Obstacles to Factorization of Partial Differential Operators into Several Factors

E. S. Shemyakova and F. Wincler

RISC-Linz, J. Kepler University Linz, A-4040 Linz, Austria
e-mail: kath@risc.uni-linz.ac.at; Franz.Winkler@risc.uni-linz.ac.at
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Abstract—In the paper, construction of algorithms for factoring linear partial differential operators is studied. New theoretical concepts—obstacle and ring of obstacles—are introduced. These are invariants and posses other interesting properties. A theorem on unique factorization extension starting from some moment, which is important from the standpoint of construction of such algorithms, is proved. Obstacles in the case of the second and third degrees are found.

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1. INTRODUCTION

We study the problem of factoring linear partial differential operators over some field. The study originates from the Grigoriev–Schwarz algorithm [1], which extends factorization (with mutually simple factors) of an operator symbol to the factorization of the operator itself. At the first step of the algorithm, only leading components of the factors are known. Then, at each subsequent step, one either finds the next component in each factor or proves that factorization of this type does not exist. In the latter case, all accumulated information about the operator is lost: at each step of the algorithm, a partial factorization of the operator is implicitly found. We suggest using this information.

In so doing, we arrive at the concepts of partial factorization and ordinary obstacle. In the case of the second-order operators, the latter coincides with the well-known Laplace invariants [2].

The partial factorizations made it possible to prove Theorem 1 on unique extension of operator factorization starting from some moment. The Grigoriev–Schwarz theorem [1] is a particular case of this theorem.

Generally, neither ordinary obstacle nor its symbol is unique or invariant (some examples were considered in [3]). We introduce a new concept—a ring of obstacles—which is defined as a quotient ring of polynomials corresponding to the ring of linear differential operators modulo some uniform ideal. Symbols of all ordinary obstacles belong to the same adjacent class in this ring of obstacles. We named this class an obstacle to factorization. One important feature of an obstacle is that it is invariant with respect to the conjugation operation. We also prove some other interesting features.

Explicit formulas for obstacles of the second- and third-degree operators of two variables are found.

It should be noted that this work has already found an application: a complete system of invariants for a third-order hyperbolic operator of two variables has been derived [4].

This study is a sequel to work [5], where factorization into two factors has been considered.

2. PRELIMINARY FACTS

Let K be a field with differentiation operations $D_1, \ldots, D_n$ defined in it. Consider a ring of linear differential operators $K[D] = K[D_1, \ldots, D_n]$. The set of all linear differential operators of order $\leq i$ equipped with the left and right multiplication is a K-bimodule denoted as $K_{\leq i}$. Thus, we have filtering $\ldots \supset K_{\leq i} \supset K_{\leq i-1} \supset \ldots \supset K_{\leq 0}$. Consider the adjoint algebra

$$Smbl_* = \sum_{i \geq 0} Smbl_i, \quad Smbl_i = K_{\leq i}/K_{\leq i-1}.$$

The K-module $Smbl_*$ is a commutative K-algebra isomorphic to the ring of polynomials $K[X] = K[X_1, \ldots, X_n]$ in n variables. The image of the operator $L \in K[D]$ upon natural projection will be considered as an element of the ring of polynomials $K[X]$ and denoted as Sym_L. In fact, an operator symbol is a homogeneous polynomial corresponding to the sum of leading terms of the operator.

We use the following notation:

$$D^{(i_1, \ldots, i_n)} := D_1^{i_1} \ldots D_n^{i_n},$$

$$\left| D^{(i_1, \ldots, i_n)} \right| = \mathrm{ord}(D^{(i_1, \ldots, i_n)}) := i_1 + \ldots + i_n.$$

By definition, the order of the zero operator is $-\infty$. For a homogeneous polynomial $S \in K[X]$, the operator obtained by substitution of D_i for X_i is denoted as $\hat{S} \in$

$K[D]$. Both the polynomial and the corresponding operator are denoted by one letter if this does not result in any confusion. The set of all operators from $K[D]$ of order i is denoted as $K_i[D]$.

Any operator $L \in K[D]$ of order d is written in the form

$$L = \sum_{|J| \le d} a_J X^J = \sum_{i=0}^{d} L_i, \qquad (1)$$

where $a_J \in K$, $J \in \mathbf{N}^n$, and L_i is the component of L of order i.

3. PARTIAL FACTORIZATIONS

We begin with several definitions presented in [5] for the case of two factors.

Definition 1. Let $L \in K[D]$ and $\mathrm{Sym}_L = S_1 \dots S_k$. The factorization $L = F_1 \circ \dots \circ F_k$ is said to have *type* $(S_1)(S_2)\dots(S_k)$ if $\mathrm{Sym}_{F_i} = S_i \, \forall i \in \{1, \dots, k\}$.

Definition 2. Let, for some operators $L, F_i \in K[D]$, $i = 1, \dots, k$, and, for some $t \in \{0, \dots, \mathrm{ord}(L)\}$,

$$\mathrm{ord}(L - F_1 \circ \dots \circ F_k) < t. \qquad (2)$$

Then, $F_1 \circ \dots \circ F_k$ is called a *partial factorization* of operator L of order t. If, additionally, $S_i = \mathrm{Sym}_{F_i}$, $i = 1, \dots, k$ (and, hence, $\mathrm{Sym}_L = S_1 \dots S_k$), then the partial factorization has type $(S_1)\dots(S_k)$.

Remark 1. Any factorization (in ordinary sense) $L \in K[D]$ is a partial factorization of order 0.

Remark 2. Let $L \in K[D]$, $\mathrm{ord}(L) = d$. Then, for any factorization of the symbol $\mathrm{Sym}_L = S_1 \dots S_k$, the corresponding operator composition $\hat{S}_1 \circ \dots \circ \hat{S}_k$ is a partial factorization of order d.

Suppose that $L \in K[D]$ and $F_1 \circ \dots \circ F_k$ is its partial factorization of order t. It is not difficult to see that condition (2) holds independent of what terms of order $< t - (d - d_i)$ are selected in the factor F_i. Hence, we obtain new partial factorizations of order equal to or less than t. Therefore, it is natural to introduce the following definition.

Definition 3. Let $L \in K[D]$, $\mathrm{Sym}_L = S_1 \dots S_k$, $\mathrm{ord}(S_i) = d_i$, $i = 1, \dots, k$, and

$$F_1 \circ \dots \circ F_k, \quad F_1' \circ \dots \circ F_k'$$

be partial factorizations of orders t and t', respectively. Let $t' < t$. Then, $F_1' \circ \dots \circ F_k'$ extends $F_1 \circ \dots \circ F_k$ if

$$\mathrm{ord}(F_i - F_i') < t - (d - d_i), \quad \forall i \in \{1, \dots, k\}.$$

Example 1. Consider the fifth-order operator

$$L = (D_1^2 + D_2 + 1) \circ (D_1^2 D_2 + D_1 D_2 + D_1 + 1).$$

Factorizations of the form

$$(D_1^2 + \dots) \circ (D_1^2 D_2 + \dots),$$

where ellipses denote arbitrarily selected terms of lower orders, are partial factorizations of order five for which the fourth-order extensions are partial factorizations of the form

$$(D_1^2 + D_2 + \dots) \circ (D_1^2 D_2 + D_1 D_2 + \dots).$$

Remark 3. Let $L \in K[D]$. Then, $F_1 \circ \dots \circ F_k$ is its partial factorization of form $(S_1)\dots(S_k)$ if and only if $F_1 \circ \dots \circ F_k$ is an extension of the partial factorization $S_1 \circ \dots \circ S_2$.

The following two propositions can easily be proved and will be used in proving the theorem below.

Proposition 1. Let S_1, S_2, and p be homogeneous polynomials in an arbitrary number of independent variables of degrees d_1, d_2, and s ($0 < s < d_1 + d_2$), respectively. Let S_1 and S_2 be mutually simple. Then, there does not exist more than one pair (u, v) of homogeneous polynomials u and v of degrees $s - d_1$ and $s - d_2$, respectively, such that

$$S_1 \cdot u + S_2 \cdot v = p.$$

The second assertion is an extension of Proposition 1 to the case of polynomials that are not mutually simple.

Proposition 2. Let S_1, S_2, and p be homogeneous polynomials in an arbitrary number of variables of degrees d_1, d_2, and s, respectively. Let polynomial S_0 of degree d_0 be the greatest common divisor of S_1 and S_2, and let $0 < s < d_1 + d_2 - d_0$. Then, there does not exist more than one pair (u, v) of homogeneous polynomials u and v of degrees $s - d_1$ and $s - d_2$, respectively, such that

$$S_1 \cdot u + S_2 \cdot v = p. \qquad (3)$$

For any factorization $S_1 \cdot S_2$ of a symbol, the corresponding composition of the operators $\hat{S}_1 \circ \hat{S}_2$ is a partial factorization of the operator L. If S_1 and S_2 are mutually simple, there does not exist more that one extension of this partial factorization to the factorization of L [1]. If there exists a nontrivial common divisor of S_1 and S_2, this is not always true. For example, consider the Blumberg–Landau operator [6]

$$L = D_x^3 + x D_x^2 D_y + 2 D_x^2 + (2x + 2) D_x D_y$$
$$+ D_x + (2 + x) D_y.$$

This is an example of the operator that has two different factorizations into different number of factors (of course, the factors are irreducible, i.e., cannot be factored into lower-order factors):

$$L = (D_x + 1) \circ (D_x + 1) \circ (D_x + x D_y)$$
$$= (D_x^2 + x D_x D_y + D_x + (2 + x) D_y) \circ (D_x + 1).$$

For the same operator L (by the way, its symbol is $X^3 + xX^2Y$), there exists a family of factorizations into two factors with the symbols $S_1 = X$ and $S_2 = X(X + XY)$:

$$L = \left(D_x + 1 + \frac{1}{x + f_1(y)} \right) \circ \left(D_x^2 + xD_xD_y \right.$$

$$+ \left(1 - \frac{1}{x + f_1(y)} \right) D_x + \left(x + 1 - \frac{x}{x + f_1(y)} \right) D_y \right),$$

where $f_1(y) \in K$ is a functional parameter. Thus, the factorization is not unique. Nevertheless, certain results can be established even in this case where the factors are not mutually simple.

Theorem 1. Let $L \in K[D]$, $\mathrm{Sym}_L = S_1 \cdot S_2$, $\mathrm{ord}(L) = d$, and polynomial S_0 of degree d_0 be the greatest common divisor of S_1 and S_2. Then, for any partial factorization of order $(d - d_0)$ of type $(S_1)(S_2)$, there does not exist more than one extension to the factorization of the operator L of the same type.

The proof is based on the following lemma.

Lemma 1. Let $L \in K[D]$, $\mathrm{Sym}_L = S_1 \cdot S_2$, $\mathrm{ord}(L) = d$, and polynomial S_0 of degree d_0 be the greatest common divisor of S_1 and S_2. Then, for any partial factorization of type $t \le (d - d_0)$ and order t, where $(S_1)(S_2)$, there does not exist more than one extension to the partial factorization of order $t - 1$ (up to lower-order terms).

Proof. If $d_0 = 0$, the lemma assertion follows from results of [1]. If $d_0 > 0$, let us write the extension of a given factorization of order t to the operator factorization in the general form as

$$L = \left(\hat{S}_1 + \sum_{j=0}^{k_1 - 1} G_j \right) \circ \left(\hat{S}_2 + \sum_{j=0}^{k_2 - 1} H_j \right), \qquad (4)$$

where $k_1 = \mathrm{ord}(S_1)$, $k_2 = \mathrm{ord}(S_2)$, $G_j \in K_j[D]$, $H_i \in K_i[D]$, $j = 0, \ldots, (k_1 - 1)$, $i = 0, \ldots, (k_2 - 1)$. Comparing coefficients of terms of power $t - 1$ on both sides of (4), we obtain

$$L_{t-1} = H_{t-k_1-1} \cdot S_1 + G_{t-k_2-1} \cdot S_2 + P_{t-1}, \qquad (5)$$

where P_{t-1} is a homogeneous polynomial of degree t that is uniquely computed by homogeneous polynomials G_i and H_j, $i > t - k_1 - 1$, $j > t - k_2 - 1$ (components of the original partial factorization of order t). For $i < 0$, the polynomials G_i and H_i are assumed equal to zero.

Now, since $t - 1 < d - d_0$, we may apply Proposition 2, which says that there does not exist more than one solution to Eq. (5). That is, there does not exist more than one extension to the partial factorization of order $t - 1$.

Corollary 1. If S_1 and S_2 are mutually simple, there does not exist more than one factorization of L of type $(S_1)(S_2)$. That is, the Grigoriev–Schwarz theorem [1] is a particular case of the above theorem.

Corollary 2. For ordinary differential operators,

$$\mathrm{GCD}(S_1, S_2) = X^{d_0},$$

where

$$d_0 = \min(\mathrm{ord}(S_1), \mathrm{ord}(S_2)).$$

Then, for any partial factorization of order

$$\max(\mathrm{ord}(S_1), \mathrm{ord}(S_2)) - 1,$$

there does not exist more than one extension to an ordinary factorization.

Corollary 3. Let $L \in K[D]$, $\mathrm{Sym}_L = S_1 \cdot S_2$, and S_1 and S_2 be mutually simple. Then, for any t, $t < \mathrm{ord}(L)$, there does not exist more than one (up to lower-order terms) partial factorization of order t.

4. RING OF OBSTACLES, OBSTACLES

The following theorem is proved by induction on the number of factors with the use of the theorem on uniqueness of factorization in the case of two factors [1].

Theorem 2. Let $L \in K[D]$, $\mathrm{Sym}_L = S_1 \cdot S_2 \ldots S_k$, and $S_1, \ldots, S_k$ be mutually simple. Then, there does not exist more than one factorization of type $(S_1)(S_2)\ldots(S_k)$.

Consider factorable operators as a submanifold of all operators from $K[D]$ with a given symbol factorization.

Theorem 3. Let K be a field. Consider the submanifold of operators from $K[D]$ with the fixed symbol $\mathrm{Sym} = S_1 \ldots S_k$, $\mathrm{ord}(S_i) = d_i$, $i = 1, \ldots, k$. Then, the codimension of the submanifold of operators having factorization of type $(S_1)(S_2)\ldots(S_k)$ is

$$\binom{n + d - 1}{n} - \sum_{i=1}^{k} \binom{n + d_i - 1}{n}.$$

Proof. Any operator L from the considered set can be written in the form

$$L = \left(S_1 + \sum_{i=0}^{d_1 - 1} G_i^1 \right) \circ \ldots \circ \left(S_k + \sum_{i=0}^{d_k - 1} G_i^k \right), \qquad (6)$$

where G_i^j denotes the ith power component in the jth factor. Comparing components of power t, $0 \le t \le \mathrm{ord}(L) - 1$, on both sides of (6), we obtain

$$P_t = (\mathrm{Sym}/S_1) \cdot G_{t-d+d_1}^1 + \ldots + (\mathrm{Sym}/S_k) \cdot G_{t-d+d_k}^k, \qquad (7)$$

where P_t is a homogeneous polynomial of degree t uniquely computable by the homogeneous polynomials G_i, H_j, $i > t - k_1$, $j > t - k_2$. Hence, if the equation is solved in the "decreasing" order (i.e., starting from $t = \mathrm{ord}(L) - 1$ with t decreasing by one at each step), this polynomial can be considered known.

The polynomials G_i, H_j, $i > t - k_1$, $j > t - k_2$, and, hence, P_t are determined uniquely, which immediately follows from Lemma 2.

Lemma 2. Let $S_1, \ldots, S_k$ be mutually simple homogeneous polynomials of degrees $d_1, \ldots, d_k$, respectively. Let us denote $S = S_1 \ldots S_k$. Then, there does not exist more than one set $(A_1, \ldots, A_k)$ such that

$$P_t = (S/S_1) \cdot A_1 + \ldots + (S/S_k) \cdot A_k, \qquad (8)$$

where $\mathrm{ord}(P_t) = t$, $t < \mathrm{ord}(S)$, and $\mathrm{ord}(A_i) + \mathrm{ord}(S/S_i) = t$.

Proof. Suppose that there exist two such sets, $(A_1', \ldots, A_k')$ and $(A_1'', \ldots, A_k'')$. Subtracting the equations corresponding to them, we obtain

$$0 = (S/S_1) \cdot B_1 + \ldots + (S/S_k) \cdot B_k, \qquad (9)$$

where $B_i = A_i' - A_i''$, $i = 1, \ldots, k$. Without loss of generality, we may assume that $B_1 \neq 0$ and rewrite Eq. (9) as

$$-(S/S_1) \cdot B_1 = (S/S_2) \cdot B_2 + \ldots + (S/S_k) \cdot B_k.$$

Now, all addends on the right-hand side of the equation are divided by S_1, whereas (S/S_1) is not divided by S_1. Hence, S_1 must divide B_1, and, thus, $\mathrm{ord}(B_1) \geq \mathrm{ord}(S_1)$.

On the other hand, we know that $\mathrm{ord}(A_i) + \mathrm{ord}(S/S_i) = t$ and $t < \mathrm{ord}(S)$; i.e., $\mathrm{ord}(A_i) < \mathrm{ord}(S_i)$, and, hence, $\mathrm{ord}(B_1) < \mathrm{ord}(S_i)$. This contradicts the conclusion of the previous paragraph and proves the lemma.

The existence of a factorization is determined by compatibility of the system of all equations (7), $t = d - 1, \ldots, 0$. The desired co-dimension is equal to the number of independent equations in the operator coefficients.

For each t, we have linear equation (7) in the polynomials $G_{t-d+d_1}^1, \ldots, G_{t-d+d_k}^k$, which is equivalent to a system of linear equations in their coefficients, $A \cdot \vec{g} = \vec{c}$, where A is the system matrix. Since the system has a unique solution, the rank of matrix A is equal to the number of variables v; i.e., the columns of A are linearly independent.

The system $A \cdot \vec{g} = \vec{c}$ is compatible when vector $\vec{c}$ belongs to the v-dimensional affine space spanned by the columns of A. The length of vector $\vec{c}$ is equal to the number of equations in the system. That is, the co-dimension of the space of solutions is equal to the difference of the number of the equations and the number of variables. The total co-dimension of the operators that have factorizations of the type $(S_1)(S_2)\ldots(S_k)$ is equal to the difference of the number of the equations and the number of variables in all steps altogether, which is found by using the well-known fact from the combinatorial analysis formulated in the following lemma.

Lemma 3. The cardinality of the set

$$\{M = x_1^{d_1} \ldots x_n^{d_n} \big| d_1 + \ldots + d_n = t\}$$

of monomials in n independent variables $x_1, \ldots, x_n$ is given by $\binom{n+t-1}{t} = \binom{n+t-1}{n-1}$.

The theorem on the co-dimension is proved.

Example 2. Consider all second-order operators of two independent variables with the symbol $S_1 \cdot S_2$, where S_1 and S_2 are mutually simple fixed homogeneous operators of the first order. By Theorem 3, the co-dimension of the manifold of operators having factorizations of the type $(S_1)(S_2)$ is equal to 1.

It is not difficult to find an explicit equation governing this manifold. For example, let $\hat{S}_1 = D_1$ and $\hat{S}_2 = D_2$. Consider all operators of the form $L = D_1 D_2 + a_{10} D_1 + a_{01} D_2 + a_{00}$. Such an operator has a factorization of type $(S_1)(S_2)$ if and only if the coefficients a_{10}, a_{01}, and a_{00} satisfy the equation

$$a_{00} - a_{10} a_{01} - \partial_x(a_{10}) = 0.$$

Example 3. Consider all third-order operators of two independent variables with the symbol $S_1 \cdot S_2$, where S_1 and S_2 are mutually simple fixed homogeneous operators of the first and second order, respectively. The co-dimension of the manifold of operators having factorizations of the type $(S_1)(S_2)$ is equal to 2.

On the other hand, if we consider a factorization of the type $(S_1)(S_2)(S_3)$, where S_1, S_2, and S_3 are mutually simple fixed operators of the first order, the co-dimension is equal to 3 by Theorem 3.

The following definition is needed for studying operators that have no factorizations of some types.

Definition 4. Let $L \in K[D]$ and $\mathrm{Sym}_L = S_1 \ldots S_k$. An operator $R \in K[D]$ is said to be *ordinary obstacle* to factorization of L of type $(S_1)(S_2)\ldots(S_k)$ if the operator $L - R$ has factorization of type $(S_1)(S_2)\ldots(S_k)$ and the order of R is minimal over all operators possessing this property.

Ordinary obstacles are closely related to partial factorizations introduced earlier.

Proposition 3. Let $L \in K[D]$ and $\mathrm{Sym}_L = S_1 \ldots S_k$. Then, an ordinary obstacle to factorization of type $(S_1)\ldots(S_k)$ has order t if and only if $t + 1$ is the minimal order of partial factorization.

Clearly, ordinary obstacles are not uniquely determined. Moreover, their symbols are not uniquely determined as well. To get a unique (in a sense, invariant) concept, we introduce the following definition.

Definition 5. Let $L \in K[D]$ and $\mathrm{Sym}_L = S_1 \cdot S_2 \cdot \ldots \cdot S_k$. A *ring of obstacles* for factorization of type $(S_1)\ldots(S_k)$ is the quotient ring

$$K(S_1, \ldots, S_k) = K[X]/I,$$

where $I = \left(\dfrac{\mathrm{Sym}_L}{S_1}, \dfrac{\mathrm{Sym}_L}{S_2}, \ldots, \dfrac{\mathrm{Sym}_L}{S_k} \right)$ is a uniform ideal.

Remark 4. In the case of two factors ($k = 2$), the ring of obstacles is given by

$$K(S_1, S_2) = K[X]/(S_1, S_2),$$

which coincides with the definition given in [5], where the case of two factors is considered.

Theorem 4. Let $L \in K[D]$ and $\mathrm{Sym}_L = S_1 \cdot S_2 \cdot \ldots \cdot S_k$, where S_i, $i \in \{1, \ldots, k\}$, are mutually simple. Then, symbols of all ordinary obstacles of type $(S_1)\ldots(S_k)$ belong to the same adjacent class of the quotient ring $K(S_1, \ldots, S_k)$.

Proof. Let us denote $d_i = \mathrm{ord}(S_i)$, $i \in \{1, \ldots, k\}$, and let t be the order of ordinary obstacles. Repeating arguments from the proof of Theorem 3, we obtain Eq. (7); i.e., the symbol of any general obstacle can be written as

$$P_t - ((\mathrm{Sym}_L/S_1) \cdot G^1_{t-d+d_1} + \ldots$$
$$+ (\mathrm{Sym}_L/S_k) \cdot G^k_{t-d+d_k}),$$

where P_t is a known uniquely determined polynomial, which is the same for all ordinary obstacles. That is, all ordinary obstacles belong to the adjacent class $[P_t]$ of the quotient ring $K(S_1, \ldots, S_k)$.

Definition 6. The adjacent class of symbols of ordinary obstacles in the ring of obstacles is called *obstacle* to factorization.

Remark 5. Any element from this adjacent class is an ordinary obstacle.

Definition 7. Factorization types $(S_1)\ldots(S_k)$ and $(b_1 S_1)\ldots(b_k S_k)$ are *similar* if $b_1, \ldots, b_k \in K$ and $b_1\ldots b_k = 1$.

Theorem 5. For a fixed operator, rings of obstacles and obstacles of similar types coincide.

Proof. Let us fix an operator $L \in K[D]$ and two similar factorization types, $(S_1)\ldots(S_k)$ and $(b_1 S_1)\ldots(b_k S_k)$, where $b_i \in K$, $i = 1, \ldots, k$. Then, the uniform ideals $(S_1, \ldots, S_k)$ and $(b_1 S_1, \ldots, b_k S_k)$, and, hence, the rings of obstacles, coincide.

Any ordinary obstacle of type $(S_1)\ldots(S_k)$ and order d_0 can be written as

$$P = L - (\hat{S}_1 + T_1) \circ \ldots \circ (\hat{S}_k + T_k), \qquad (10)$$

where T_i is the sum of components of orders $d_i - 1, \ldots, d - d_i - d_0 + 1$ and $\mathrm{ord}(P) = d_0$.

There exist $T'_1, \ldots, T'_k$ such that T'_i, like T_i, is the sum of components of orders $d_i - 1, \ldots, d - d_i - d_0 + 1$, and

$$(S_1 + T_1) \circ \ldots \circ (S_k + T_k)$$
$$= (b_1 S_1 + T'_1) \circ \ldots \circ (b_k S_k + T'_k).$$

Therefore, P is ordinary obstacle of order d_0 of type $(b_1 S_1)\ldots(b_k S_k)$. On the other hand, we have already proved that the obstacle rings $K(S_1, \ldots, S_k)$ and $(b_1 S_1, \ldots, b_k S_k)$ coincide. Hence, the adjacent classes coincide as well; i.e., obstacles of similar types coincide.

Remind the following definition:

Definition 8. Let $g \in K^*$ be an invertible element from a ring K and $L \in K[D]$. Then, the operator $g^{-1} \circ L \circ g$ is called conjugate of L.

Theorem 6. Ordinary obstacles for conjugate operators are conjugate: if P is an ordinary obstacle for $L \in K[D]$, then $g^{-1} P g$ is an ordinary obstacle for $g^{-1} L g$, where $g \in K^*$.

Proof. Let us consider an ordinary obstacle for L of order d_0 given by (10). Let us find the conjugate obstacle using g:

$$g^{-1} P g = g^{-1} L g - g^{-1} \circ (S_1 + T_1)$$
$$\circ \prod_{j=2}^{k-1} (S_i + T_i) \circ (S_k + T_k) \circ g.$$

There exist $T'_1, \ldots, T'_k$ such that T'_i, like T_i, is the sum of components of orders $d_i - 1, \ldots, d - d_i - d_0 + 1$, and

$$g^{-1} P g = g^{-1} L g - (g^{-1} S_1 + T'_1)$$
$$\circ \prod_{j=2}^{k-1} (S_i + T'_i) \circ (g S_k + T'_k).$$

Corollary 4. Obstacles to conjugate operators coincide.

Proof. Ordinary obstacles to conjugate operators are conjugate; hence, the symbol of ordinary obstacles is not changed.

Theorem 7. Let $n = 2$, $L \in K[D]$, $\mathrm{ord}(L) = d$, and $\mathrm{Sym}_L = S_1\ldots S_k$, where S_i, $i \in \{1, \ldots, k\}$, are mutually simple. Then, the obstacle ring $K(S_1, \ldots, S_k)$ raised to power $d - 1$ is equal to zero (i.e., actual obstacles can be of order $d - 2$ or less).

Proof. Let us denote $d_i = \mathrm{ord}(S_i)$, $i \in \{1, \ldots, k\}$. Repeating arguments from the proof of Theorem 3, we obtain Eq. (7) for $t = d - 1$:

$$P_{d-1} = (\mathrm{Sym}_L/S_1) \cdot G^1_{d_1 - 1} + \ldots + (\mathrm{Sym}_L/S_k) \cdot G^k_{d_k - 1},$$

which has at most one solution with respect to $G^1_{d_1 - 1}, \ldots, G^k_{d_k - 1}$. Let us consider the corresponding system of linear equations in their coefficients. By Lemma 3, the number of equations in such a system is equal to d, and the number of variables is also equal to d, which means that the system has a unique solution. Hence, we have found a partial factorization of order $d - 1$.

We remind that an operator $L \in K[D]$, $\mathrm{ord}(L) = d$, is called strictly hyperbolic if its symbol is factored into d different factors.

Theorem 8. Let $n = 2$ and $L \in K[D]$ be a strictly hyperbolic operator of order d. Then, an ordinary obsta-

cle of a fixed type to factorization into first-order factors is determined uniquely.

Proof. Let $(S_1)\dots(S_d)$ be some factorization type and P be an ordinary obstacle for this factorization type. Let the order of obstacles be p. Suppose that there is another ordinary obstacle of this type. Then, it can be written in the form

$$P + (\text{Sym}_L/S_1) \cdot A_1 + \dots + (\text{Sym}_L/S_d) \cdot A_d,$$

where A_i are homogeneous polynomials of degrees $p_i = p - \text{ord}(\text{Sym}_L/S_i) = p - (d - 1)$; i.e., $p \geq d - 1$. On the other hand, by Theorem 7, the ring of obstacles raised to power $d - 1$ is equal to zero; therefore, $p \leq d - 2$.

5. SECOND-ORDER OPERATORS OF TWO VARIABLES

Consider a second-order hyperbolic operator $L \in K[D_x, D_y]$ in the coordinate system where its symbol has the form XY. Then, by Theorems 7 and 8, both ordinary obstacles to this operator have order 0 and are uniquely determined. Let us find explicit formulas for them.

Theorem 9. Let

$$L = D_x \cdot D_y + aD_x + bD_y + c,$$

where $a_{10}, a_{01}, a_{00} \in K$. Then, obstacles of types $(X)(Y)$ and $(Y)(X)$ are given by

$$c - ab - \partial_x(a),$$

$$c - ab - \partial_y(b),$$

respectively.

Proof. The factorization of L of type $(X)(Y)$ has the form

$$L = (D_x + g_{00}) \circ (D_y + h_{00}),$$

where g_{00} and h_{00} are some elements of K. Comparing the first-order components on the left-hand and right-hand sides of the equation, we obtain

$$(a - h_{00})D_x + (b - g_{00})D_y = 0; \qquad (11)$$

i.e., $a = h_{00}$ and $b = g_{00}$. Now, we calculate the obstacle as

$$L - (D_x + b) \circ (D_y + a) = c - ab - \partial_x(a).$$

The obstacle of type $(Y)(X)$ is found similarly.

Remark 6. The obstacles found coincide with the Laplace invariants [2].

6. THIRD-ORDER OPERATORS OF TWO VARIABLES

Consider a third-order operator $L \in K[D_1, D_2]$. Let its symbol be $S_1 \cdot S_2 \cdot S_3$. Then, we have six factorization types when factoring into three factors,

$$(S_1)(S_2)(S_3), \quad (S_1)(S_3)(S_3), \quad (S_2)(S_1)(S_3),$$

$$(S_2)(S_3)(S_1), \quad (S_3)(S_1)(S_2), \quad (S_3)(S_2)(S_1),$$

and six types of factorization into two factors,

$$(S_1)(S_2S_3), \quad (S_2)(S_1S_3), \quad (S_3)(S_1S_2),$$

$$(S_1S_2)(S_3), \quad (S_1S_3)(S_2), \quad (S_2S_3)(S_1).$$

6.1. Two Factors

Our theory works in the case of mutually simple factor symbols. That is, if the considered type is $(S_1)(S_2S_3)$, then S_1 and S_2S_3 must be mutually simple. Taking this into account and noting that symmetric types are considered similarly, we may confine ourselves to the following two important cases: factorization type $(X)(X^2 + XY)$ for the operator with the symbol $X^2Y + XY^2$ and type $(X)(Y^2)$ for the operator with the symbol XY^2.

Note that, by Theorem 7, such ordinary obstacles may be only of orders one or zero. In the former case, an ordinary obstacle is not uniquely determined.

Theorem 10. Let

$$L = \widehat{\text{Sym}}_L + a_{20}D_{xx} + a_{11}D_{xy}$$
$$+ a_{02}D_{yy} + a_{10}D_x + a_{01}D_y + a_{00},$$

where $a_{ij} \in K$. Let $\text{Sym}_L = XY(X + Y)$. Then,

$$\text{Obst}_{(X)(YX + YY)}$$

$$= (a_{02}^2 - a_{11}a_{02} + a_{01} + \partial_x(a_{02} - a_{11}))D_y$$

$$+ a_{00} - a_{02}a_{10} + a_{02}^2a_{20} + 2a_{02}\partial_x(a_{20})$$

$$- \partial_x(a_{10}) + a_{20}\partial_x(a_{02}) + \partial_{xx}(a_{20})$$

is an ordinary obstacle to factorization of L of type $(X)(YX + YY)$. Let $\text{Sym}_L = X^2Y$. Then,

$$\text{Obst}_{(Y)(XX)} = (a_{10} - a_{20}a_{11} - \partial_y(a_{11}))D_x$$

$$+ a_{00} - a_{20}a_{01} + a_{20}^2a_{02} + 2a_{20}\partial_y(a_{02})$$

$$- \partial_y(a_{01}) + a_{02}\partial_y(a_{20}) + \partial_{yy}(a_{02})$$

is an ordinary obstacle to factorization of L of type $(Y)(XX)$.

Proof. All factorizations of type $(X)(YX + YY)$ are of the form

$$L = (D_x + G_0) \circ (D_{xy} + D_{yy} + H_1 + H_0), \qquad (12)$$

where $G_0 = g_{00} \in K$, $H_1 = h_{10}D_x + h_{01}D_y \in K[D_x, D_y]$, and $H_0 = h_{00} \in K$. Comparing the second-order components on both sides of Eq. (12), we obtain the following system of linear equations in the coefficients h_{10}, h_{01}, and g_{00}:

$$\begin{cases} a_{20} = h_{10} \\ a_{11} = h_{01} + g_{00} \\ a_{02} = g_{00}. \end{cases}$$

Finding the only solution to the system and comparing coefficients of D_x on both sides of (12), we obtain

$$h_{00} = a_{10} - a_{20}a_{02} - \partial_x(a_{20}).$$

Now, we can calculate the ordinary obstacle as $P = L - (D_x + G_0) \circ (D_{xy} + D_{yy} + H_1 + H_0)$.

The ordinary obstacle for the factorization type $(Y)(XX)$ is found similarly.

6.2. Three Factors

Here, it will suffice to consider the case of a hyperbolic operator with the symbol $XY(X + Y)$ and the factorization type $(X)(Y)(X + Y)$. In this case, an ordinary obstacle may be only of order one or zero (by Theorem 7) and is unique (by Theorem 8).

Theorem 11. Let

$$L = D_x D_y (D_x + D_y) + a_{20}D_{xx} + a_{11}D_{xy}$$
$$+ a_{02}D_{yy} + a_{10}D_x + a_{01}D_y + a_{00},$$

where $a_{ij} \in K$. The ordinary obstacle of type $(X)(Y)(X + Y)$ is given by

$$\mathrm{Obst}_{(X)(Y)(X+Y)}$$
$$= (a_{10} - a_{20}a_{11} - a_{20}^2 - \partial_x(a_{20}) + \partial_y(s_2))D_x$$
$$+ (a_{01} - a_{02}a_{11} + a_{02}^2 + \partial_x(-a_{11} + a_{02}))D_y$$
$$+ a_{00} + a_{20}a_{02}s_2 + s_2\partial_x(a_{20})$$
$$+ (a_{20}\partial_x + \partial_{xy} + a_{02}\partial_y)(s_2),$$

where $s_2 = a_{20} - a_{11} + a_{02}$.

Proof. Let us write factorization of type $(X)(Y)(X + Y)$ in the general form as

$$L = (D_x + g_0) \circ (D_y + h_0) \circ (D_x + D_y + f_0). \quad (13)$$

Comparing the second-order components on both sides of the equation, we obtain the unique solution

$$h_0 = a_{20}, \quad g_0 = a_{02}, \quad f_0 = a_{11} - a_{02} - a_{20}.$$

Now, the ordinary obstacle can be calculated as the difference of the left-hand and right-hand sides of (13).

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