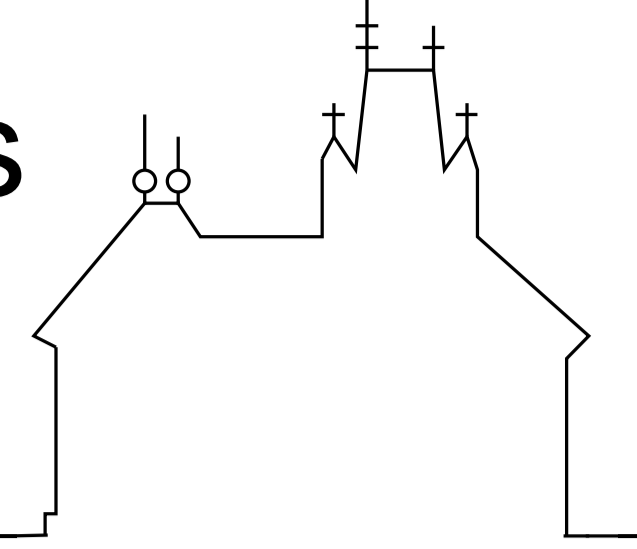


Parameterized telescoping proves algebraic independence of sums

Carsten Schneider



Abstract: Usually creative telescoping is used to derive recurrences for sums. In this article we show that the non-existence of a creative telescoping solution, and more generally, of a parameterized telescoping solution, proves algebraic independence of certain types of sums. Combining this fact with summation-theory shows transcendence of whole classes of sums.

Parameterized telescoping

Given $f_1(k), \dots, f_d(k)$ over a field^a \mathbb{K} ;

find constants $c_1, \dots, c_d \in \mathbb{K}$ and $g(k)$ such that

$$g(k+1) - g(k) = c_1 f_1(k) + \dots + c_d f_d(k). \quad (1)$$

If one succeeds, one gets the sum-relation

$$g(n+1) - g(0) = c_1 \sum_{k=0}^n f_1(k) + \dots + c_d \sum_{k=0}^n f_d(k).$$

Telescoping: Restrict to $d = 1$.

Zeilberger's creative telescoping: Take a bivariate sequence $f(m, k)$ and set $f_i(k) := f(m+i-1, k)$ in (1).

In the summation package Sigma [Sch07b] parameterized telescoping can be solved in Karr's $\Pi\Sigma^*$ -fields: the $f_i(k)$ can be indefinite nested sums and products.

^a All rings and fields contain \mathbb{Q} .

$\Pi\Sigma^*$ -extensions and sequences

Example. Let $\mathbb{F} := \mathbb{Q}(m)(k)(h)(b)$ be a rational function field and define the field automorphism σ by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}(m), \\ \sigma(k) &= k+1, \\ \sigma(h) &= h + \frac{1}{k+1}, & H_{k+1} &= H_k + \frac{1}{k+1}, \\ \sigma(b) &= \frac{m-k}{k+1}b, & \binom{m}{k+1} &= \frac{m-k}{k+1} \binom{m}{k}. \end{aligned}$$

(\mathbb{F}, σ) is a difference field, more precisely, a $\Pi\Sigma^*$ -field.

Difference rings and fields: A *difference field* (\mathbb{F}, σ) is a ring (resp. field) \mathbb{F} together with a ring (resp. field) automorphism $\sigma: \mathbb{F} \rightarrow \mathbb{F}$; the *constant ring* (resp. *constant field*) is given by $\text{const}_{\mathbb{F}} := \{f \in \mathbb{F} \mid \sigma(f) = f\}$.

$\Pi\Sigma^*$ -field: A difference field $(\mathbb{F}(t), \sigma)$ is a Σ^* -extension (resp. Π -extension) of a difference field $(\mathbb{F}, \sigma) \Leftrightarrow$

- 1 t is **transcendental** over \mathbb{F} ,
- 2 $\sigma'(f) = \sigma(f)$ for all $f \in \mathbb{F}$,
- 3 $\sigma'(t) = t + f$ (resp. $\sigma'(t) = ft$) for some $f \in \mathbb{F}^*$,
- 4 the constant field remains unchanged, i.e., $\text{const}_{\mathbb{F}}(\mathbb{F}(t)) = \text{const}_{\mathbb{F}}$.

A $\Pi\Sigma^*$ -extension is either a Π - or a Σ^* -extension. $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ is a $\Pi\Sigma^*$ -extension (resp. Σ^* -extension or Π -extension) if it is a tower of such extensions. $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ is a $\Pi\Sigma^*$ -field over \mathbb{F} , if $\mathbb{F} = \text{const}_{\mathbb{F}}$.

Example. Each of the extensions k, h, b forms a $\Pi\Sigma^*$ -extension over the field below. In particular, $\text{const}_{\mathbb{Q}}(\mathbb{Q}(m)(k)(h)(b)) = \mathbb{Q}(m)$.

Ring of sequences: The set of sequences over a field \mathbb{K} is denoted by

$$S(\mathbb{K}) := \{(a_n)_{n \geq 0} \mid a_i \in \mathbb{K}\};$$

we identify two sequences if they agree from a certain point on. The difference ring $(S(\mathbb{K}), S)$ with the shift operation (ring automorphism)

$$S: (a_0, a_1, a_2, \dots) \mapsto (a_1, a_2, a_3, \dots)$$

is called the *ring of sequences*.

Goal: Embed, e.g., $\mathbb{Q}(m)(k)(h, b)$ into $(S(\mathbb{Q}(m)), S)$.

Embedding example

We construct step by step an embedding $(\mathbb{Q}(m)(k)(h)[b], \sigma)$ into $(S(\mathbb{Q}(m)), S)$.

• Start with $\tau_0: \mathbb{Q}(m) \rightarrow S(\mathbb{Q}(m))$ where

$$\tau_0(c) = \langle c, c, c, \dots \rangle \quad \forall c \in \mathbb{Q}(m).$$

• Next, define the ring homomorphism $\tau_1: \mathbb{Q}(m)(k) \rightarrow S(\mathbb{Q}(m))$ with $\tau_1(\frac{p}{q}) = (F(k))_{k \geq 0}$ where

$$F(k) = \begin{cases} 0 & q(k) = 0 \\ \frac{p(k)}{q(k)} & q(k) \neq 0. \end{cases}$$

Note that

$$\tau_1(\sigma(f)) = S(\tau_1(f)), \quad \forall f \in \mathbb{Q}(m)(k).$$

τ_1 is **injective**: Since $p(k), q(k)$ have only finitely many roots, $\tau_1(\frac{p}{q}) = 0$ if and only if $\frac{p(k)}{q(k)} = 0$. Hence τ_1 is injective.

• Define the ring homomorphism $\tau_2: \mathbb{Q}(m)(k)(h) \rightarrow S(\mathbb{Q}(m))$ with $\tau_2(h) = (H_n)_{n \geq 0}$ and

$$\tau_2(\sum_{i=0}^d f_i h^i) = \sum_{i=0}^d \tau_1(f_i) \tau_2(h)^i.$$

τ_2 is **injective**: If not, take $f = \sum_{i=0}^d f_i h^i \in \mathbb{Q}(m)(k)(h)^*$ with $\deg(f) = d$ minimal such that $\tau_2(f) = 0$. Note that $f \notin \mathbb{Q}(m)(k)$ (otherwise, $0 = \tau_2(f) = \tau_1(f)$; since τ_1 is injective, $f = 0$). Define

$$g := \sigma(f_d) f - f_d \sigma(f) \in \mathbb{Q}(m)(k)(h).$$

Note: $\deg(g) < d$ by construction. Moreover,

$$\tau_2(g) = \tau_1(\sigma(f_d)) \tau_2(f) - \tau_1(f_d) \tau_2(\sigma(f)) = 0.$$

By the minimality of $\deg(f)$, $g = 0$, i.e.,

$$\sigma(f_d) f - f_d \sigma(f) = 0.$$

Equivalently,

$$\frac{\sigma(f)}{f} = \frac{\sigma(f_d)}{f_d} \in \mathbb{Q}(m)(k).$$

With $f \notin \mathbb{Q}(m)(k)$ this contradicts to [Kar81].

• To this end, take the ring homomorphism $\tau_3: \mathbb{Q}(m)(k)(h)[b] \rightarrow S(\mathbb{Q}(m))$ with $\tau_3(b) = (\binom{m}{n})_{n \geq 0}$ and

$$\tau_3(\sum_{i=0}^d f_i b^i) = \sum_{i=0}^d \tau_2(f_i) \tau_3(b)^i.$$

By similar arguments, τ_3 is injective.

Result 1: The embedding into the ring of sequences

A *generalized d'Alembertian extension* $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ of (\mathbb{F}, σ) is a $\Pi\Sigma^*$ -extension such that for all $1 \leq i \leq e$,

$$\sigma(t_i) - t_i \in \mathbb{F}[t_1, \dots, t_{i-1}] \quad \text{or} \quad \sigma(t_i)/t_i \in \mathbb{F};$$

note that the t_i are transcendental and $\mathbb{K} := \text{const}_{\mathbb{F}}(\mathbb{F}(t_1) \dots (t_e)) = \text{const}_{\mathbb{F}}$.

Embedding: Suppose that (\mathbb{F}, σ) describes the rational case ($\mathbb{F} = \mathbb{K}(k)$ with $\sigma(k) = k+1$), the q -rational case or the mixed case.

Then there is an injective ring homomorphism

$$\tau: \mathbb{F}[t_1, \dots, t_e] \rightarrow S(\mathbb{K})$$

with $\tau(c) = \langle c, c, c, \dots \rangle \quad \forall c \in \mathbb{K}$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{F}[t_1, \dots, t_e] & \xrightarrow{\tau} & S(\mathbb{K}) \\ \downarrow \sigma & & \downarrow S \\ \mathbb{F}[t_1, \dots, t_e] & \xrightarrow{\tau} & S(\mathbb{K}) \end{array}$$

We call such an embedding a \mathbb{K} -*monomorphism*.

Consequence:

$$\mathbb{F}[t_1, \dots, t_e] \cong \tau(\mathbb{F})[\tau(t_1), \dots, \tau(t_e)].$$

In particular, the sequences $\tau(t_1), \dots, \tau(t_e)$ are algebraically independent over $\tau(\mathbb{F})$.

Result 2: Parameterized telescoping and $\Pi\Sigma^*$ -extension

Let (\mathbb{F}, σ) be a difference field with constants \mathbb{K} and $f_1, \dots, f_d \in \mathbb{F}^*$. Then:

There are no $0 \neq (c_1, \dots, c_d) \in \mathbb{K}^d$ and $g \in \mathbb{F}$ with

$$\sigma(g) - g = c_1 f_1 + \dots + c_d f_d. \quad (2)$$

\Leftrightarrow

There is a Σ^* -extension $(\mathbb{F}(t_1) \dots (t_d), \sigma)$ of (\mathbb{F}, σ) with $\sigma(t_i) = t_i + f_i$ for $1 \leq i \leq d$.

Remark. The case $d = 1$ (telescoping) has been worked out in [Kar81].

As a consequence, one can check with telescoping that, e.g., the difference field $(\mathbb{Q}(m)(k)(h)(b), \sigma)$ is a $\Pi\Sigma^*$ -field.

There are no $0 \neq (c_1, \dots, c_d) \in \mathbb{Z}^d$ and $g \in \mathbb{F}^*$ with

$$\frac{\sigma(g)}{g} = f_1^{c_1} \dots f_d^{c_d}. \quad (3)$$

\Leftrightarrow

There is a Π -extension $(\mathbb{F}(t_1) \dots (t_d), \sigma)$ of (\mathbb{F}, σ) with $\sigma(t_i) = f_i t_i$ for $1 \leq i \leq d$.

Result 3: A criterion for algebraic independence of sums and products

Combining the ideas from Result 1 and Result 2 gives the following main result:

Let $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ be a generalized d'Alembertian-extension of (\mathbb{F}, σ) with $\mathbb{K} := \text{const}_{\mathbb{F}}$.

Let $\tau: \mathbb{F}[t_1, \dots, t_e] \rightarrow S(\mathbb{K})$ be a \mathbb{K} -monomorphism.

Let $f_1, \dots, f_d \in \mathbb{F}[t_1, \dots, t_e]^*$ with $(F_i(k))_{k \geq 0} := \tau(f_i)$. Then:

There are no $g \in \mathbb{F}[t_1, \dots, t_e], 0 \neq (c_1, \dots, c_d) \in \mathbb{K}^d$ with (2).

The sequences $\{(S_1(n))_{n \geq 0}, \dots, (S_d(n))_{n \geq 0}\}$ given by^a

$$S_1(n) := \sum_{k=r}^n F_1(k), \dots, S_d(n) := \sum_{k=r}^n F_d(k) \quad (4)$$

are algebraically independent over $\tau(\mathbb{F}[t_1, \dots, t_e])$.

^a The lower bound r is chosen big enough.

Let $f_1, \dots, f_d \in \mathbb{F}^*$ with $(F_i(k))_{k \geq 0} := \tau(f_i)$. Then:

There are no $g \in \mathbb{F}^*$ and $0 \neq (c_1, \dots, c_d) \in \mathbb{Z}^d$ with (3).

The sequences $\{(S_1(n))_{n \geq 0}, \dots, (S_d(n))_{n \geq 0}\}$ given by

$$S_1(n) := \prod_{k=r}^n F_1(k), \dots, S_d(n) := \prod_{k=r}^n F_d(k) \quad (5)$$

are algebraically independent over $\tau(\mathbb{F})$.

The $(q-)$ rational case

Theorem. Let $f_1(k), \dots, f_d(k) \in \mathbb{K}(k)$. If there are no $g(k) \in \mathbb{K}(k)$ and $c_1, \dots, c_d \in \mathbb{K}$ with (1), then the sequences (4) are algebraically independent over $\mathbb{K}(n)$; i.e., $\exists P(x_1, \dots, x_d) \in \mathbb{K}(n)[x_1, \dots, x_d]^*$ with

$$P(S_1(n), \dots, S_d(n)) = 0 \quad \forall n \geq 0.$$

Corollary. Let $p_1(k), p_2(k), \dots \in \mathbb{K}[k]^*$, $u_1(k), u_2(k), \dots \in \mathbb{K}[k]^*$ and $v \in \mathbb{K}[k]$ with $\deg(v) > 0$ where

$$\gcd(p_i, v) = \gcd(u_i, v) = 1 \quad \forall i \geq 1,$$

$$\gcd(v(k), v(k+r)) = 1 \quad \forall r \in \mathbb{N}^*,$$

$$v(r) \neq 0 \quad \forall r \in \mathbb{N}^*.$$

Then the sums

$$S_1(n) := \sum_{k=1}^n u_1(k) \left(\frac{p_1(k)}{v(k)} \right), S_2(n) := \sum_{k=1}^n u_2(k) \left(\frac{p_2(k)}{v(k)} \right)^2, \dots$$

are algebraically independent over $\mathbb{K}(n)$.

Proof. Denote $f_i(k) := u_i \binom{m}{k}^i$. Suppose there are $g(k) \in \mathbb{K}(k)$ and $c_i \in \mathbb{K}$ with (1) where $d \geq 1$ is minimal. Then

$$\begin{aligned} g(k+1) - g(k) &= c_1 u_1 p_1 v^{d-1} + c_2 u_2 p_2^2 v^{d-2} + \dots + c_d u_d p_d^d \\ &=: \frac{w}{v^d}. \end{aligned}$$

Since $c_d \neq 0$, $\gcd(v, c_d u_d p_d^d) = 1$. Hence $\gcd(w, v) = 1$, and thus $\gcd(w, v^d) = 1$. By [Abr71, Pau95] such a $g(k) \in \mathbb{K}(k)$ cannot exist; a contradiction. \square

Example. Choosing $p_i = u_i = 1, v = k$ in the Corollary proves that the generalized harmonic numbers $\{H_n^{(i)} \mid i \geq 1\}$ are algebraically independent over $\mathbb{K}(n)$.

Example. Similarly, the q -harmonic numbers $\{\sum_{i=1}^n \frac{1}{(1-q^i)^i} \mid i \geq 1\}$ (or $\{\sum_{i=1}^n (\frac{q^i}{1-q^i})^i \mid i \geq 1\}$) are algebraically independent over $\mathbb{K}(q^n)$.

The $(q-)$ hypergeometric case

Let $f(k)$ be hypergeometric, i.e., for all r big enough,

$$\alpha(r) = \frac{f(r+1)}{f(r)}$$

for some $\alpha(k) \in \mathbb{K}(k)$.

We restrict to the case that there are no $v(k) \in \mathbb{K}(k)$ and no root of unity $\gamma \in \mathbb{K}$ with $f(k) = \gamma^k v(k)$. Then there is a $\Pi\Sigma^*$ -field $(\mathbb{K}(k)(t), \sigma)$ over \mathbb{K} with

$$\sigma(k) = k+1 \quad \text{and} \quad \sigma(t) = \alpha t,$$

and a \mathbb{K} -monomorphism $\tau: \mathbb{K}(k)(t) \rightarrow S(\mathbb{K})$.

Theorem. For $1 \leq i \leq d$, let $v_i \in \mathbb{K}(k)$ and $f_i := v_i t$.

If there are no $c_i \in \mathbb{K}$ and $w \in \mathbb{K}(k)$ with $g := wt$ such that (2), then the sequences

$$f(n), S_1(n) = \sum_{k=r}^n v_1(k) f(k), \dots, S_d(n) = \sum_{k=r}^n v_d(k) f(k)$$

(r big enough) are algebraically independent over $\mathbb{K}(n)$; i.e., $\exists P(x_0, x_1, \dots, x_d) \in \mathbb{K}(n)[x_0, x_1, \dots, x_d]^*$ with

$$P(f(n), S_1(n), \dots, S_d(n)) = 0 \quad \forall n \geq 0.$$

Corollary Let $f(m, k)$ be hypergeometric in m, k where $f \neq \gamma^k v$ for all $v \in \mathbb{K}(m, k)$ and all roots of unity $\gamma \in \mathbb{K} \setminus \{1\}$. If **Z's algorithm** [Zei91] fails to solve (1) with $f_i := f(m+i-1, k)$, then $S_0(n) = f(m, n)$ and

$$S_1(n) = \sum_{k=r}^n f(m, k), \dots, S_d(n) = \sum_{k=r}^n f(m+d-1, k)$$

(r big enough) are transcendental over $\mathbb{K}(m)(n)$.

Example. For the Apéry-sum

$$A(m) = \sum_{k=0}^m \binom{m}{k}^2 \binom{m+k}{k}$$

Zeilberger's algorithm finds a recurrence of order 2, but not a smaller one. Hence, the sequences

$$\begin{aligned} S_0(n) &= \binom{m}{n}^2 \binom{m+n}{n}, \\ S_1(n) &= \sum_{k=0}^n \binom{m}{k}^2 \binom{m+k}{k}, \\ S_2(n) &= \sum_{k=0}^n \binom{m+1}{k}^2 \binom{m+k+1}{k} \end{aligned}$$

are algebraically independent over $\mathbb{Q}(m)(n)$.

Example. In [Abr03] a criterion is given when Zeilberger's algorithm fails to find a creative telescoping solution for a hypergeometric input summand $f(m, k)$. If $f(m, k)$ satisfies this criterion, then all the sequences

$$\{f(m, n)\} \cup \left\{ \sum_{k=r}^n f(m+i, k) \mid i \geq 0 \right\}$$

in n (r big enough) are algebraically independent over $\mathbb{K}(m)$. A typical example [Abr03, Exp. 2] is

$$f(m, k) = \frac{1}{mk+1} (-1)^k \binom{m+1}{k} \binom{2m-2k-1}{m-1}.$$

Remark. Analogously, all ideas can be carried over to the q -hypergeometric case.

The nested sum case

Consider the following sum from [PS03]:

$$S(m, n) := \sum_{k=0}^n f(m, k) = \sum_{k=0}^n (1 + 5H_k(m-2k)) \binom{m}{k}^5.$$

Then the package Sigma shows that the sequences

$$\binom{m}{n}, H_n, S(m, n), S(m+1, n), S(m+2, n), S(m+3, n)$$

in n are algebraically independent over $\mathbb{K}(m)(n)$.

Proof. Sigma constructs the $\Pi\Sigma^*$ -field $(\mathbb{Q}(m)(k)(h)(b), \sigma)$ and designs the $\mathbb{Q}(m)$ -monomorphism from above.

Then it sets

$$\begin{aligned} f_1 &= b^5 (1 + 5h(m-2k)), \\ f_2 &= \frac{b^5 (m+1)^5 (5h(-2k+m+1) + 1)}{(-k+m+1)^5}, \\ f_3 &= \frac{b^5 (m+1)^5 (m+2)^5 (5h(-2k+m+2) + 1)}{(-k+m+1)^5 (-k+m+2)^5}, \end{aligned}$$

$$f_4 = \frac{b^5 (m+1)^5 (m+2)^5 (m+3)^5 (5h(-2k+m+3) + 1)}{(-k+m+1)^5 (-k+m+2)^5 (-k+m+3)^5};$$

this is motivated by $\binom{m+1}{k} = \frac{m+1}{m+1-k} \binom{m}{k}$ which shows that

$$\tau(f_i) = \langle f(m+i-1, k) \rangle_{k \geq 0}.$$

Finally, Sigma proves algorithmically that there are no $g \in \mathbb{Q}(m)(k)(h)(b)$ and $c_i \in \mathbb{Q}(m)$ with (2). Hence the transcendence of the sequences follows by Result 3. \square

Remark. Note that the situation $m = n$ for the sum $S(m, n)$ is completely different. We get

$$S(m, m) = \sum_{k=0}^m \binom{m}{k}^2 \binom{m+k}{k}.$$

In particular, $S(m, m)$ satisfies a recurrence relation of minimal order 2.

Some references

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