorization of the considering type:

$$L_1 \circ L_2 = (D_{xy} + a_1 D_x + b_1 D_y + c_1) \circ (D_{yy} + a_2 D_x + b_2 D_y + c_2)$$

with all coefficients in K, as the initial factorization of a family of factorizations. By the gauge transformations we may assume $c_2 = 0$. The linearized problem is the equation $F_1 \circ L_2 + L_1 \circ F_2 = 0$ w.r.t. $F_1, F_2 \in K[D]$ and $\mathrm{ord}(F_1) = 1,\ \mathrm{ord}(F_2) = 1$. The equation has a non-trivial solution, which depends on two parameter functions $f_1(x), f_2(x) \in K$, and the existence is provided by the conditions

$$a_2 = 0, \quad c_1 = b_1 a_1 + \partial_x(a_1).$$

Thus, a family may exist only if the considered operator L has the form

$$L = (D_{xy} + a_1 D_x + b_1 D_y + b_1 a_1 + \partial_x(a_1)) \circ (D_y + b_2) \circ D_y$$

for some $a_1, b_1, b_2 \in K$. Then, one may prove that in this case any family of factorization has the form

$$L = (D_{xy} + \dots) \circ (D_y + \dots) \circ D_y\ ,$$

meaning that is there is no irreducible fourth-order family of the type $(XY)(Y^2)$.

Now, consider a factorization into three or four factors. Then either the factor on the left, or that on the right of this factorization is of order one. Therefore, by Remark 4.4, such a factorization cannot exist as they cannot exist in the cases 1) and 3).

IV. Case of no different factors of the symbol. Then there exist variables such that the symbol is X^4. Consider factorizations into two factors. Then, by Theorem 4.5 and because of the symmetry, it is enough to consider types of factorizations $(X)(X^3)$ and $(X^2)(X^2)$.

1) Case of the type $(X)(X^3)$. Prove that there is no irreducible family of the type $(X)(X^3)$. Consider a factorization of the type $(X)(X^3)$:

$$L_1 \circ L_2 = (D_x + c_1) \circ (D_{xxx} + aD_{xx} + bD_{xy} + cD_{yy} + dD_x + eD_y + f) ,$$

where all coefficients are in K. Solving the linearized problem, we get that a family in this case may be parameterized by only one function in one variable, and such a family may exist provided two conditions on the initial coefficients hold (one of them is just $c = 0$). Then, when we look for a family in general form, one may prove that such families indeed can exist, but all such families are reducible.

2) Case of the type $(X^2)(X^2)$. Here we have the Example 4.11 of a family of factorizations depended on one functional parameter in one variable. In fact the maximal number of parameters in this type of factorization is four [33].

Finally, by Remark 4.4, an irreducible families into four and three factors cannot exist as they cannot exist in the case 1). □

4.7 Conclusion

For second, third and fourth order LPDOs on the plane with completely factorable symbols, we have completely investigated what factorization types admit irreducible parametric factorizations. For these factorization types, examples are given. Note the our method is general and we cover the case of ordinary operators as a particular case. For operators of orders two and three, we describe in addition the structure of their families of factorizations. For the partial operators of order four, the question remains open (for ordinary operators, the possible number of parameters in a family of factorizations has been investigated in [33]). For the case of partial differential operators we would surmise that no more than two or one parameters (which could be functions) are possible. Generalizations to LPDOs with arbitrary symbols (without the complete factorization assumption), to high order LPDOs, and to those in multiple-dimensional space are of interest also.

5 Generalized Laplace Transformations

5.1 Introduction

Though the problem of solving Partial Differential Equations (PDEs) has been intensively studied since Newton's time, yet the number of PDEs that can be solved analytically remains small. Even solutions expressed as quadratures are not common. For example, the simple-looking hyperbolic PDE

$$z_{xy} + az_x + bz_y + c = 0, \quad a = a(x,y), b = b(x,y), c = c(x,y) \tag{51}$$

can be solved only in some particular cases (see for example [22]). One of the methods for finding an analytical solution of an equation such as (51) requires the factorization of the corresponding Linear Partial Differential Operator (LPDO).

Another method for expanding the number of analytically solvable PDEs consists in transforming the PDEs and obtaining the corresponding transformations of their solutions. The oldest transformation algorithm for equation (51) is the Laplace Cascade method (see in the section 1.5), known since the 18th century.

The method tries to apply transformations to a given LPDO until the operator becomes factorable, in which case we can solve the corresponding equation. In the same time, there is a relation between the solution spaces of the initial and this transformed factorable operator. In that way, some PDEs may be solved. In the general case, the process does not terminate, and we have an infinite sequence of transformed operators, none of which is factorable. In that way, the Laplace Cascade Method is a method to solve PDEs of the form $z_{xy} + az_x + bz_y + c = 0$, which is one of the important methods of symbolic integration. However, it leads to solutions not very often.

The classical Laplace Method has been the subject of many generalizations: to non-linear PDEs [1, 7, 2], high-order PDEs [35], to PDEs in multi-dimensional spaces [9], to systems of PDEs [4], etc. All of these generalizations of the Laplace Transformations extend the class of the considering PDEs. However, as a different direction, one may think about some new transformations for the original type of equation (51) .

In the present chapter of the thesis, we introduce such new transformations, which generalize the Laplace ones. The idea comes from the following observation: for an operator

$$L = D_x \circ D_y + aD_x + bD_y + c \ , \tag{52}$$

where $a = a(x,y)$, $b = b(x,y)$, $c = c(x,y)$, the classical Laplace transformations $L \to L_1$, $L \to L_{-1}$ can be defined by the equalities

$$\begin{aligned} \Big(D_y + a - ln(h)_y\Big) \circ L &= L_1 \circ \Big(D_y + a\Big) \ , \\ \Big(D_x + b - ln(k)_x\Big) \circ L &= L_{-1} \circ \Big(D_x + b\Big) \ , \end{aligned}$$

where $h = c - a_x - ab$ and $k = c - b_y - ab$ are two Laplace invariants.

Now, it seems natural to look for more general transformations $L \to \widetilde{L}_1$ defined by some equality of the form

$$\Big(pD_x + qD_y + r\Big) \circ L = \widetilde{L}_1 \circ \Big(p_1 D_x + q_1 D_y + r_1\Big), \tag{53}$$

where all the coefficients depend on two variables x and y.

Here, we present some such transformations, which immediately lead us to new classes of analytically solvable PDEs. Some interesting examples are given explicitly.

The results of this chapter has been submitted for publishing [29].

5.2 Generalized Laplace Transformations for LPDOs of Arbitrary Order

Here we consider Linear Partial Differential Operators (LPDOs) of arbitrary order in a multi-dimensional space.

Definition 5.1. Let $L, L_1, M, M_1 \in K[D]$. We say that L_1 is the result of a *Right Generalized Laplace transformation (R-transformation)* of an operator L and write

$$L_1 = \varphi_1(L, M, M_1) \ ,$$

if $\mathrm{Sym}(L_1) = \mathrm{Sym}(L)$, $\mathrm{ord}(M) = \mathrm{ord}(M_1) = 1$ and

$$M \circ L = L_1 \circ M_1 \ . \tag{54}$$

Definition 5.2. Let $L, L_{-1}, M, M_{-1} \in K[D]$. We say that L_1 is the result of a *Left Generalized Laplace transformation (L-transformation)* of an operator L and write

$$L_{-1} = \varphi_{-1}(L, M, M_{-1}),$$

if $\mathrm{Sym}(L_{-1}) = \mathrm{Sym}(L)$, $\mathrm{ord}(M) = \mathrm{ord}(M_{-1}) = 1$ and

$$L \circ M = M_{-1} \circ L_{-1}. \tag{55}$$

Remark 5.3. There is a certain symmetry between L- and R-transformations, namely, if $L_1 = \varphi_1(L, M, M_1)$, then $L = \varphi_{-1}(L_1, M_1, M)$.

Remark 5.4. The assumption that the symbols of L and L_1 are equal immediately implies $\mathrm{Sym}(M) = \mathrm{Sym}(M_1)$ and $\mathrm{Sym}(M) = \mathrm{Sym}(M_{-1})$.

Remark 5.5. By the Theorem [10], if Sym(L) is coprime with the symbol Sym(M), then there is at most one pair (L_1, M_1), such that $M \circ L = L_1 \circ M_1$. If those symbols are not coprime, there is an example

$$D_{xy} + \frac{1}{x+k} = \varphi_1\Big(D_{xy}, D_x, D_x - \frac{1}{x+k}\Big),$$

where k is a parameter.

Remark 5.6. The kernel of L is mapped to the kernel of L_1 by the substitution $z \to M_1(z)$, while the kernel of L_{-1} is mapped to the kernel of L_1 by the substitution $z \to M(z)$.

We use the notation

$$L^\alpha = \alpha \circ L \circ \alpha^{-1}$$

for any invertible function $\alpha \in K$ and any operator $L \in K[D]$.

Theorem 5.7. *If $L_1 = \varphi_1(L, M, M_1)$ in $K[D]$, then*

$$\begin{array}{ll} a)\ L_1^\beta & = \varphi_1(L^\alpha,\ \ \beta M \alpha^{-1},\ \ \beta M_1 \alpha^{-1})\ , \\ b)\ L_1^\alpha & = \varphi_1(L^\alpha,\ \ M^\alpha,\ \ M_1^\alpha) \end{array}$$

for any invertible functions $\alpha, \beta \in K$. If $L_{-1} = \varphi_{-1}(L, M, M_{-1})$ in $K[D]$, then

$$\begin{array}{ll} a)\ L_{-1}^\beta & = \varphi_{-1}(L^\alpha,\ \ \alpha M \beta^{-1},\ \ \alpha M_{-1} \beta^{-1})\ , \\ b)\ L_{-1}^\alpha & = \varphi_{-1}(L^\alpha,\ \ M^\alpha,\ \ M_{-1}^\alpha) \end{array}$$

for any invertible functions $\alpha, \beta \in K$.

Proof. As $L_1 = \varphi_1(L, M, M_1)$, we have $M \circ L = L_1 \circ M_1$. Then

a) $(\beta \circ M \circ \alpha^{-1}) \circ (\alpha \circ L \circ \alpha^{-1}) = (\beta \circ L_1 \circ \beta^{-1}) \circ (\beta \circ M_1 \circ \alpha^{-1})$, and so we are done;

b) $(\alpha^{-1} \circ M \circ \alpha) \circ (\alpha^{-1} \circ L \circ \alpha) = (\alpha^{-1} \circ L_1 \circ \alpha) \circ (\alpha^{-1} \circ M_1 \circ \alpha)$.

The statements about φ_{-1} are proved analogously. □

Corollary 5.8. *The existence of an R- or L- transformation of L with the result $\widetilde{L}$ depends on the equivalence classes (w.r.t. the operation $L \to L^\alpha$)of those operators only. Therefore this existence property can be expressed in terms of operator's invariants.*

Remark 5.9. Let us consider bivariate case, that is the operator M has the form

$$M = pD_x + qD_y + r\ .$$

Then the quantity p/q is unaltered under transformations of the form $M \to Ma^{-1}$, $M \to aM$.

The direct computation proves the following result.

Theorem 5.10. *Let $L, L_1, M, M_1 \in K[D_x, D_y]$. If*

$$\varphi_1(L, M, M_1) = L_1 \ ,$$

then

$$\varphi_1(L + A \circ M_1, M, M_1) = L_1 + M \circ A \ ,$$

where $A \in K[D_x, D_y]$, $\mathrm{ord}(A) \le \mathrm{ord}(L) - 2$.

Theorem 5.11. *Let the operators $L_1, L_{-1} \in K[D_x, D_y]$ are the result of the classical Laplace transformations of an operator $L \in K[D]$, $L = D_x \circ D_y + aD_x + bD_y + c$. Then*

$$\begin{aligned} L_1 &= \varphi_1(L, D_y + a_1, D_y + a), \\ L_{-1} &= \varphi_1(L, D_x + b_1, D_x + b), \end{aligned}$$

where $a_1 = a - h_y/h$, $b_1 = b - k_x/k$, and $h = c - a_x - ab$, $k = c - b_y - ab$ are the Laplace invariants of L.

There exist analogous formulas in terms of φ_{-1}.

Proof. Using the expressions for coefficients of L_1 (see (10)) and L_{-1} (see (11)) for the classical Laplace transformations, one checks that

$$\begin{aligned} (D_y + a_1) \circ L &= L_1 \circ (D_y + a), \\ (D_x + b_1) \circ L &= L_{-1} \circ (D_x + b). \end{aligned}$$

□

In that way the classical Laplace Transformations are two particular examples of the L- and R- transformations of the hyperbolic operators $L = D_x \circ D_y + aD_x + bD_y + c$. In the next section we look for the other L- and R-transformations of those operators.

5.3 L- and R-transformations of $D_x \circ D_y + aD_x + bD_y + c$

Consider a second-order hyperbolic partial differential operators $L \in K[D_x, D_y]$ in the canonical form:

$$L = D_x \circ D_y + aD_x + bD_y + c \ . \tag{56}$$

Since the properties of the L-transformations are completely symmetric to those of the R-transformations, we formulate most of results for the R-transformations only.

Direct computations prove the following theorem.

Theorem 5.12. *Let $M, M_1 \in K[D]$, and $\varphi_1(L, M, M_1)$ be an R-transformation of the operator L of the form* (56)*. If*

$$M = pD_x + qD_y + r \ , \tag{57}$$

then

$$M_1 = pD_x + qD_y + r + q\frac{p_y}{p} + p\frac{q_x}{q} - p_x - q_y \ . \tag{58}$$

Therefore, in this case the operator M_1 is completely determined by the operator M, and so if we consider L- or R-transformations of the operators of the form (56), we may omit M_1 and write

$$L_1 = \varphi_1(L, M).$$

In order to find new R-transformations of the operator L (56), we look for some operators $L_1, M \in K[D_x, D_y]$ of the forms $L_1 = D_{xy} + a_1 D_x + b_1 D_y + c_1$ and (57), such that

$$M \circ L = L_1 \circ M_1 \ . \tag{59}$$

The cases $p = 0$ and $q = 0$ will be considered later in the section 5.5. Now we consider the case $p \neq 0$, $q \neq 0$. Suppose $p \neq 1$ and there is an R-transformation $L_1 = \varphi_1(L, M, M_1)$, then by the Theorem 5.7, there is the R-transformation $L_1^\beta = \varphi_1(L, \beta M, \beta M_1)$, where $\beta = 1/p$. Then without loss of generality we may assume

$$p = 1.$$

Substitute the expressions for L, L_1, M_1, M into the equality (59) and expand the products. Then from the coefficients at D_x^2, D_y^2 and D_x we have in order

$$\begin{aligned}
a_1 &= a \ , \\
b_1 &= b - \tfrac{q_x}{q} \ , \\
c_1 &= c + \tfrac{b_x}{q} + b_y - \tfrac{q_x}{q}\left(a + \tfrac{b}{q} - \tfrac{r}{q} - 2\tfrac{q_x}{q^2}\right) - \tfrac{r_x}{q} - \tfrac{q_{xx}}{q^2} \ .
\end{aligned}$$

Therefore, the operators L_1 and M_1 are determined for given operators L and F. However, two conditions to be satisfied are still left (these conditions are obtained from the coefficient at D_y and from the free one). They are algebraic-differential expressions in a, b, c, q, r with derivatives up to the second order.

It is not trivial to solve these two equations, however, it is not obligatory to do this in the general case. Indeed, the main purpose of the current investigations is to extend the class of PDEs with known analytical solution. Thus, below we concentrate on the special case

$$q = 1,$$

that is the case of R-transformations, where the operator M has the symbol $X + Y$.

Definition 5.13. We call such R-transformations *$(X + Y)$-R-transformations.*

5.4 $(X+Y)$-R-Transformations

Here we use all the notation of the previous section. Consider the $(X+Y)$-R-transformations

$$L_1 = \varphi_1(L, D_x + D_y + r), \quad r \in K.$$

By the corollary 5.8, the property of existence of a R-transformation is invariant w.r.t. the operation $L \to L^\alpha$. Therefore, we may always choose a function α, such that the coefficients at D_x and D_y in the operator L^α are equal. Thus, without loss of generality we can assume

$$b = a.$$

In this case,

$$\begin{array}{rcl} M_1 &=& M \ , \\ L_1 &=& L + a_x + a_y - r_x \ , \end{array}$$

and the condition $M \circ L = L_1 \circ M_1$ implies

$$D_x(r) = D_y(r) \ , \tag{60}$$

$$\begin{array}{rl} (D_x + D_y)c - r(D_x + D_y)a - a(D_x + D_y)r & \\ +rD_x(r) - D_{xy}(r) &= \ 0 \ . \end{array} \tag{61}$$

The first equality (60) implies that $r(x,y)$ is a function of $(x+y)$, or equivalently, of $(x+y)/2$:

$$r(x,y) = R(u) = R((x+y)/2)$$

for some function $R = R(u)$ of one variable. Consider the linear change of variables

$$u = (x+y)/2, \ v = (x-y)/2 \ .$$

It implies $D_u = D_x + D_y, \ D_v = D_x - D_y$. Thus, the last condition (61) has the form

$$D_u(c) - RD_u(a) - aD_u(R) + RD_u(R)/2 - D_{uu}(R)/4 = 0,$$

where a, c are understood as functions of the variables u and v. This implies

$$D_u(c - Ra + R^2/4 - D_u(R)/4) = 0.$$

Therefore,

$$c - Ra + R^2/4 - D_u(R)/4 = K(v).$$

for some function $K(v)$ of one variable, and so

$$c(u,v) = a(u,v)R(u) - R(u)^2/4 + \dot{R}(u)/4 + K(v).$$

where $\dot{R}(u) = D_u(R)$.

Now by the operation $L \to L^{\alpha}$, we may transform the obtained results to the general case, that is the coefficient a is not obligatory equal the coefficient b. In that way, we proved the following theorem:

Theorem 5.14. *An $(X+Y)$-R-transformation of an operator $L = D_{xy} + aD_x + bD_y + c \in K[D_x, D_y]$ exists if and only if*

$$\begin{aligned} a &= a_0 + \alpha_y \ , \\ b &= a_0 + \alpha_x \ , \\ c &= a_0 R(u) - R(u)^2/4 + \dot{R}(u)/4 + K(v)/2 \\ &\quad + a_0\alpha_x + a_0\alpha_y + \alpha_{xy} + \alpha_x\alpha_y \ , \end{aligned}$$

where $u = (x+y)/2, v = (x-y)/2$, and

$$a_0 = a_0(x,y) \ , \quad \alpha = \alpha(x,y)$$

are arbitrary functions of two variables, and $R(u), K(v)$ are arbitrary functions of one variable.

In this case the operators $M = M_1$ has the form

$$D_x + D_y + R(u) + \alpha_x + \alpha_y.$$

Remark 5.15. Let a given operator $L = D_{xy} + aD_x + bD_y + c$ there is an $(X+Y)$-transformation, then in general (but not always) it is unique.

Straightforward computation shows that the following theorem holds.

Theorem 5.16. *For the operator L and the $(X+Y)$-transformed operator described in the Theorem 5.14, we may compute their Laplace invariants h, k (7) and the invariant $m = h - k$. For the initial operator L they are*

$$\begin{aligned} h &= a^2 - R(u)a + (R^2(u) - \dot{R}(u))/4 - K(v) + a_x \ , \\ k &= a^2 - R(u)a + (R^2(u) - \dot{R}(u))/4 - K(v) + a_y \ , \\ m &= a_x - a_y \ , \end{aligned}$$

while those of the transformed operator L_1 are

$$\begin{aligned} h_1 &= a^2 - R(u)a + (R^2(u) + \dot{R}(u))/4 - K(v) - a_y \ , \\ k_1 &= a^2 - R(u)a + (R^2(u) + \dot{R}(u))/4 - K(v) - a_x \ , \\ m_1 &= a_x - a_y \ , \end{aligned}$$

that is

$$\begin{aligned} h_1 &= h + \dot{R}(u)/2 - a_x - a_y \ , \\ k_1 &= k + \dot{R}(u)/2 - a_x - a_y \ , \\ m_1 &= m \ . \end{aligned}$$

Example 5.17. There is an R-transformation of the operator $L = D_{xy} + (1/2 - xy)$:

$$\varphi_1(L, D_x + D_y + x + y) = D_{xy} + (-1/2 - xy) \ .$$

In the theorem 5.14 this transformation corresponds to the choice

$$\begin{aligned} R(u) &= 2u \ , \\ K(v) &= 2v^2 \ . \end{aligned}$$

The invariants of the transformed operator are

$$\begin{aligned} h_1 &= h + 1 \ , \\ k_1 &= k + 1 \ , \\ m_1 &= m = 0 \ , \end{aligned}$$

where h, k, m are the invariants of the initial operator L.

Example 5.18. There is a LT-transformation of the operator $L = D_{xy} + 1 + (x-y)^4 - (x+y)^2$:

$$\varphi_1(L, D_x + D_y + 2(x + y)) = D_{xy} - 1 + (x-y)^4 - (x+y)^2 \ .$$

In the theorem 5.14 this transformation corresponds to the choice

$$\begin{aligned} R(u) &= 4u \ , \\ K(v) &= 8v^2 \ . \end{aligned}$$

The invariants of the transformed operator are

$$\begin{aligned} h_1 &= h + 2 \ , \\ k_1 &= k + 2 \ , \\ m_1 &= m = 0 \ , \end{aligned}$$

where h, k, m are the invariants of the initial operator L.

5.5 $X-$ and $Y-$transformations

Consider now the cases $p = 0$ and $q = 0$, which were not considered in the section 5.3.

Definition 5.19. An R-transformation $\varphi_1(L, M)$ is an *X-transformation* if $Sym_M = X$. Analogously, we define an *Y-transformation.*

Consider the operators in $K[D_x, D_y]$, that are of the form

$$L = D_x \circ D_y + aD_x + bD_y + c \ .$$

By the Theorem 5.11, whenever the Laplace invariant $k = ab - c + b_y$ of the operator L does not vanish, there exist an operator L_{-1}, such that

$$(D_x + b - k_x/k) \circ L = L_{-1} \circ (D_x + b) \ .$$

Thus, the Laplace transformation $L \to L_{-1}$ is an X-transformation of the operator L. However, some other X-transformations of the same operator L may exist. Let us look for some functions $a_1, b_1, c_1, r, r_1 \in K$ such that

$$(D_x + r) \circ L = (D_{xy} + a_1 D_x + b_1 D_y + c_1) \circ (D_x + r_1). \tag{62}$$

Expand the products on the both sides of the equality (62), and compare the coefficients. Then from the equality of the coefficients at D_{xx}, one immediately gets

$$a_1 = a.$$

Now since for the X-transformation of Laplace we have $r_1 = b$, we suggest to represent r_1 as

$$r_1 = b + \rho \ ,$$

where ρ is a function in K. We substitute this for b_1 into the equality 62, and obtain

$$\begin{aligned} b_1 &= r - \rho \ , \\ c_1 &= a(r - \rho - b) + a_x - b_y + c - \rho_y \ , \\ r &= b + \rho - \rho_x/\rho \ , \end{aligned}$$

and the condition

$$\rho \left(\frac{c}{\rho}\right)_x - (ab)_x + R = 0, \tag{63}$$

where

$$R = \frac{\rho_x}{\rho}(ab + b_y + \rho_y) + \rho(b_y - a_x + \rho_y) - (b + \rho)_{xy} \ ,$$

to be satisfied. The equation (63) can be solved w.r.t. c:

$$c = -\rho(\int(-ba_x - ab_x + \frac{\rho_x(ab + b_y + \rho_y)}{\rho} + \rho(b_y - a_x + \rho_y) - b_{xy} - \rho_{xy})\rho^{-1}dx + R\,(y))$$

In that way, we describe all possible X-transformations (Y-transformations can be described analogously) in the following theorem:

Theorem 5.20. *Let*

$$L_1 = \varphi_1(D_x \circ D_y + aD_x + bD_y + c, D_x + r),$$

where all the coefficients belong to K. Then

$$\begin{aligned} c &= -\rho(\int(-ba_x - ab_x + \frac{\rho_x(ab + b_y + \rho_y)}{\rho} + \rho(b_y - a_x + \rho_y) - b_{xy} - \rho_{xy})\rho^{-1}dx + R\,(y)) \ , \\ r &= b + \rho - \rho_x/\rho \ , \\ L_1 &= D_{xy} + aD_x + (b - ln(\rho)_x)D_y + c + a_x - b_y - \rho_y - aln(\rho)_x \ , \end{aligned}$$

where a, b, ρ are arbitrary functions in $K[D_x, D_y]$.

For the operator D_{xy} the classical Laplace transformations are not defined, while their generalizations, X-transformations (Y-transformations) exist.

Example 5.21. Let

$$L_1 = \varphi_1(D_x \circ D_y, D_x + r),$$

where all the coefficients belong to K. Then

$$\begin{aligned} r &= -ln(B(x))_x \ , \\ L_1 &= D_x \circ D_y + \frac{B(x)A(y)_y}{(A(y) - \int B(x)dx)^2} \ (D_y + 1) \ , \end{aligned}$$

where $A(y), B(x)$ are arbitrary one-variable functions in $K[D_x, D_y]$. Note that all these L_1 are factorable.

5.6 New Integrable PDEs

One of the main applications of transformations methods is an extension of the number of PDEs with known analytical solution. Namely, by the remark 5.6, the solution of the transformed equation $L_1(z_1) = 0$ can be easily computed from that of $L(z) = 0$. In that way, we may apply the R-transformations φ_1 to all PDEs with known analytical solutions (see them for example in [22]), and thus get new classes of solvable PDEs.

On the other hand, the L-transformations φ_{-1} provides us with a method of PDEs' integration. Indeed, we may apply these transformations to a given PDE as long as we get an PDE with known solution. Then the analytical solution of the initial PDE can be found.

1. The classical example of a successive application of the Laplace Cascade method are equations of the form

$$u_{xy} - \frac{c}{(x+y)^2} = 0 \ , \tag{64}$$

where c is a constant. It is stated [34] that the process will be infinite unless

$$c = n(n+1) \ .$$

Therefore, whenever $c = n(n+1)$ the analytical solution of (64) may be found.

It is astonishing, that all these PDEs are the members of just one sequence of our $(X+Y)$-transformations! At the same time they all cannot be members of some sequence of Laplace Transformations.

The sequence of $(X+Y)$-transformations, which enumerates all the equations of the form (64) with $c = n(n+1)$, starts with the clearly integrable equation $\partial_{xy}(u) = 0$:

$$D_{xy} \overset{R=-\frac{1}{u}}{\longrightarrow} D_{xy} - \tfrac{2}{(x+y)^2} \overset{R=-\frac{2}{u}}{\longrightarrow} D_{xy} - \tfrac{6}{(x+y)^2} \overset{R=-\frac{3}{u}}{\longrightarrow}$$
$$D_{xy} - \tfrac{20}{(x+y)^2} \overset{R=-\frac{4}{u}}{\longrightarrow} \dots , \tag{65}$$

where $R = R(u)$ is a function from the theorem 5.14, which is required to define an $(X+Y)$-transformation. The solution of the initial equation $\partial_{xy}(u) = 0$ is $u = F(x) + G(y)$, where $F(x)$ and $G(y)$ are arbitrary functions of one variable. Then the solution of the equation $(\partial_{xy} - \frac{2}{(x+y)^2})(u) = 0$ is computed as follows:

$$\begin{aligned} & M_1(F(x) + G(y)) \\ & \quad = (D_x + D_y + R((x+y)/2))(F(x) + G(y)) \\ & \quad = -2\frac{F(x) + G(y)}{x+y} + F(x)_x + G(y)_y \ . \end{aligned}$$

The solutions of the equation corresponding to any other operator of the sequence (65) is computed analogously.

2. In fact, the sequence (65) of $(X+Y)$-transformations of the operator D_{xy} is just a particular case of a parameterized one:

$$D_{xy} \overset{R=-\frac{1}{u-k_1}}{\longrightarrow} D_{xy} - \frac{2}{(x+y)^2 - 4k_1(x+y) + 4k_1^2} \overset{R=R(u,k_2)}{\longrightarrow} \dots$$

where k_1, k_2, etc. are constant parameters. Again the solutions of all the operators of the sequence can be computed explicitly. In other words, we have a new class of PDEs, which can be solved analytically.

3. The following is an example of an X-transformation:

$$\begin{aligned} D_{xy} + \Big(b - \frac{1}{x+y}\Big) D_y - W \ &= \\ & \varphi_1\Big(D_{xy} + bD_y \quad , \quad D_x + b + x + y - \frac{1}{x+y}\Big) \ , \end{aligned}$$

where $A(x)$ is an arbitrary one-variable function in K, and

$$\begin{aligned} b \ &= \ A(x) - y + \int W dy \ , \\ W \ &= \ (x+y) e^{\frac{x(x+2y)}{2}} \ . \end{aligned}$$

The initial operator $D_{xy} + bD_y$ is factorable, that is the corresponding PDE is analytically integrable. Therefore, the analytical solution of the transformed PDE can be found also. Note that the transformed PDE is not factorable.

4. Consider the operator

$$L = D_{xy} + AD_x + AD_y \ ,$$

where A is a constant in K. The corresponding equation $L(z) = 0$ can be solved analytically [22]. There are $X+Y$-transformations of the operator L:

$$\varphi_1\Big(L, D_x + D_y + R\Big((x+y)/2\Big)\Big) = L - \frac{\dot{R}((x+y)/2)}{2} \ ,$$

where

$$R\,(u) = 2A - B\tanh\left(Bu + C\right) \ ,$$

B and C are arbitrary constants. Therefore, the analytical solution of the equation $(L - \frac{1}{2}\dot{R}((x+y)/2))(z) = 0$ can be computed.

5.7 Conclusions

In view of important possible applications, such as extending the number of analytically solvable LPDEs, or the construction of a new integration method, or the normalization of LPDOs, GL-transformations are very interesting as a subject for further investigations. In the present chapter the basic properties of GL-transformations have been proved. Based on these properties, one can conjecture some new hypotheses. For example, it seems likely that for LPDOs of some (not very restricted) forms, there always exist an entire family of GL-transformations parameterized by functions in one variable.

6 A Maple-Package for Linear Partial Differential Operators with Parametric Coefficients

6.1 Introduction

The previous chapters in this thesis have been used to expound the new theoretical results obtained. The present chapter is devoted to an exposition of a new computational package, which was used for obtaining the theoretical results of the thesis. What is more, the new results have now themselves been incorporated into the package.

Formally speaking, it would be possible to obtain all of the results under discussion by hand. Although it is not a particularly difficult task to calculate the obstacles to factorizations of LPDOs of order two, nor are the results of one or two applications of the Laplace transformations. However, the computation of only slightly longer sequences of transformations is already an almost impossible task, if it is done by hand. In particular, large expressions were immediately in evidence when I made the computations for the Generalized Laplace transformations theory in chapter 5. Equally challenging were the expressions generated during the finding of the fifth invariant of the complete system of invariants (chapter 3).

I have used the Computer Algebra system MAPLE [20]. Although symbolic algorithms for LPDOs have been quite actively investigated as a research area of Computer Algebra, there is no convenient package for the purposes of this thesis in MAPLE. For this reason, I made a developed a new package, *LPDO manipulations*, which provides the basic manipulations and some algorithms for LPDOs of arbitrary order and in an arbitrary number of variables. It is important that the LPDOs may have symbolic coefficients. It is hoped that beyond the confines of this thesis, the package will be helpful for scientists working in the present topic or in related areas.

The theoretical results of this thesis constitute a special part of the package.

6.2 Other Packages

Consider the existing packages available in MAPLE.

6.2.1 Package *Ore_Algebra*

The package *Ore_Algebra* is one of the main tools for LPDOs in MAPLE. Unfortunately, it is mostly concentrated on the cases in which the coefficients are polynomials or rational functions. For example, an attempt to compute the composition of such two simple operators as D_x and $D_y + \sin(x)$ proceeds as follows.

```
> restart:
> with(Ore_algebra):
> # declare an Ore algebra:
> A:=skew_algebra(diff=[Dx,x],diff=[Dy,y]):
> # declare the operators L1, L2:
> L1:=Dx:
> L2:=Dy+sin(x):
> # compute the composition of L1 and L2:
> M:=skew_product(L1,L2,A);

Error, (in Ore_algebra:-skew_product)
skew polynomials must be members of the algebra
```

The procedure *skew_product* outputs an error because the sine function is being used. To treat LPDOs whose coefficients contain some special functions — such as sine — or some parameters in the *Ore_algebra* package, one needs to describe them in advance, when one declares an Ore algebra. For example, the following code computes the composition of D_x and $D_y + q(x, y)$, where $q(x, y)$ is some parameter function.

```
> restart:
> with(Ore_algebra):
> A:=skew_algebra(diff=[Dx,x],diff=[Dy,y], func=[q]):
> L1:=Dx:
> L2:=Dy+q(x,y):
> # compute the composition of the two operators:
> M:=skew_product(L1,L2,A);

                        M := Dx Dy+Dx q(x,y)+diff(q(x,y),x)
```

Nevertheless, the problem of treating LPDOs with parametric coefficients in *Ore_algebra* is not completely solved, because in most symbolic-algebraic algorithms for LPDOs, one cannot predict what parameters will appear during the execution of the algorithms. For instance, a typical cause of the appearance of new parameters is the solution of differential equations. Consider, for example, the following code, which solves the equation $q_x = q_y$.

```
> Eq1:=diff(q(x,y),x)=diff(q(x,y),y):
> pdsolve(Eq1,q(x,y));
```

```
q(x,y) = _F1(y+x)
```

Thus, the new parameter function $\_F1(y+x)$ has been introduced. On the one hand, one could in principle determine what new parameters have been introduced and redeclare the Ore algebra. On the other hand, new parameters appear in the program very often, and it is not very convenient (and not efficient) to continually search for new parameters and frequently have to re-declare the Ore algebra.

6.2.2 Package *Ore_Tools*

This package treats univariate differential operators only. Just as in the case of the *Ore_Algebra* package, this one concentrates on the cases in which the coefficients are polynomials or rational functions.

The important feature of this package is the *OrePoly* structure, which keeps a differential operator as a list of its coefficients:

```
> restart:
> with(OreTools):
> # define the differential algebra:
> A := SetOreRing(x, 'differential');
                A := UnivariateOreRing(x, differential)
> L := OrePoly(1/x, -2/(x*(1+x)), 1);
                                           2
                   L := OrePoly(1/x, - ---------, 1)
                                       x (1 + x)
```

Convert this OrePoly structure to the corresponding linear functional equation, by applying it to a function $f(x)$:

```
> Apply(L, f(x), A);
                             /d      \
                           2 |-- f(x)|   / 2       \
                   f(x)      \dx     /   |d        |
                   ---- - ----------- + |--- f(x)|
                    x      x (1 + x)     |  2      |
                                         \dx       /
```

6.2.3 Package PDEtools

The package PDEtools treats partial differential equations, rather than operators. It serves as a collection of commands and routines for finding analytical solutions for partial differential equations, both for linear and for non-linear equations. Unfortunately, there is lack of special techniques for linear-equation methods in this package.

6.3 Description of the New Package

6.3.1 General Description

The new package, *LPDO manipulations*, serves as a tool for investigations in the area of symbolic-algebraic algorithms for LPDOs. Here is a list of the basic features of the package.

- The number of independent variables is arbitrary, their names being listed at the beginning of each worksheet. Usually one knows the independent variables in advance.

- LPDOs of arbitrary orders are allowed.

- Arbitrary parameters are allowed. One need not declare them in advance.

- Easy access to the coefficients of LPDOs. Some packages keep the LPDOs as an expression, that is as a polynomial-like structure. If one needs to extract a certain coefficient, the operation can be made by means of some supplied procedure. However, when one treats LPDOs, one mostly manipulates the coefficients, that is one writes the formulae in terms of coefficients, etc. So it is more efficient to maintain coefficients, rather than expressions.

  For example, package *Ore_Algebra* keeps LPDOs as expressions, while package *Ore_Tools* keeps LODOs as an array of their coefficients.

- The basic arithmetic (addition, subtraction, composition) of LPDOs is implemented.

- The operations of transposition and conjugation of LPDOs is implemented.

- Application to a function is implemented. Thus, one can convert an LPDO to the corresponding differential equation, which can be treated then by other packages of MAPLE.

Now we list some more advanced features, which are in the package already.

- Laplace invariants.

- Invariants w.r.t. gauge transformations $L \to g^{-1}Lg$ for operators of the form

  $$L_3 = D_x D_y (pD_x + qD_y) + a_{20}D_x^2 + a_{11}D_{xy} + a_{02}D_y^2 + a_{10}D_x + a_{01}D_y + a_{00} \ , \tag{66}$$

  where all the coefficients are functions in the variables x and y. See them in the chapter 3.

- Equivalence test for LPDOs of the forms (66) and

  $$D_x D_y + aD_x + bD_y + c \ ,$$

  where all the coefficients are functions in the variables x and y. Namely, a test to judge whether two LPDOs are equivalent w.r.t. the gauge transformations $L \to g^{-1}Lg$ is implemented.

- Obstacles to factorizations of second- and third- order LPDOs (see chapter 2).

- Factorization of second- and third- order LPDOs (Grigoriev-Schwarz algorithm, see section 1.7).

- Laplace transformations $L \to L_1$ and $L \to L_{-1}$(see section 1.5).

6.3.2 List of External Procedures

Here I list all the exported (available for usual users) procedures of my package.

- LPDO__set_vars(vars::list) – declares the independent variables. It is obligatory to declare them at the beginning of each worksheet. The argument is the list of the independent variables.

- LPDO__nvars() – returns the number of variables.

- LPDO__create() – creates an operator of degree 0 with the free coefficient equaled 0.

- LPDO__create0(a_{00}) – creates an LPDO of degree 0 with the free coefficient equaled to a_{00}.

- LPDO__degree(L) – returns the degree of the operator L.

- LPDO__simplify(L) – simplifies (in Maple sense) the coefficients of LPDO L.

- LPDO__expand(L) – expands (in Maple sense) the coefficients of L.

- LPDO__factor(L) – factors (in Maple sense) the coefficients of L.
- LPDO__print(L) – prints non-zero coefficients of the operator L. Each of them is given in the form

$$[i_1, \ldots, i_n] \qquad A$$

which means that the coefficient at D^J in L is the expression A.

- LPDO__inc(L_1,L_2) – sets the operator $L1$ to be the sum of the operators $L1$ and $L2$.
- LPDO__set_value(L,f,J::list) – sets the coefficient of D^J in the operator L equaled f.
- LPDO__add_value(L,f,J::list) – adds to the coefficient of D^J in the operator L the function f.
- LPDO__value(L,J::list) – returns the coefficient at D^J in the operator L.
- LPDO__add(L_1, L_2) – returns the sum of two LPDOs $L1$ and $L2$.
- LPDO__minus(L_1, L_2) – returns the difference of two LPDOs $L1$ and $L2$.
- LPDO__mult1(f, L) – returns the result of the multiplication of the operator L by the function f on the left.
- LPDO__mult(L_1, L_2) – returns the composition of operators $L1$ and $L2$.
- LPDO__conj(L, f) – returns the result of the gauge transformation: $f^{-1}Lf$.
- LPDO__apply(L, f) – applies the operator L to the function f.
- LPDO__transpose(L) – returns the transposed to L operator.

The following procedures address the important case of bivariate LPDOs. To describe some of them we use the notations x, y for variables, though any other notations can be used (just declare them by *LPDO__set_vars*).

- LPDO__create2($a_{20}, a_{11}, a_{02}, a_{10}, a_{01}, a_{00}$) – creates the operator

$$a_{20}D_{xx} + a_{11}D_{xy} + a_{02}D_{yy} + a_{10}D_x + a_{01}D_y + a_{00} \ .$$

This is the short form of declaration of second-order bivariate operators.

- LPDO__create1(a_{10}, a_{01}, a_{00}) – creates the operator

$$a_{10}D_x + a_{01}D_y + a_{00} \ .$$

This is the short form of declaration of first-order bivariate operators.

- LPDO__Lapl_h(L) – returns the Laplace invariant h (see section 1.5). The argument L is required to be an LPDO with the symbol XY.

- LPDO__Lapl_k(L) – returns the Laplace invariant k (see section 1.5). The argument L is required to be a LPDO with the symbol XY.

- LPDO__Inv(n, L) – for $n = 1, \ldots, 5$ returns the invariant number n of the full system of invariants of the operator L (see chapter 3 for details). The operator L should have the symbol of the form

$$XY(pX + qY).$$

- LPDO__obstacle($r_1, s_1, r_2, s_2, r_3, s_3, L$) – computes common obstacles to factorizations into three factors. The procedure returns the list of four elements, where the first three elements are the factors of the incomplete factorization of L of the factorization type $(S_1)(S_2)(S_3)$ given in order, while the last element of the list is the obstacle to factorizations of L of that type. Here

$$S_i = r_iX + s_iY \ , \ i = 1, 2, 3 \ .$$

Naturally it is required that

$$Sym_L = S_1 \cdot S_2 \cdot S_3 \ ,$$

and S_i are coprime. Also L is expected to be bivariate. Note that by Theorem 2.35, the obstacle and the corresponding incomplete factorization are unique in this case.

When the last element of the list that the procedure outputs zero, then one can conclude that the operator L is factorable, and the first three elements of the list are the factors of the factorization of L of the factorization type $(S_1)(S_2)(S_3)$ given in order.

- LPDO__obstacle2($r_1, s_1, p_{20}, p_{11}, p_{02}, L, i$) – computes common obstacles to factorizations into two factors. The factorization type is $(S_1)(S_2)$, if $i = 12$, and $(S_2)(S_1)$, if $i = 21$, where

$$S_1 = r_1X + s_1Y \ , \ S_2 = p_{20}X^2 + p_{11}XY + p_{02}Y^2 \ .$$

The procedure returns the list of three elements, where the first two of those are the factors of the incomplete factorization given in order, while the last element of the list is the obstacle to factorizations.

It is required that

$$Sym_L = S_1 \cdot S_2 \ ,$$

S_1, S_2 are coprime, and L is bivariate.

When the last element of the list that the procedure outputs zero, then one can conclude that the operator L is factorable, and the first two elements of the list are the factors of the factorization of L given in order.

- LPDO__Laplace_trans(i,L) – returns the result of the Laplace transformation

  $$L \to L_1 \ ,$$

  if $i = 1$, and the result of the Laplace transformation

  $$L \to L_{-1} \ ,$$

  if $i = -1$ (see section 1.5). The argument L is required to be an LPDO with the symbol XY.

6.4 Examples of Worksheets

6.4.1 Declaration of an LPDO

The package works with an arbitrary number of variables, arbitrary parameters, and coefficients. The only restriction is that the number of independent variables, and their names should be declared at the very beginning. For example, declare the independent variables x, y:

```
> LPDO__set_vars([x,y]):
```

To work with an operator, we declare its name:

```
> L2:=LPDO__create():
```

At this stage, $L2$ is a zero-operator, that is the operator, that multiplies by zero. Then one can change the operator, describing the coefficients of its standard representation, that is of the representation of the form

$$L2 = \sum_{|J| \le d} a_J D^J \ .$$

For example, declare the operator

$$L2 = D_y + q(x, y) \ .$$

```
> LPDO__set_value(L2,1,[0,1]):
> LPDO__set_value(L2,q(x,y),[0,0]):
```

Print the operator:

```
> LPDO__print(L2);
                              [0, 0], q(x, y)
                                 [0, 1], 1
```

Also there are short forms of declarations of operators of orders zero, one, two. For example, the code

```
> LPDO__create2(a20(x,y),a11(x,y),a02(x,y),a10(x,y),a01(x,y),a00(x,y)):
```

declares the operator

$$a20(x,y)D_x^2+a11(x,y)D_xD_y+a02(x,y)D_x^2+a10(x,y)D_x+a01(x,y)D_y+a00(x,y)\ ,$$

and the code

```
> L1:=LPDO__create1(1,0,0):
> LPDO__print(L1);

                                 [1, 0], 1
```

declares the operator

$$D_x\ .$$

6.4.2 Basic Arithmetics

Now consider an example from section 6.2.1, which shows that package *Ore_Algebra* requires the declaration of parametric coefficients in advance. Now, in the new package we are not bound by those requirements:

```
> M:=LPDO__mult(L1,L2):
> LPDO__print(M);

                                    d
                            [0, 0], -- q(x, y)
                                    dx
                             [1, 0], q(x, y)
                                [1, 1], 1
```

Other basic arithmetical operations as addition and subtraction can be executed in the following way:

```
> A:=LPDO__add(L1,L2):
> LPDO__print(A);

                              [0, 0], q(x, y)
                                 [1, 0], 1
                                 [0, 1], 1

> S:=LPDO__minus(L1,L2):
> LPDO__print(S);

                              [0, 0], -q(x, y)
                                 [1, 0], 1
                                 [0, 1], -1
```

Perform the conjugation by the function $f(x, y) = x + y$ (the gauge transformation) and the transposition of the operator $L2$:

```
C:=LPDO__conj(L2,f(x,y)):LPDO__print(C);

                                          /d           \
                     q(x, y) f(x, y) + |-- f(x, y)|
                                          \dy          /
             [0, 0], ------------------------------
                                 f(x, y)
                              [0, 1], 1

> T:=LPDO__transpose(L2):
> LPDO__print(T);

                              [0, 0], q(x, y)
                                 [0, 1], -1
```

Remark 6.1. An operator is kept as a one-dimensional array of coefficients, and so there is easy access to any coefficient of the standard representation of the operator.

For example, show the coefficient of the operator M (we defined it above when demonstrated the multiplication of LPDOs) at D_{xy}:

```
>LPDO__value(M,[1,1]);
                              1
```

Apply the operator $L1$ to a function $f(x, y)$:

```
LPDO__apply(L1,f(x,y));
                              d
                              -- f(x, y)
                              dx
```

Thus, one gets the corresponding to $L1$ differential expression written in the standard MAPLE way.

6.4.3 Computing of Invariants

Compute five invariants (chapter 3) w.r.t. gauge transformations for the operator

$$L3 := D_x D_y (x D_x + y D_y) + D_x + 1 \ .$$

```
> L3:= LPDO__create():
> LPDO__add_value(L3, x, [2,1]):
> LPDO__add_value(L3, y, [1,2]):
> LPDO__add_value(L3, 1, [1,0]):
> LPDO__add_value(L3, 1, [0,0]):
> I1 := LPDO__Inv(1,L3);
                               I1 := 0
> I2 := LPDO__Inv(2,L3);
                               I2 := 0
> I3 := LPDO__Inv(3,L3);
                                      2
                               I3 := x
> I4 := LPDO__Inv(4,L3);
                               I4 := 0
> I5 := LPDO__Inv(5,L3);
                                     3
                              I5 := x  y
```

Now, every operator $L33$, that can be obtained from $L3$ by conjugation has the same set of invariants. For example, consider the conjugation of $L3$ by the function $f(x, y) = x$.

```
> C3:=LPDO__conj(L3,x);
                                   x + 1
                            [0, 0], -----
                                     x
                              [1, 0], 1
                              [1, 1], 2
                              [0, 2], y/x
                              [2, 1], x
                              [1, 2], y
```

```
> I1 := LPDO__Inv(1,C3);
                               I1 := 0
> I2 := LPDO__Inv(2,C3);
                               I2 := 0
> I3 := LPDO__Inv(3,C3);
                                      2
                               I3 := x
> I4 := LPDO__Inv(4,C3);
                               I4 := 0
> I5 := LPDO__Inv(5,C3);
                                     3
                              I5 := x  y
```

In much the same way, the equivalence's test procedure *LPDO__equiv* compares the invariants and output "false" or "true".

6.4.4 Computing of Obstacles to Factorizations

Compute the common obstacles from the examples 2.41 and 2.39. There we consider the operator

$$L = D_x D_y (D_x + D_y) + aD_x + bD_y + c \ ,$$

where $a = a(x, y), b = b(x, y), c = c(x, y)$ - parameters. Compute a common obstacle to factorizations into two factors, for example, a common obstacle to factorizations of the type

$$(X)(Y)(X + Y) \ .$$

```
> L:= LPDO__create2(0, 1, 0, a(x,y),b(x,y), c(x,y)):
> LPDO__add_value(L, 1, [2,1]):
> LPDO__add_value(L, 1, [1,2]):
> res:=LPDO__obstacle2(1,0,0,1,1,L,12):
> LPDO__print(res[1]);
                              [1, 0], 1
> LPDO__print(res[2]);
                           [0, 0], a(x, y)
                              [0, 1], 1
                              [1, 1], 1
                              [0, 2], 1
> LPDO__print(res[3]);
                                        /d          \
                     [0, 0], c(x, y) - |-- a(x, y)|
                                        \dx         /
                           [0, 1], b(x, y)
```

Then the following code produces common obstacles to factorizations of the types

$$(Y)(XX + XY), \quad (X + Y)(XY) \ .$$

```
> res:=LPDO__obstacle2(0,1,1,1,0,L,12):
> LPDO__print(res[1]);
> LPDO__print(res[2]);
> LPDO__print(res[3]);
> res:=LPDO__obstacle2(1,1,0,1,0,L,12):
> LPDO__print(res[1]);
> LPDO__print(res[2]);
> LPDO__print(res[3]);
```

If the first factor is of order two (and, therefore, the second is of order one), then we execute the procedure with the same arguments except the last one, which gets the value 21. Thus, the following code produces common obstacles to factorizations of the types

$$(XY + YY)(X), \quad (XX + XY)(Y) \ , (XY)(X + Y) \ .$$

```
> res:=LPDO__obstacle2(1,0,0,1,1,L,21):
> LPDO__print(res[1]);
> LPDO__print(res[2]);
> LPDO__print(res[3]);
> res:=LPDO__obstacle2(0,1,1,1,0,L,21):
> LPDO__print(res[1]);
> LPDO__print(res[2]);
> LPDO__print(res[3]);
> res:=LPDO__obstacle2(1,1,0,1,0,L,21):
> LPDO__print(res[1]);
> LPDO__print(res[2]);
> LPDO__print(res[3]);
```

Compute the common obstacle to factorizations of the type $XY(X + Y)$ for the operator L:

```
> res:=LPDO__obstacle(1,0,0,1,1,1, L):
> LPDO__print(res[1]);
```

```
[1, 0], 1
```

```
> LPDO__print(res[2]);
```

```
[0, 1], 1
```

```
> LPDO__print(res[3]);

                              [0, 0], 1
                              [1, 0], 1
                              [0, 1], 1

> LPDO__print(res[4]);

                              [0, 0], c
                              [1, 0], a
                              [0, 1], b
```

Therefore, we have

$$L = D_x D_y (D_x + D_y + 1) + aD_x + bD_y + c\ .$$

Generalize the problem: consider the operator

$$L = D_x D_y (D_x + qD_y) + aD_x + bD_y + c$$

with $q = q(x, y)$. The following code computes the common obstacle to factorizations of the factorization type $(X)(Y)(X + qY)$:

```
> LPDO__set_value(L, q(x,y), [1,2]):
> LPDO__print(L):
                           [0, 0], c(x, y)
                           [1, 0], a(x, y)
                           [0, 1], b(x, y)
                              [1, 1], 1
                              [2, 1], 1
                           [1, 2], q(x, y)
> res:=LPDO__obstacle(1,0,0,1,1,q(x,y), L):
LPDO__print(res[1]);
                                  d
                                  -- q(x, y)
                                  dx
                         [0, 0], - ----------
                                   q(x, y)
                              [1, 0], 1
> LPDO__print(res[2]);

                              [0, 1], 1

> LPDO__print(res[3]);
```

```
                    /d          \   /d          \
  -q(x, y) - |-- q(x, y)| + |-- q(x, y)| q(x, y)
              \dx         /   \dy         /
[0, 0], - ------------------------------------------------
                              q(x, y)
                        [1, 0], 1
                     [0, 1], q(x, y)
```

```
> LPDO__print(res[4]);
```

The last command produces

$$
\begin{aligned}
Obst_{(X)(Y)(X+qY)} = \quad & (a - q_{xy}/q + q_{yy} + q_x q_y/q^2)D_x + \\
& (b + q_x/q + 2q_x^2/q^2 - q_{xx}/q)D_y + \\
& c + 3q_x q_{xy}/q^2 - q_x q_{yy}/q - 2q_y q_x^2/q^3 - q_{xxy}/q + q_{xyy} + q_y q_{xx}/q^2 \ .
\end{aligned}
$$

6.4.5 Factorization of LPDOs

Naturally, when a common obstacle is zero, one concludes that the considered LPDO is factorable. Consider the example from the section 1.7, where the Grigoriev-Schwarz algorithm is applied to compute factorization

$$L = (D_x + 1) \circ (D_{yy} + D_x + D_y + x - 1) \ .$$

The following shows how one can compute it by means of only one procedure:

```
> L:= LPDO__create2(1, 1, 1, x, 1, x):
> LPDO__add_value(L, 1, [1,2]):
> res:=LPDO__obstacle2(1,0,0,0,1,L,12):
> LPDO__print(res[1]);

                        [0, 0], 1
                        [1, 0], 1
> LPDO__print(res[2]);

                      [0, 0], x - 1
                        [1, 0], 1
                        [0, 1], 1
                        [0, 2], 1
> LPDO__print(res[3]);
                         0
```

6.4.6 Computing of Invariants and Transformations of Laplace

Consider the code computing the example 1.1. At first, we declare the operator L and compute its Laplace invariants h and k.

```
> L:= LPDO__create2(0, 1, 0, 0, 0, -2/(x+y)^2):
> h:=LPDO__Lapl_h(L);
                                             2
                                      h := --------
                                                  2
                                            (x + y)
> k:=LPDO__Lapl_k(L);
                                             2
                                      k := --------
                                                  2
                                            (x + y)
```

Then, since both h and k are not zero, we can apply the both Laplace transformations to the operator L, and compute their Laplace invariants h_1, k_1 and h_{-1}, k_{-1}:

```
> L1:=LPDO__Laplace_trans(1,L):
> LPDO__print(L1):
                                                2
                                   [0, 0], - --------
                                                    2
                                              (x + y)
                                               2
                                     [1, 0], -----
                                             x + y
                                       [1, 1], 1
> h1:=LPDO__Lapl_h(L1);
                                        h1 := 0
> k1:=LPDO__Lapl_k(L1);
                                              2
                                      k1 := --------
                                                   2
                                             (x + y)
> L_1:=LPDO__Laplace_trans(-1,L): LPDO__print(L_1);
                                                2
                                   [0, 0], - --------
                                                    2
                                              (x + y)
                                               2
```

```
                    [0, 1], -----
                            x + y
                      [1, 1], 1
> h_1:=LPDO__Lapl_h(L_1);
                              2
                  h_1 := --------
                                2
                         (x + y)
> k_1:=LPDO__Lapl_k(L_1);
                     k_1 := 0
```

Since $h_1 = 0$ and $k_{-1} = 0$, both L_1 and L_{-1} are factorable, and, therefore, solvable. The formula (8) that one uses in such the case is implemented in *pdsolve*:

```
> EqL1:=LPDO__apply(L1,z1(x,y)):
> res:=pdsolve(EqL1,z1(x,y)); assign(res);
```

The procedure *pdsolve* returns the solution

$$z_1 = \frac{1}{(x+y)^2}\Big(\int \_F1(y)(x+y)^2 dy + \_F2(x)\Big) ,$$

where $\_F1(y) = B(y)$ and $\_F2(x) = A(x)$ are some parameter functions.

Finally, compute the solution of $L(z) = 0$ from the solution of $L_1(z_1) = 0$:

```
> z:=collect(simplify((1/h)*diff(z1(x,y),x)),diff);
```

The last line gives us

$$z = \frac{1}{2}\_F2(x) + \frac{1}{(x+y)}\Big((x+y)\int (x+y)\_F1(y)dy - \int (x+y)^2\_F1(y)dy - \_F2(x)\Big) .$$

Verify that this is the solution of $L(z) = 0$:

```
> LPDO__apply(L,z);
                     0
```

6.5 Conclusion

The package *LPDO manipulations* is a new tool to work with LPDOs with parametric coefficients. For future I plan to add to the package the following:

- the Laplace Cascade method of integration (section 1.5),
- the factorization algorithm of Grigoriev and Schwarz for arbitrary order LPDOs (section 1.7),
- some other algorithms for LPDOs.

Another important and necessary direction of development of the package is the implementation of some transformations methods for LPDOs. Indeed, most of the known methods works for LPDOs of certain form, and for a given operator we need to find a related to it (in some sense) operator that has the form required by implemented the methods.

7 Conclusion

One of the main goals of the whole theory of LPDOs is the finding of solutions of LPDEs and systems of LPDEs. Therefore, it is important to think about ways in which one can apply the results obtained here and elsewhere to the solution of LPDEs, even though this is broaching a very general and abstract problem of the theory.

The factorization of LPDOs is one important technique for solving LPDEs, which is the reason why, for different kinds of LPDOs, studies of the properties of factorizations and then the construction of factorization algorithms have been at the center of attention during the last decades. Two chapters of the present thesis have been devoted to factorization problems. In chapter 2, I have studied factorizations of LPDOs, having an arbitrary order and having an arbitrary number of independent variables, that satisfy the condition that the symbols of the factors of the factorizations are pairwise coprime. In other words, if one considers some operator L with the symbol Sym_L, then only factorizations

$$L = F_1 \circ \cdots \circ F_k \ ,$$

where $F_i, i = 1, \ldots, k$ are pairwise coprime are considered. This requirement is necessary because most of the results of the chapter essentially use the algorithm of Grigoriev-Schwarz (see section 1.7), which requires that any considered factorization has such a property. In contrast, one advantage of introducing the rings of obstacles, as has been defined above, is that no such requirement is needed, and it is worthwhile to try to extract some properties of factorizations in the general case. Also, for both general and the special cases which have been considered here, one can study the functorial properties of rings of obstacles.

The notion of obstacle rings has already been fruitful, but generalizations are possible. For example, the notion of factorization has itself been generalized in, for example, [33]. This generalization is very interesting, and seems to be a promising path to future extensions of existing results. Specifically, it seems that it will be interesting to explore the notion of obstacle rings for these generalized factorizations.

The advantages of generalizing obstacles to cases in which the symbols of the factors are not coprime have already been pointed out. However, there is another possibility, which I suggested in chapter 4. The main feature of this case lays in the fact that no uniqueness of factorizations can be expected, in contradistinction to the case that was considered in the theory of obstacle, where there is at most one factorization (see section 1.7) of an operator. In addition, in the case of non coprime symbols, parametric factorizations can appear. In chapter 4, I have considered the linearization of the space of parameters that appear in a factorization, so as to study the possible number of parameters and to describe LPDOs that admit parametric factorizations.

Factorization, it must be remembered, is only one method for solving LPDEs. Other methods include transformational ones, which consist in applying certain transformations to a given operator. Once we obtain, as a result, an LPDO whose corresponding LPDE has a known analytical solution, then the properties of the transformations allow us to compute the solutions of the initial LPDE from those of the transformed operator. In chapter 5, I have introduced new transformations, called GL-transformations. These transformations apply to operators of an arbitrary order and in an arbitrary number of independent variables. I described the properties of such transformations when they are applied to operators of the form

$$L = D_{xy} + aD_x + bD_y + c \ , \tag{67}$$

where $a = a(x, y), b = b(x, y), c = c(x, y)$. It is worth noting that not much has been established about the general case. Here is an opportunity of future work, in which the properties of transformations applied more generally might be established. Also for a future, there is the interesting possibility of generalizing the idea of GL-transformations to cases in which even the operator M, which is involved in the defibrination of such transformations, is of order higher than one.

The idea of generalized factorizations was pointed out above. Since this generalization of factorization (see for example [33]) is based on the classical Laplace transformations, it is then a natural idea to attempt similar generalizations based on the GL-transformations, rather than the Laplace ones.

Also on the topic of transformations, one may recall the gauge transformations given by $L \to g^{-1}Lg$ of an LPDO L. The classical Laplace invariants (7) are invariants with respect to these transformations. Moreover, these invariants form a full system of invariants of operators of the form given in (67), that is to say, any other invariant can be expressed in terms of those two. In chapter 3, I have given explicit formulae for five invariants for operators of the form

$$L = D_xD_y(pD_x + qD_y) + a_{20}D_x^2 + a_{11}D_{xy} + a_{02}D_y^2 + a_{10}D_x + a_{01}D_y + a_{00} \ ,$$

where again all the coefficients depend on independent variables x and y. The invariants also form a full system. The question of extending the methods and results to obtaining similar full systems of invariants for operators of higher orders remains open.

Finally, the package described in chapter 6 has been an indispensable tool for exploring the properties of LPDOs and obtaining the results given in this thesis. In addition to enhancing the capabilities of Maple in the area of LPDOs, the package offers the possibility of implementing all well-known algorithms in this area of research. For example, the important algorithm of Grigoriev—Schwarz can be made available to all MAPLE users, as can algorithms developed in this thesis. The development of computer algebra systems in general has revolutionized mathematical research. These days, many mathematicians routinely speak of “experimental mathematics”, and the capacity to experiment with LPDOs, to extend algorithms such as Grigoriev—Schwarz to high orders, etc., could yield a great harvest of new results.

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Index

9 Curriculum Vitae

Education

- OCTOBER 2004 - JUNE 2007

  PhD student in *Research Institute for Symbolic Computation (RISC)*,

  Johannes Kepler University, Linz, Austria

  Scientific adviser: Franz Winkler.

- SEPTEMBER 1999 - JUNE 2004

  Undergraduate and graduate study at *Lomonosov Moscow State University*, Russia,
  Faculty of Mechanics and Mathematics, Division of Mathematics,
  Department of Higher Algebra, Scientific supervisor - E. V. Pankratiev,

  M.Sc. Diploma with honour (Cum Laude) in Mathematics and Applied Mathematics

- SEPTEMBER 1989 – JUNE 1999

  Physics and Mathematics Lyceum 33, Ivanovo, Russia,

  High School Diploma with honour (Cum Laude)

Employment

- OCTOBER 2002 - SEPTEMBER 2004

  Engineer, Laboratory of Computational Methods,
  Lomonosov Moscow State University

- APRIL 2000 - JUNE 2004
  Private tutor in Mathematics

Review work
I am a reviewer for ISSAC 2007.

Computer software languages

MAPLE PROGRAMMING LANGUAGE, SQL, C/C++, PASCAL, BASIC, HTML

Languages spoken

Russian (native), English (fluent), German (medium level), French (basic).

Awards, Prizes, Scholarships

1999-2004 State scholarship for "A" student to study at MSU.

1999 First place, All-Russian Mathematical Olympics (Ivanovo oblast, age cat.17).

1999 Third place, All-Russian Physical Olympics (Ivanovo oblast, age category 17).

1998 First Place, All-Russian Mathematical Olympics (Ivanovo oblast, age cat.16)

1997 Third Place, All-Russian Mathematical Olympics (Ivanovo oblast, age cat.15)

1996 Third Place, All-Russian Mathematical Olympics (Ivanovo oblast, age cat.14)

Publications

[1] E. Shemyakova, F. Winkler. *A Full System of Invariants for Third-Order Linear Partial Differential Operators in General Form.*, submitted, 2007.

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[8] E.S. Shemyakova. *Involutive divisions for effective involutive algorithms*, J. of Mathematical Sciences, 135(5), 3425–3436, 2006.

Invited Talks

Jan. 2007 "Symbolic Computation for Solutions of Partial Differential Equations."

Brock University, St. Catharines, Ontario, Canada. Faculty of Mathematics and Science.

Jan. 2007 "Symbolic Computation for Solutions of Partial Differential Equations."

Symbolic Computation Group, David R. Cheriton School of Computer Science University of Waterloo Waterloo, Ontario, Canada

Presentations

- **June 2006** Applications of Computer Algebra: ACA'06, Bulgaria, Varna.
  Talk: *Obstacles to Factorizations of LPDOs into Several Factors.*

- **May 2006** International Workshop on Computer Algebra and its Application to Physics, Dubna (Russia).

  Talk: *Obstacles to Factorizations of Linear Partial Differential Operators. General Case.* (joint with Franz Winkler)

- **April 2006** TC'06 (Transgressive Computing, conf. in honor Jean Della Dora), Granada, Spain.

  Talk: *Obstacle to Factorization of LPDOs.*

- **December 2005** ASCM'05 (Asian Symposium on Computer Mathematics), Seoul, South Korea, the Korea Institute for Advanced Study (KIAS).

  Talk: *Families Of Factorizations Of Linear Partial Differential Operators in an Arbitrary Number of Variables*

- **July-August 2005** ACA'2005 (Conference on Applications of Computer Algebra), Nara Women's University, Nara, Japan.

  Talk: *Families Of Factorizations Of Linear Partial Differential Operators.*

- **July 2005** SNC'05 (Symbolic-Numeric Computation, satellite of ISSAC'05).

  Talk: *Implementation of a factorization algorithm for LPDOs* (joint with E.Kartaschova, F.Winkler)

- **June 2005** SCEMMP'05 (Symmetry in Nonlinear Mathematical Physics), Workshop: Symbolic Calculations and Exact Methods in

Mathematical Physics, Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv (Kiev), Ukraine.

Talk: *A Maple program for factorization of LPDOs in two variables.*

(joint with E.Kartaschova, F.Winkler)

- **May 2005** International Workshop on Computer Algebra and its Application to Physics, Dubna, Russia.

  Talk: *Applications of BK method of LPDOs factorization.*

- **May 2004** International Workshop on Computer Algebra and its Application to Physics, Dubna (Russia).

  Talk: *Generalizations and Completion of Some Involutive Divisions.*

- **May 2004** International Algebraic Conference (IAC 2004), Moscow, Russia.

  Talk: *Involutive Divisions. Graphs. Applications.*

- **April 2004** RISC-Linz, Austria. Invited talk:

  Completions and Generalizations of Some Involutive Divisions.

- **March 2004** 66th Workshop on General Algebra 19th Conference for Young Algebraists, University of Potsdam (German).

  Talk: *Completion and Generalization of Involutive Divisions.*

- **March 2003** 65th Workshop on General Algebra 18th Conference for Young Algebraists, University of Potsdam, Germany.

  Talk: *Projections and graphs of involutive divisions.*

- **May 2003** International Workshop on Computer Algebra and its Application to Physics, Dubna, Russia.

  Talk: *Involutive divisions. Graphs.*

- **July 2003** ACA'03, USA, section *Groebner bases and applications.*

  Talk: *Graphs in the theory of involutive divisions.*

Refereed Posters

- **July 2006** ISSAC'06, Italy, Genova.

  Poster: *Approximate Factorization of LPDOs. A Full System of Invariants for Order Three.*