Non-commutative Computer Algebra and its Applications with the Computer Algebra System SINGULAR:PLURAL

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Introduction

What do we call Non-commutative Computer Algebra?

Potentially computer–assisted symbolic algebraic manipulation with constructive (presented by finite data) objects

- elements of algebras (polynomials etc.)
- ideals and submodules of free modules
- factor modules
- subalgebras
- ring homomorphisms
- module homomorphisms

Key instrument: Gröbner bases

G. Bergman, Teo Mora, Edward Green, Victor Ufnarovski, . . .
Applications of Gröbner bases

Gröbner Basics are, according to Buchberger, Sturmfels et.al.

...the most important and fundamental applications of Gröbner Bases.

- Ideal (resp. module) membership problem
- Intersection with subrings (elimination of variables)
- Intersection of ideals (resp. submodules)
- Quotient and saturation of (two–sided) ideals
- Kernel of a module homomorphism
- Kernel of a ring homomorphism
- Algebraic relations between pairwise commuting polynomials
- Hilbert polynomial of graded ideals and modules

This list is universal for both commutative and non–commutative situations.
Origins Of Non-commutativity

Let $C$ be some algebra of functions ($C^\infty$ etc).
For any function $f \in C$, we introduce an operator

$$F : C \to C, \ F(t) = f \cdot t.$$  

We call $f$ a representative of $F$. $\forall f, g \in C$ we have $F \circ G = G \circ F$.

**Definition**

A map $\partial : C \to C$ is called a **differential** if $\partial$ is $C$–linear and $\forall f, g \in C,$

$$\partial(fg) = \partial(f)g + f\partial(g).$$

In particular, $\partial_i = \frac{\partial}{\partial t_i}$ on $C$ are differentials.

**News**

Bad news: operators $F$ and $\partial_i$ do not commute.
Good news: $\partial_j \circ \partial_i = \partial_i \circ \partial_j$ and there is a relation between $F$ and $\partial_i$. 
Non–commutative Relations

Lemma

For any differential $\partial$ and $f \in C$, \[\partial \circ F = F \circ \partial + \partial(f).\]

Proof.

$\forall h \in C$, we have the following:

\[(\partial \circ F)(h) = \partial(f \cdot h) = f \cdot \partial(h) + \partial(f) \cdot h = (F \circ \partial)(h) + \partial(f) \cdot (h) = (F \circ \partial + \partial(f))(h).\]

Example

Let $C = \mathbb{K}[t_1, \ldots, t_n]$ and $\partial_i = \frac{\partial}{\partial t_i}$. Then there is a $n$–th Weyl algebra $\mathbb{K}\langle t_1, \ldots, t_n, \partial_1, \ldots, \partial_n \mid \{ t_j t_i = t_i t_j, \partial_j \partial_i = \partial_i \partial_j, \\ \partial_k t_k = t_k \partial_k + 1 \} \cup \{ \partial_j t_i = t_i \partial_j \}_{i \neq j} \rangle$, an algebra of linear differentional operators with polynomial coefficients.
More Non–commutative Relations

**Shift Algebra**

For small $\triangle t \in \mathbb{R}$, we define a shift operator

$$\sigma_t : C \rightarrow C, \quad \sigma_t(f(t)) = f(t + \triangle t).$$

Then, since $\sigma_t(f \cdot g) = \sigma_t(f) \cdot \sigma_t(g)$, we define a real shift algebra

$$\mathbb{K}(\triangle x) \langle x, \sigma_x \mid \sigma_x x = x\sigma_x + \triangle x\sigma_x \rangle.$$

**The Center of an Algebra**

For a $\mathbb{K}$–algebra $A$, we define the center of $A$ to be

$$Z(A) = \{ a \in A \mid a \cdot b = b \cdot a \ \forall b \in A \}.$$  

It is a subalgebra of $A$, containing constants of $\mathbb{K}$. 

\(q\)-Calculus and Non–commutative Relations

Let \(k\) be a field of char 0 and \(\mathbb{K} = k(q)\).

**\(q\)-dilation operator**

\[ D_q : C \rightarrow C, \quad D_q(f(x)) = f(qx): \]

\[ \mathbb{K}(q)\langle x, D_q \mid D_q \cdot x = q \cdot x \cdot D_q \rangle. \]

**Continuous \(q\)-difference Operator**

\[ \Delta_q : C \rightarrow C, \quad \Delta_q(f(x)) = f(qx) - f(x): \]

\[ \mathbb{K}(q)\langle x, \Delta_q \mid \Delta_q \cdot x = q \cdot x \cdot \Delta_q + (q - 1) \cdot x \rangle. \]

**\(q\)-differential Operator**

\[ \partial_q : C \rightarrow C, \quad \partial_q(f(x)) = \frac{f(qx) - f(x)}{(q - 1)x}: \]

\[ \mathbb{K}(q)\langle x, \partial_q \mid \partial_q \cdot x = q \cdot x \cdot \partial_q + 1 \rangle. \]
Major Directions. Algebras and Systems

**Free and path algebras, their factor-algebras**

- **OPAL, GRB**, the group of Edward Green (Blacksburg)
- **NCGB/NCALGEBRA, MATHEMATICA** package
- **GBNP/GROBNER**, GAP package

**Functionality**

Full or reduced Gröbner Bases w.r.t. some orderings, membership problems, basic ideal arithmetics.

Wanted: projective resolutions and cohomology.
Major Directions. Algebras and Systems

Algebras, close to commutative.
Noetherian integral domains with PBW bases

- **MACAULAY2**, Weyl algebras, exterior algebras
- **KAN/SM1**, rings of differential and \((q-)\)difference operators
- **MAS2**, \(GR\)-algebras and beyond
- **FELIX**, \(G\)-algebras
- **SINGULAR:PLURAL**, \(G\)- and \(GR\)-algebras.
What is PLURAL?

PLURAL is the kernel extension of SINGULAR, providing a wide range of symbolic algorithms with non–commutative polynomial algebras (GR–algebras).

- Gröbner bases, Gröbner basics, non–commutative Gröbner basics
- more advanced algorithms for non–commutative algebras,

- PLURAL is distributed with SINGULAR (from version 3-0-0 on)
- freely distributable under GNU Public License
- available for most hardware and software platforms
Preliminaries

Let $\mathbb{K}$ be a field and $R$ be a commutative ring $R = \mathbb{K}[x_1, \ldots, x_n]$. 

$$\text{Mon}(R) \ni x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mapsto (\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha \in \mathbb{N}^n.$$

Definition

1. A total ordering $\prec$ on $\mathbb{N}^n$ is called a **well–ordering**, if 
   - $\forall F \subseteq \mathbb{N}^n$ there exists a minimal element of $F$, 
     in particular $\forall \ a \in \mathbb{N}^n$, $0 \prec a$.
2. An ordering $\prec$ is called a **monomial ordering on** $R$, if 
   - $\forall \alpha, \beta \in \mathbb{N}^n \ \alpha \prec \beta \Rightarrow x^\alpha \prec x^\beta$ 
   - $\forall \alpha, \beta, \gamma \in \mathbb{N}^n$ such that $x^\alpha \prec x^\beta$ we have $x^{\alpha+\gamma} \prec x^{\beta+\gamma}$.
3. Any $f \in R \setminus \{0\}$ can be written uniquely as $f = cx^\alpha + f'$, with $c \in \mathbb{K}^*$ and $x^{\alpha'} \prec x^\alpha$ for any non–zero term $c'x^{\alpha'}$ of $f'$. We define 
   - $\text{lm}(f) = x^\alpha$, the **leading monomial** of $f$
   - $\text{lc}(f) = c$, the **leading coefficient** of $f$.
Towards $GR$–algebras

Suppose we are given the following data

1. a field $\mathbb{K}$ and a commutative ring $R = \mathbb{K}[x_1, \ldots, x_n]$,
2. a set $C = \{c_{ij}\} \subset \mathbb{K}^*$, $1 \leq i < j \leq n$
3. a set $D = \{d_{ij}\} \subset R$, $1 \leq i < j \leq n$

Assume, that there exists a monomial well–ordering $\prec$ on $R$ such that

$$\forall 1 \leq i < j \leq n, \ \text{lm}(d_{ij}) \prec x_i x_j.$$

The Construction

To the data $(R, C, D, \prec)$ we associate an algebra

$$A = \mathbb{K}\langle x_1, \ldots, x_n | \{x_j x_i = c_{ij} x_i x_j + d_{ij} \} \forall 1 \leq i < j \leq n \rangle.$$
PBW Bases and $G$–algebras

Define the $(i, j, k)$–nondegeneracy condition to be the polynomial

$$NDC_{ijk} := c_{ik}c_{jk} \cdot d_{ij}x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} \cdot d_{ik}x_j + d_{jk}x_i - c_{ij}c_{ik} \cdot x_i d_{jk}.$$ 

**Theorem**

$A = A(R, C, D, \prec)$ has a PBW basis $\{x_1^{a_1}x_2^{a_2} \ldots x_n^{a_n}\}$ if and only if

$$\forall 1 \leq i < j < k \leq n, \ NDC_{ijk} \text{ reduces to } 0 \ w.r.t. \ relations$$

**Definition**

An algebra $A = A(R, C, D, \prec)$, where nondegeneracy conditions vanish, is called a $G$–algebra (in $n$ variables).
Definition

Let $A$ be an associative $K$–algebra and $M$ be a left $A$–module.

1. The **grade** of $M$ is defined to be $j(M) = \min\{i \mid \text{Ext}_A^i(M, A) \neq 0\}$, or $j(M) = \infty$, if no such $i$ exists or $M = \{0\}$.

2. $A$ satisfies the **Auslander condition**, if for every fin. gen. $A$–module $M$, for all $i \geq 0$ and for all submodules $N \subseteq \text{Ext}_A^i(M, A)$ the inequality $j(N) \geq i$ holds.

3. $A$ is called an **Auslander regular** algebra, if it is Noetherian with $\text{gl. dim}(A) < \infty$ and the Auslander condition holds.

4. $A$ is called a **Cohen–Macaulay** algebra, if for every fin. gen. nonzero $A$–module $M$, $j(M) + \text{GKdim}(M) = \text{GKdim}(A) < \infty$. 
We collect the properties in the following Theorem.

**Theorem (Properties of $G$–algebras)**

Let $A$ be a $G$–algebra in $n$ variables. Then

- $A$ is left and right Noetherian,
- $A$ is an integral domain,
- the Gel’fand–Kirillov dimension $\text{GKdim}(A) = n + \text{GKdim}(\mathbb{K})$,
- the global homological dimension $\text{gl. dim}(A) \leq n$,
- the Krull dimension $\text{Kr.dim}(A) \leq n$,
- $A$ is Auslander-regular and a Cohen-Macaulay algebra.

We say that a $GR$–algebra $\mathcal{A} = A/ T_A$ is a factor of a $G$–algebra in $n$ variables $A$ by a proper two–sided ideal $T_A$. 
Examples of $GR$–algebras

Mora, Apel, Kandri–Rody and Weispfenning, . . .

- algebras of solvable type, skew polynomial rings
- univ. enveloping algebras of fin. dim. Lie algebras
- quasi–commutative algebras, rings of quantum polynomials
- positive (resp. negative) parts of quantized enveloping algebras
- some iterated Ore extensions, some nonstandard quantum deformations
- many quantum groups
- Weyl, Clifford, exterior algebras
- Witten’s deformation of $U(\mathfrak{sl}_2)$, Smith algebras
- algebras, associated to $(q–)$differential, $(q–)$shift, $(q–)$difference and other linear operators
Left, right and twosided structures

It suffices to have implemented
- left Gröbner bases
- functionality for opposite algebras $\mathcal{A}^{op}$
- functionality for enveloping algebras $\mathcal{A}^{env} = \mathcal{A} \otimes_K \mathcal{A}^{op}$
- mapping $\mathcal{A} \rightarrow \mathcal{A}^{op} \rightarrow \mathcal{A}$

Then
1. for a finite set $F \subset \mathcal{A}$, $\text{RGB}_{\mathcal{A}}(F) = \left( \text{LGB}_{\mathcal{A}^{op}}(F^{op}) \right)^{op}$
2. the two–sided Gröbner can be computed, for instance, with the algorithm by Manuel and Maria Garcia Roman in $\mathcal{A}^{env}$. 
Gröbner Trinity

With essentially the same algorithm, we can compute

1. GB left Gröbner basis $G$ of a module $M$
2. SYZ left Gröbner basis of the 1st syzygy module of $M$
3. LIFT the transformation matrix between two bases $G$ and $M$

The algorithm for Gröbner Trinity must be able to compute ...

- with submodules of free modules
  - accept monomial module orderings as input
  - distinguish preferred module components
- within factor algebras
- with extra weights for the ordering / module generators
- and to use the information on Hilbert polynomial
**Implementation of Gröbner Basics**

### Universal Gröbner Basics
- Ideal (resp. module) membership problem (NF, REDUCE)
- Intersection with subrings (elimination of variables) (ELIMINATE)
- Intersection of ideals (resp. submodules) (INTERSECT)
- Quotient and saturation of ideals (QUOT)
- Kernel of a module homomorphism (MODULO)
- Kernel of a ring homomorphism (NCPREIMAGE.LIB)
- Algebraic relations between pairwise commuting polynomials
- Hilbert polynomial of graded ideals and modules
Anomalies With Elimination

Admissible Subalgebras

Let \( A = \mathbb{K}\langle x_1, \ldots, x_n \mid \{ x_j x_i = c_{ij} x_i x_j + d_{ij} \}_{1 \leq i < j \leq n} \rangle \) be a \( G \)-algebra. Consider a subalgebra \( A_r \), generated by \( \{ x_{r+1}, \ldots, x_n \} \). We say that such \( A_r \) is an admissible subalgebra, if \( d_{ij} \) are polynomials in \( x_{r+1}, \ldots, x_n \) for \( r + 1 \leq i < j \leq n \) and \( A_r \subsetneq A \) is a \( G \)-algebra.

Definition (Elimination ordering)

Let \( A \) and \( A_r \) be as before and \( B := \mathbb{K}\langle x_1, \ldots, x_r \mid \ldots \rangle \subset A \). An ordering \( \prec \) on \( A \) is an elimination ordering for \( x_1, \ldots, x_r \) if for any \( f \in A \), \( \text{lm}(f) \in B \) implies \( f \in B \).
Constructive Elimination Lemma

"Elimination of variables $x_1, \ldots, x_r$ from an ideal $I$" means the intersection $I \cap A_r$ with an admissible subalgebra $A_r$. In contrast to the commutative case:

- not every subset of variables determines an admissible subalgebra
- there can be no admissible elimination ordering $\prec_{A_r}$ on $A$

Lemma

Let $A$ be a $G$–algebra, generated by $\{x_1, \ldots, x_n\}$ and $I \subset A$ be an ideal. Suppose, that the following conditions are satisfied:

- $\{x_{r+1}, \ldots, x_n\}$ generate an admissible subalgebra $B$,
- $\exists$ an admissible elimination ordering $\prec_B$ for $x_1, \ldots, x_r$ on $A$.

Then, if $S$ is a left Gröbner basis of $I$ with respect to $\prec_B$, we have $S \cap B$ is a left Gröbner basis of $I \cap B$. 
Consider the algebra $A = \mathbb{K}\langle a, b \mid ba = ab + b^2 \rangle$. It is a $G$–algebra with respect to any well–ordering, such that $b^2 \prec ab$, that is $b \prec a$. Any elimination ordering for $b$ must satisfy $b \succ a$, hence $A$ is not a $G$–algebra w.r.t. any elimination ordering for $b$.

The Gröbner basis of a two–sided ideal, generated by $b^2 - ba + ab$ in $\mathbb{K}\langle a, b \rangle$ w.r.t. an ordering $b \succ a$ is infinite and equals to

$$\{ ba^{n-1}b - \frac{1}{n}(ba^n - a^nb) \mid n \geq 1 \}.$$ 

Finding an admissible elimination ordering can be done by solving a linear programming problem (ongoing work with J. Lobillo, Granada).
Non-commutative Gröbner basics

For the noncommutative PBW world, we need even more basics:

- Gel’fand–Kirillov dimension of a module (GKDIM.LIB)
- Two–sided Gröbner basis of a bimodule (e.g. twostd)
- Annihilator of finite dimensional module
- Preimage of one–sided ideal under algebra morphism
- Finite dimensional representations
- Graded Betti numbers (for graded modules over graded algebras)
- Left and right kernel of the presentation of a module
- Central Character Decomposition of a module (NCDECOMP.LIB)

Very Important

- Ext and Tor modules for centralizing bimodules (NCHOMOLOG.LIB)
- Hochschild cohomology for modules
Non-commutative Gröbner basics in \textsc{Plural} \\

Unrelated to Gröbner Bases, but Essential Functions 

Center of an algebra and centralizers of polynomials 
Operations with opposite and enveloping algebras \\

\textsc{Plural} as a Gröbner engine 

- implementation of all the universal Gröbner basics available 
- \texttt{slimgb} is available for \texttt{Plural} 
- \texttt{janet} is available for two–sided input 
- non–commutative Gröbner basics: 
  - as kernel functions (\texttt{twostd, opposite etc}) 
  - as libraries (\texttt{NCDECOMP.LIB}, \texttt{NCTOOLS.LIB}, \texttt{NCPREIMAGE.LIB} etc)
Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and $f \in R$. We are interested in

\[ R[f^{-s}] = \mathbb{K}[x_1, \ldots, x_n, \frac{1}{fs}] \]

as an $R$–module for $s \in \mathbb{N}$.

On the one hand, $R[f^{-s}] \cong R[y]/\langle yf^s - 1 \rangle$.

On the other hand, $R[f^{-s}]$ is a $D$–module, where $D$ is the $n$–th Weyl algebra

\[ \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n | \{ \partial_j x_i = x_i \partial_j + \delta_{ij} \} \rangle. \]

The algorithm ANNFS computes a $D$–module structure on $R[f^{-s}]$, that is a left ideal $I \subset D$, such that $R[f^{-s}] \cong D/I$.

Especially interesting are cases when $f$ is irreducible singular (among other, a reiffen curve), reducibly singular or when $f$ is a hyperplane arrangement (arrange).
Challenge: Ann $F_s$ for different $F$
Let char $\mathbb{K} = 0$ and $F \in \mathbb{K}[x_1, \ldots, x_n]$.

Problem Formulation
Compute the ideal $\text{Ann } F_s \in \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n | \partial_i x_j = x_j \partial_i + \delta_{ij} \rangle$ ($n$–th Weyl algebra).
Both algorithms available use two complicated eliminations.

- Oaku–Takayama approach (ANNFSOT command)
- Briançon–Maisonobe approach (ANNFSBM command)
- Bernstein polynomial is computed within both approaches
- many convenient auxiliary tools

- polynomial singularities
- very hard: Reiffen curves $x^p + y^q + xy^{q-1}$, $q \geq p + 1 \geq 5$
- generic and non–generic hyperplane arrangements
- further examples by F. Castro and J.-M. Ucha

Systems: \texttt{KAN/SM1, RISASIR, MACAULAY2, SINGULAR:PLURAL}.
Let $A$ be a $K$–algebra and $M$ be a left $A$–module.

**Autonomy Degree**

The **autonomy degree** of $M$, $ad(M)$ is the first natural $i \geq 0$, such that $\forall 0 \leq j < i$, $\text{Ext}^j_A(M, A) = 0$ and $\text{Ext}^i_A(N, A) \neq 0$.

If $A$ is a Cohen–Macaulay algebra, 

$$ad(M) = j(M) = d(A) - d(M),$$

hence we need only to compute $d(M)$ to determine the $ad(M)$. 
Cohen–Macaulay and Controllability Degree

For an $A$–module $M$ (a system module), one defines $N$ to be the $\text{Hom}_A(M, A)$ (an adjoint or a dual module).

**Controllability Degree**

The **controllability degree** of $M$ is the first natural $i > 0$, that $\forall 0 < j < i$, $\text{Ext}^j_A(N, A) = 0$ and $\text{Ext}^i_A(N, A) \neq 0$.

Suppose that $\text{Ext}^0_A(N, A) = 0$, that is $\text{Hom}_A(\text{Hom}_A(M, A), A) = 0$. Then, if $A$ is a Cohen–Macaulay algebra, we have

$$cd(M) = j(N) = d(A) - d(N)$$

In general, we test the condition $\text{Hom}_A(\text{Hom}_A(M, A), A) = 0$ (it is satisfied by many system modules) and compute $d(N)$. 
Application of Preimage Algorithm to $D$–modules

The algorithm of Oaku and Takayama (1999)

The 1st step of the OT algorithm requires to compute the preimage of the left ideal

$$L = \langle \{ t_j - f_j, \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j + \partial_i \} \rangle, \quad 1 \leq j \leq p, \quad 1 \leq i \leq n$$

in the subalgebra $\mathbb{K}\langle \{ t_j \cdot \partial t_j \} \rangle \langle \{ x_i, \partial x_i \mid [\partial x_i, x_i] = 1 \} \rangle$ of

$$\mathbb{K}\langle \{ t_j, \partial t_j \} \mid [\partial t_j, t_j] = 1 \rangle \otimes_{\mathbb{K}} \mathbb{K}\langle \{ x_i, \partial x_i \mid [\partial x_i, x_i] = 1 \} \rangle$$

Moreover, in the preimage, $t_j \cdot \partial t_j$ will be replaced by $-s_j - 1$. 
Symmetric Deformation: Theorem

Theorem

Let \( A, B \) be GR–algebras and \( \Phi \in \text{Mor}(A, B) \).
Let \( I_\Phi \) be the \((A, A)\)–bimodule \( A\langle \{ x_i - \Phi(x_i) \mid 1 \leq i \leq n \} \rangle \subset A \otimes_K B \)
and \( f_i := \Phi(x_i) \). Suppose there exists an elimination ordering for \( B \) on \( A \otimes_K B \), such that

\[
1 \leq i \leq n, 1 \leq j \leq m, \quad \text{lm}(lc(f_iy_j)f_i - lc(y_jf_i)f_iy_j) \prec x_iy_j.
\]

Then

1) \( A \otimes^\Phi_K B \) is a G–algebra (resp. \( A \otimes^{\Phi}_K B \) is a GR–algebra).
2) Let \( \mathcal{J} \subset B \) be a left ideal, then

\[
\Phi^{-1}(\mathcal{J}) = (I_\Phi + \mathcal{J}) \cap A.
\]
Application of Preimage Algorithm to $D$–modules

Setup with the Symmetric Deformation

\[ A := \mathbb{K}\langle s_j, X_i, D_i \mid D_i X_i = X_i D_i + 1 \rangle \]

\[ B := \mathbb{K}\langle t_j, D t_j, x_i, d_i \mid d_i x_i = x_i d_i + 1, D t_j t_j = t_j D t_j + 1 \rangle \]

Consider the map $\phi : A \to B$, where $s_j \mapsto -t_j D t_j - 1$, $X_i \mapsto x_i$, $D_i \mapsto d_i$.

Hence, $I_\phi = \langle \{ X_i - x_i, D_i - d_i, t_j D t_j + s_j + 1 \} \rangle \subset A \otimes_\mathbb{K} B =: E$.

Due to the structure, we replace $E$ with $E' = \mathbb{K}\langle t_j, D t_j, x_i, d_i, s_j \rangle$ subject to the relations

\[ \{[d_i, x_i] = 1, [D t_j, t_j] = 1, s_j t_j = t_j s_j - t_j, s_j D t_j = D t_j s_j + D t_j \}. \]

Respectively, $I_\phi \subset E$ becomes $I'_\phi = \langle \{ t_j D t_j + s_j + 1 \} \rangle \subset E'$.

Any ordering $\prec$ satisfying $\{ t_j, D t_j \} \succ \{ x_i, d_i, s_j \}$ (which is very easy to find) satisfies the conditions of the Theorem.
Dancing Flamenco

By the Theorem, for any \( L \subset B \), \( \phi^{-1}(L) = \left( l_\phi + L \right) \cap A \). Hence,

\[
l_\phi + L = \langle \{ t_j - f_j, \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_j} \partial t_j + \partial_i, t_j \partial t_j + s_j \} \rangle =
\]

\[
= \langle \{ t_j - f_j, \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_j} \partial t_j + \partial_i, f_j \partial t_j + s_j \} \rangle
\]

Citing Gago–Vargas, Hartillo and Ucha JSC paper from 2005...

"...As far as we know, the example \( f = (x^2 + y^3) \cdot (x^3 + y^2) \) is intractable for available computer algebra systems."

→ Demonstration.
Applications

- **Systems and Control Theory (VL, E. Zerz et. al.)**
  - CONTROL.LIB, NCONTROL.LIB, RATCONTROL.LIB
  - algebraic analysis tools for System and Control Theory
  - In progress: non–commutative polynomial algebras (NCONTROL.LIB)

- **Algebraic Geometry (W. Decker, C. Lossen and G. Pfister)**
  - SHEAFCOH.LIB
  - computation of the cohomology of coherent sheaves
  - In progress: direct image sheaves (F. - O. Schreyer)

- **$D$–Module Theory (VL and J. Morales)**
  - DMOD.LIB
  - Ann $F^s$ algorithms: OT (Oaku and Takayama), BM (Briançon and Maisonobe)
Applications In Progress

- Homological algebra in $GR$–algebras (with G. Pfister)
  - NCHOMOLOG.LIB
  - Ext and Tor modules for centralizing bimodules
  - Hochschild cohomology for modules

- Clifford Algebras (VL, V. Kisil et. al.)
  - CLIFFORD.LIB
  - basic algorithms and techniques of the theory of Clifford algebras

- Annihilator of a left module (VL)
  - NCANN.LIB
  - the original algorithm of VL for $\text{Ann}(M)$ for $M$ with $\dim_K M = \infty$
  - the algorithm terminates for holonomic modules, i.e. for a module $M$, such that $\text{GKdim}(M) = 2 \cdot \text{GKdim}(\text{Ann}(M))$
  - high complexity, a lot of tricks and improvements needed
Perspectives

Gröbner bases for more non–commutative algebras

- tensor product of commutative local algebras with certain non–commutative algebras (e.g. with exterior algebras for the computation of direct image sheaves)
- different localizations of $G$–algebras
  - localization at some ”coordinate” ideal of commutative variables (producing e.g. local Weyl algebras $\mathbb{K}[x]_\langle x \rangle_\langle D \mid Dx = xD + 1 \rangle$)
  - local orderings and the generalization of standard basis algorithm, Gröbner basics and homological algebra
  - localization as field of fractions of commutative variables (producing e.g. rational Weyl algebras $\mathbb{K}(x)_\langle D \mid Dx = xD + 1 \rangle$), including Ore Algebras (F. Chyzak, B. Salvy)
  - global orderings and a generalization Gröbner basis algorithm. Gröbner basics require distinct theoretical treatment!
Software from RISC Linz

Algorithmic Combinatorics Group, Prof. Peter Paule

- most of the software are packages for Mathematica

The Software is freely available for non-commercial use

www.risc.uni-linz.ac.at/research/combinat/software/
Symbolic Summation

Hypergeometric Summation
- FASTZEIL, Gosper’s and Zeilberger’s algorithms
- ZEILBERGER, Gosper and Zeilberger alg’s for MAXIMA
- MULTISUM, proving hypergeometric multi-sum identities

$q$–Hypergeometric Summation
- QZEIL, $q$–analogues of Gosper and Zeilberger alg’s
- BBIBASIC TELESCOPE, generalized Gosper’s algorithm to bibasic hypergeometric summation
- QMULTISUM, proving $q$–hypergeometric multi-sum identities

Symbolic Summation in Difference Fields
- SIGMA, discovering and proving multi-sum identities
More Software from RISC Linz

**Sequences and Power Series**
- **ENGEL**, $q$–Engel Expansion
- **GENERATINGFUNCTIONS**, manipulations with univariate holonomic functions and sequences
- **RLANGGFUN**, inverse Schützenberger methodology in **MAPLE**

**Partition Analysis, Permutation Groups**
- **OMEGA**, Partition Analysis
- **PERMGROUP**, permutation groups, group actions, Polya theory

**Difference/Differential Equations**
- **DIFFTOOLS**, solving linear difference eq’s with poly coeffs
- **ORESYS**, uncoupling systems of linear Ore operator equations
- **RATDIFF**, rat. solutions of lin. difference eq’s after van Hoeij
- **SUMCRACKER**, identities and inequalities, including summations
Thank you for your attention!

Please visit the SINGULAR homepage

http://www.singular.uni-kl.de/
Criteria for detecting useless critical pairs

**Generalized Product Criterion**

Let $A$ be a $G$–algebra of Lie type (that is, all $c_{ij} = 1$). Let $f, g \in A$. Suppose that $\text{lm}(f)$ and $\text{lm}(g)$ have no common factors, then $\text{spoly}(f, g) \rightarrow \{f, g\} [g, f]$, where $[g, f] := gf - fg$ is the Lie bracket.

**Chain Criterion**

If $(f_i, f_j)$, $(f_i, f_k)$ and $(f_j, f_k)$ are in the set of pairs $P$ and $x^{\alpha_j} | \text{lcm}(x^{\alpha_i}, x^{\alpha_k})$, then we can delete $(f_i, f_k)$ from $P$.

The Chain Criterion can be proved with the Schreyer’s construction of the first syzygy module of a given module, which generalizes to the case of $G$–algebras.