Invariant Form of BK-factorization and its Applications[1]

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Abstract

We present invariant formulation of the Beals-Kartashova factorization procedure (BK-factorization) which allows to factorize simultaneously classes operators equivalent to the initial one under gauge transformations. We compare two factorization methods - Hensel descent and Beals-Kartashova factorization procedure (BK-factorization) aiming to emphasize the constructiveness of BK-factorization. We also show the possibility to use the same procedure for the construction of the approximate factorization of LPDE in the case when corresponding LPDO is not exactly factorizable and point out the problems to solve while factoring approximately noncommutative polynomials.

Keywords: linear partial differential operator, Hensel descent, BK-factorization, invariant transformations, approximate factorization

1   Introduction

Factorization of ordinary and partial linear differential operators (LODOs and LPDOs) is a very well-studied problem and a lot of pure existence theorems are known. For LODOs it is proven that a factorization is unique up to factor permutation while for LPDOs even uniqueness is not true any more and in fact parametric families of factorizations can be constructed for a given LPDO as will be demonstrated below.

First **constructive method** of factoring second order hyperbolic LPDO in the form

$$\partial_x \partial_y + a\partial_x + b\partial_y + c, \tag{1}$$

belong to Laplace who formulated it in terms of invariants $\hat{a} = c - ab - a_x$ and $\hat{b} = c - ab - b_y$ now called Laplace invariants. An operator (1) is factorizable if at least one of its Laplace invariants is equal to zero. Various algorithms are known now for factoring LPDOs over different differential fields beginning with the simplest field of rational functions [1].

Recently two papers ([2], [3]) on factoring arbitrary order LPDOs have been published. In [2] a modification of well-known Hensel lifting algorithm (see, for instance, [4]) is presented and **sufficient conditions** for the existence of intersection of principal ideals are given. These results are applied then to re-formulate the factorization formulae for second and third order operators from the ring $D = \mathbb{Q}(x, y)[\partial_x, \partial_y]$ obtained by Miller (1932) in terms of principal intersections.

In [3] **necessary and sufficient conditions** are given for factoring of bivariate LPDOs of arbitrary order with coefficients being arbitrary smooth functions. In [5] it was shown that this procedure called now BK-factorization principally can not be generalized on the case of more than two variables. In was also shown that conditions of factorization found in [3] are invariants under gauge transformation and classical Laplace invariants are particular case of this generalized invariants.

In this paper we re-formulate BK-factorization in more suitable for applications invariant form and illustrate it with a few examples, give a sample of the symbolical implementation of this method in MATHEMATICA and also discuss some possibilities to use this method for approximate factorization of LPDOs.

2   Hensel descent and BK-factorization

Hensel descent, the latest known to us new method published before BK-factorization, has been considered as constructive. There also exist an

opinion that BK-factorization is a minor generalization of the Hensel descent. To clarify the matter, we begin this paper with comparative analysis of these two methods in order to show merits and draw-backs of each of them.

- In [3] factorization of a bivariate LPDO is looked for in the form

$$A_n = \sum_{j+k \le n} a_{jk} \partial_x^j \partial_y^k = (p_1 \partial_x + p_2 \partial_y + p_3) \Big(\sum_{j+k < n} p_{jk} \partial_x^k \partial_y^j \Big) \quad (2)$$

  and in [2] - operator of $m \ge 2$ independent variables is regarded and for $m = 2$ factorization is looked for in the form

$$A_n = \sum_{j+k \le n} a_{jk} \partial_x^j \partial_y^k = \Big(\sum_{j+k \le l} p_{jk} \partial_x^k \partial_y^j \Big) \Big(\sum_{j+k \le r} p_{jk} \partial_x^k \partial_y^j \Big) \text{ with } l+r = n.$$

$$(3)$$

- In [3] coefficients a_{jk} are arbitrary smooth functions, for instance trigonometric functions; in [2] conditions for reducibility of an operator are studied when "coefficients are from a universal field of zero characteristic" , while "studying factorization algorithms we will assume that the input operators are from the ring $\mathbb{Q}(x_1, ..., x_m)[\partial_1, ..., \partial_m]$" This suggestion is necessary:"From now on the coefficients of a given second-order operator are assumed to be from the base field $\mathbb{Q}(x, y)$. This is necessary if the goal is to obtain constructive answers allowing to factorize large classes of operator" ([2], Sec.3);"In this section we study third-order LPDOs from the ring $D = \mathbb{Q}(x, y)[\partial_x, \partial_y]$." ([2], Sec.4).

- In [3] it was shown that in generic case factorization can be constructed explicitly and algebraically, while in [2] (Sec.5) it is concluded that "the factorization problem for second- and third-order differential operators in two variables has been shown to require the solution of a partial Riccati equation, which in turn requires to solve a general first-order ODE and possibly ordinary Riccati equation.

The bottleneck for designing a factorization algorithm for a LPDO is general first-order ODE which make the full problem intractable at present because in general there are no solution algorithm available"

- In [5] it is pointed out that BK-factorization procedure has to be modified in some way (presently unknown to authors) in order to proceed with operators of more than 2 independent variables, while in [2] (Sec.5) it is written that "some of the results described in this article may be generalized to any number of independent variables"

- In [5] it shown that BK-factorization procedure gives rise to construction of the whole class of generalized invariants particular case of them being classical Laplace invariants. This leads to a possibility to factorize simultaneously the whole class of operators equivalent under gauge transformation (see next Section) while Hensel descent is used for factoring of a one specific operator.

We summarize all this in the Table below.

Property / Method	BK-factorization	Hensel descent
Order of operator	n	n
Coefficients of operator	arbitrary smooth functions	rational functions
Number of variables	2	possibly > 2
Conditions	necessary and sufficient	sufficient
Form of factors	of order 1 and $(n-1)$	of order k and $(n-k)$
Formulated as	explicit formulae	ideals intersection

In the next Sections we demonstrate some other interesting properties of BK-factorization - first of all, that it has invariant form and can be used therefore to factorize simultaneously the whole classes of equivalent LPDOs. Second, the use of this invariant form of BK-factorization for construction of approximate factorization for LPDEs to be solved numerically.

4

3 Invariant Formulation

We present here briefly main ideas presented in [3], [5] beginning with the definition of equivalent operators.

Definition. *The operators A, $\tilde{A}$ are called equivalent if there is a gauge transformation that takes one to the other:*

$$\tilde{A}g = e^{-\varphi}A(e^{\varphi}g).$$

BK-factorization is then pure algebraic procedure which allows to to construct explicitly a factorization of an arbitrary order LPDO A in the form

$$A := \sum_{j+k\leq n} a_{jk}\partial_x^j\partial_y^k = L \circ \sum_{j+k\leq(n-1)} p_{jk}\partial_x^j\partial_y^k$$

with first-order operator $L = \partial_x - \omega\partial_y + p$ where ω is **an arbitrary simple root** of the characteristic polynomial

$$\mathcal{P}(t) = \sum_{k=0}^{n} a_{n-k,k}t^{n-k}, \quad \mathcal{P}(\omega) = 0. \tag{4}$$

Factorization is possible then for each simple root $\tilde{\omega}$ of (4) **iff**
for $n=2 \quad \Rightarrow \quad l_2 = 0$,
for $n=3 \quad \Rightarrow \quad l_3 = 0, \ \& \ l_{31} = 0$,
for $n=4 \quad \Rightarrow \quad l_4 = 0, \ \& \ l_{41} = 0, \ \& \ l_{42} = 0$,
and so on. All functions $l_2, \ l_3, \ l_{31}, \ l_4 \ l_{41}, \ l_{42}, ...$ are explicit functions of a_{ij} and $\tilde{\omega}$.

Theorem. *All $l_2, l_3, l_{31}, ...$ are* **invariants** *under gauge transformations.*

Definition. *Invariants $l_2, l_3, l_{31}, ...$ are called generalized invariants of a bivariate operator of arbitrary order.*

In particular case of the operator (1) its generalized invariants coincide with Laplace invariants.

Corollary. *If an operator A is factorizable, then all operators equivalent to it, are also factorizable.*

As the first step of BK-factorization, coefficients p_{ij} are computed as solutions of some system of algebraic equations. At the second step, equality to zero of all generalized invariants $l_{ij} = 0$ has to be checked so that **no differential equations are to be solved** in generic case. Generic case corresponds to a simple root of characteristic polynomial, and each simple root generates corresponding factorization. Moreover, putting some restrictions on the coefficients of the initial LPDO $a_{i,j}$ as functions of x and y, one can describe *all* factorizable operators in a given class of functions (see Example 5.3 in [3]). The same keeps true for all operators equivalent to a given one. Equivalent operators are easy to compute:

$$e^{-\varphi}\partial_x e^{\varphi} = \partial_x + \varphi_x, \quad e^{-\varphi}\partial_y e^{\varphi} = \partial_y + \varphi_y,$$

$$e^{-\varphi}\partial_x\partial_y e^{\varphi} = e^{-\varphi}\partial_x e^{\varphi}e^{-\varphi}\partial_y e^{\varphi} = (\partial_x + \varphi_x) \circ (\partial_y + \varphi_y)$$

and so on. Some examples:

- $A_1 = \partial_x\partial_y + x\partial_x + 1 = \partial_x(\partial_y + x), \quad l_2(A_1) = 1 - 1 - 0 = 0;$

- $A_2 = \partial_x\partial_y + x\partial_x + \partial_y + x + 1, \quad A_2 = e^{-x}A_1 e^x; \quad l_2(A_2) = (x+1) - 1 - x = 0;$

- $A_3 = \partial_x\partial_y + 2x\partial_x + (y+1)\partial_y + 2(xy + x + 1), \quad A_3 = e^{-xy}A_2 e^{xy};$ $l_2(A_3) = 2(x + 1 + xy) - 2 - 2x(y+1) = 0;$

- $A_4 = \partial_x\partial_y + x\partial_x + (\cos x + 1)\partial_y + x\cos x + x + 1, \quad A_4 = e^{-\sin x}A_2 e^{\sin x};$ $l_2(A_4) = 0.$

Generic case which can be treated pure algebraically by BK-factorization corresponds to **a simple root of characteristic polynomial**. Each multiple root leads to necessity of solving some Ricatti equation(s) (RE). If appeared RE happens to be solvable, such a root generates a parametric family of factorizations for a given operator. For instance, well-known Landau operator

$$\partial_{xxx}^3 + x\partial_{xxy}^3 + 2\partial_{xx} + (2x+2)\partial_{xy}^2 + \partial_x + (2+x)\partial_y$$

has characteristic polynomial with one distinct root $\omega_1 = -x$ and one double root $\omega_{2,3} = 0$. Factorization then has form

$$(\partial_x + r)(\partial_x - r + 2)(\partial_x + x\partial_y)$$

where r is a solution of Ricatti equation

$$1 - 2r + \partial_x(r) + r^2 = 0$$

which is easily solvable:

$$r = 1 + \frac{1}{x + Y(y)}$$

with arbitrary smooth function $Y(y)$ of one variable y so that factorization has form

$$A = (\partial_x + 1 + \frac{1}{x + Y(y)})(\partial_x + 1 - \frac{1}{x + Y(y)})(\partial_x + x\partial_y).$$

Notice that to factorize **an ordinary differential operator** it is always necessary to solve some RE. Nevertheless, just formal application of BK-factorization will produce all the linear factors in the case when corresponding RE are solvable. For instance, the factorization has been constructed in [6]

$$x\partial_{xxx} + (x^2 - 1)\partial_{xx} - x\partial_x + \frac{2}{x^2} - 1 = (\partial_x + \frac{x^2 - 1}{x})(x\partial_x - \sqrt{2})(\partial_x + \frac{\sqrt{2} - 1}{x}).$$

while both RE appearing at the intermediate steps are solvable.

These two last examples show the main difference between factorizing of ordinary and partial differential operators - LODO has always unique factorization while LPDO may have many. An interesting question here would be to compute the exact number of all possible factorizations of a given LPDO into all linear factors (its upper bound is, of course, trivial: $n!$). A really challenging task in this context would be to describe some additional conditions on the coefficients of an initial operator which lead to solvable RE.

4   Left and Right Factors

Factorization of an operator is the first step on the way of solving corresponding equation. But for solution we need **right** factors and BK-factorization constructs **left** factors which are easy to construct. On the other hand, the existence of a certain right factor of a LPDO is equivalent to the existence of a corresponding left factor of the *transpose* of that operator. Moreover taking transposes is trivial algebraically, so there is also nothing lost from the point of view of algorithmic computation. In our paper [3] we just used the transpose in one example assuming that the operation is well known. Apparently, it is not, and some new works appeared quite recently on the construction of a complicated new right-factor algorithms (i.e. [7] and others) for bivariate operators of order 2 and 3 though our left-factor algorithm gives the explicit formulae for arbitrary order n. This is the reason why we decided to include the explicit formula for the adjoint in our present text.

Definition. *The transpose A^t of an operator $A = \sum a_\alpha \partial^\alpha$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. is defined as*

$$A^t u = \sum (-1)^{|\alpha|} \partial^\alpha (a_\alpha u).$$

and the identity

$$\partial^\gamma (uv) = \sum \binom{\gamma}{\alpha} \partial^\alpha u \, \partial^{\gamma - \alpha} v$$

implies that

$$A^t = \sum (-1)^{|\alpha + \beta|} \binom{\alpha + \beta}{\alpha} (\partial^\beta a_{\alpha + \beta}) \partial^\alpha.$$

Now the coefficients are

$$A^t = \sum \widetilde{a}_\alpha \partial^\alpha,$$

$$\widetilde{a}_\alpha = \sum (-1)^{|\alpha + \beta|} \binom{\alpha + \beta}{\alpha} \partial^\beta (a_{\alpha + \beta}).$$

with a standard convention for binomial coefficients in several variables, e.g. in two variables

$$\binom{\alpha}{\beta} = \binom{(\alpha_1, \alpha_2)}{(\beta_1, \beta_2)} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}.$$

In particular, for order 2 in two variables the coefficients are

$$\widetilde{a}_{jk} = a_{jk}, \quad j+k = 2; \widetilde{a}_{10} = -a_{10}+2\partial_x a_{20}+\partial_y a_{11}, \widetilde{a}_{01} = -a_{01}+\partial_x a_{11}+2\partial_y a_{02},$$

$$\widetilde{a}_{00} = a_{00} - \partial_x a_{10} - \partial_y a_{01} + \partial_x^2 a_{20} + \partial_x \partial_x a_{11} + \partial_y^2 a_{02}.$$

For instance, the operator

$$\partial_{xx} - \partial_{yy} + y\partial_x + x\partial_y + \frac{1}{4}(y^2 - x^2) - 1 \tag{5}$$

is factorizable as

$$\left[\partial_x + \partial_y + \tfrac{1}{2}(y - x)\right]\left[...\right]$$

and its transpose A_1^t is factorizable then as

$$\left[...\right]\left[\partial_x - \partial_y + \tfrac{1}{2}(y + x)\right].$$

Implementation of the BK-factorization for bivariate operators of order $n \leq 4$ is therefore quite straightforward and has been done in MATHE-MATICA while all roots of characteristic polynomial are known in radicals. For instance, for the operator (5) with 2 simple roots we get one factorization

$$\left[\partial_x - \partial_y + \tfrac{1}{2}(y + x)\right]\left[\partial_x + \partial_y + \tfrac{1}{2}(y - x)\right].$$

corresponding to the first root while in the case of the second root, generalized invariant is equal to 2.

If $n \geq 5$ the problem is generally not solvable in radicals and very simple example of non-solvable case is: $x^5 - 4x - 2 = 0$. Thus, to find solutions in radical for $n > 4$ one needs some constructive procedure of finding solvable Galois group but this lies beyond the scope of the present paper.

9

5   Approximate Factorization

An interesting possible application of the invariant form of BK-factorization is to use it for construction of approximate factorization of a given LPDE, in the case when exact factorization of corresponding LPDO does not exists. Indeed, as a results of BK-factorization one gets

(1) **factorization coefficients** $\{p_{ij}^{(i)}\}$ for the i-th factorization of a given operator, and

(2) **generalized invariants** $l_2^{(i)} p_{ij}^{(i)}$, with all $p_{ij}^{(i)}, l_{(k_j)}^{(i)}$ being explicit functions of the coefficients of initial operator a_{ij}.

In numerical simulations coefficients a_{ij} of the equation are always given with some non-zero accuracy, say $\varepsilon > 0$, which means that it is enough to construct an approximate factorization in the following sense. One has to find restrictions on the coefficients a_{ij} of an initial LPDO which provide $|l_{k_j}^{(j)}| < \varepsilon$ with a given accuracy $0 < \varepsilon << 1$. Many different strategies are possible here, we just give a brief sketch of two approaches we are working on right now:

5.1   Quantifier Elimination

We illustrate this idea on the simple example of a hyperbolic operator $\partial_{xx} - \partial_{yy} + a_{10}\partial_x + a_{01}\partial_y + a_{00}$ with linear polynomial coefficients.

What we have is:

$$a_{00}(x, y) = b_3 x + b_2 y + b_1,$$

$$a_{10}(x, y) = c_3 x + c_2 y + c_1,$$

$$a_{01}(x, y) = d_3 x + d_2 y + d_1;$$

a function constructed from general invariants

$$\mathcal{R} = \frac{s_3 - s_2}{2} + \frac{(s_3 x + s_2 y + s_1)^2}{4}$$

with $s_i = c_i - d_i$.

What we need is:
To find some function(s) $F = F(a_{ij})$ such that if $F(a_{i,j}) = 0$, then

$$-\varepsilon < a_{00} - \mathcal{R} < \varepsilon, \quad \text{for some constant} \ \ 0 < \varepsilon << 1,$$

i.e. to find some conditions on the initial polynomials which provide that function $\mathcal{R}$ differs not too much from one these polynomials, namely a_{00}.

Notice that simple symmetry considerations allowed us to reduce number of variables needed for CAD calculations. Initially we had 9 variables $b_3, b_2, b_1, c_3, c_2, c_1, d_3, d_2, d_1$ but in fact it is enough to regard only 6 variables $s_1, s_2, s_3, b_1, b_2, b_3$. Nevertheless, the computation time may become crucial while using this approach due to the substantial number of variables. On the other hand, this approach allows us to work generally on the operator level including initial and/or boundary conditions first at some later stage.

5.2   Auxiliary Operator

Another approach is to construct a new auxiliary operator with coefficients $\widetilde{a}_{ij} = f(x, y))a_{ij}$ for all or for some of the coefficients a_{ij} of the initial operator, keep invariants (almost) equal to zero and find function(s) $f(x, y)$ minimizing the differences between the coefficients of initial and new operators. In this way an auxiliary operator is constructed which can be regarded as an approximate operator for the initial operator. Of course, it does not mean that solutions of the initial and approximate operators will be also close but simple properties of linear operators show that it is necessary (but not sufficient!) step on the way of construction of a good approximate solution of a given LPDE - in the case of a well-posed problem, of course. In particularly, it means that one has to introduce proper metrics in the space of operators **and** in the space of solutions. Choice of the both metrics and of a function f will depend on (1) coefficients of the initial operator; (2) class of functions in which we are looking for a solution; (3) initial and/or boundary conditions.

To demonstrate all this let us regard two different un-factorizable modifications of the operator (5):

$$A = \partial_{xx} - \partial_{yy} + y\partial_x + x\partial_y + \frac{1}{2}(y^2 - x^2) - 1 \tag{6}$$

with $l_2(A) = \frac{1}{4}(y^2 - x^2)$ and

$$B = \partial_{xx} - \partial_{yy} + \sin y \ \partial_x + \cos x \ \partial_y + \frac{1}{2}(\sin^2 y - \cos^2 x) \tag{7}$$

with $l_2(B) = \frac{1}{2}(\cos y - \sin x)$ (see Fig.1). One can see immediately that $l_2(B)$ is a bounded function of two variables and $l_2(A)$ is an unbounded. This means that quite different choice of function f is needed for these two cases in order to minimize the invariants. Influence of initial/boundary conditions is now also very clear - for instance, best approximation of $l_2(B)$ can be obtained in the narrow belts of the lines parallel to one of the coordinate axis while for $l_2(A)$ these directions are in no way special.
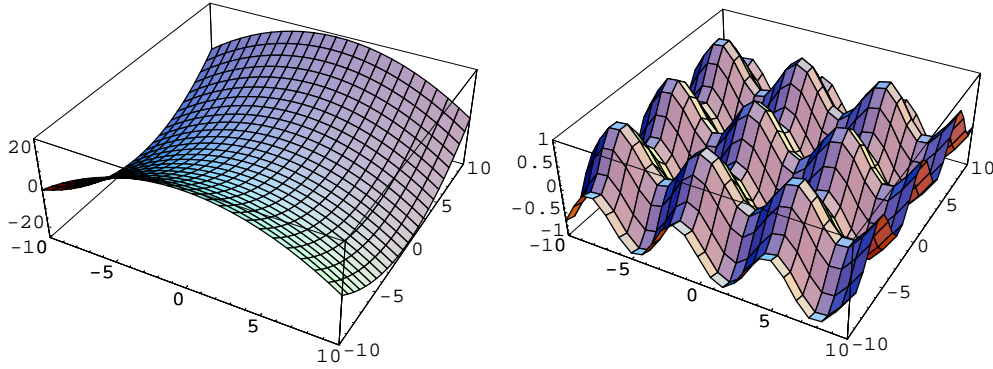

Figure 1: Invariant $l_2(A) = \frac{1}{4}(y^2 - x^2)$ (left) and invariant $l_2(B) = \frac{1}{2}(\cos y - \sin x)$ (right), in the domain $-10 \le x, y \le 10$
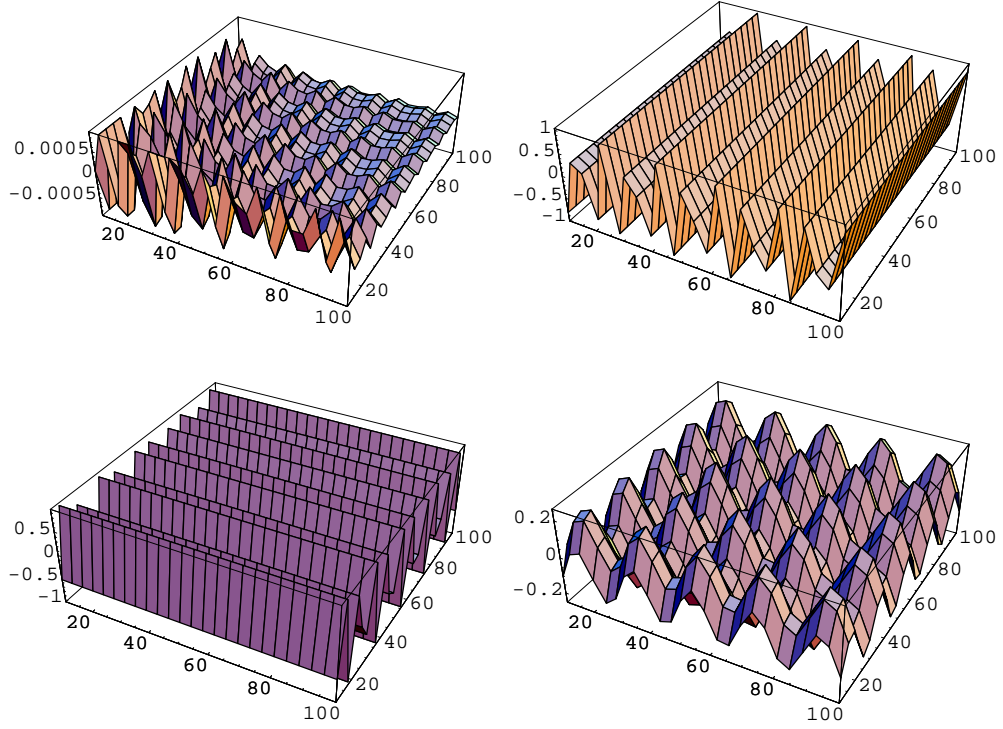
Figure 2: Upper panel: $l_2(\tilde{B})$ (left) and $a_{10} - \tilde{a}_{10}$ (right); lower panel: $a_{01} - \tilde{a}_{01}$ (left) and $a_{00} - \tilde{a}_{00}$ (right); in the domain $10 \leq x, y \leq 100$

To construct a sample of such an approximate factorization for the operator (7) we just suppose intuitively that auxiliary operator $\tilde{B}$ is "good" if its coefficients differ from the coefficients of (7) not much, and its invariant is small. Our MATHEMATICA implementation of the BK-factorization includes simple graphic functions to display the differences between all the parameters of the initial and auxiliary operators. A choice of the function $f(x, y) = \sin \frac{1}{xy}$ gives an auxiliary operator $\tilde{B}$ of the form

$$\tilde{B} = \partial_{xx} - \partial_{yy} + \sin y \sin \frac{1}{xy} \, \partial_x + \cos x \sin \frac{1}{xy} \, \partial_y \qquad (8)$$

$$+ \frac{1}{2}(\sin^2 y - \cos^2 x) \sin \frac{1}{xy}.$$

It is demonstrated at the Fig.2 that for $10 \leq x, y \leq 100$ operator $\tilde{B}$ gives good enough approximation and correspondingly approximate factorization of the initial operator B has form

$$B \sim \left[\frac{1}{2} \left(-\cos x \sin \frac{1}{xy} + \sin \frac{1}{xy} \sin y \right) + \partial_x + \partial_y \right]$$
$$\cdot \left[\frac{1}{2} \left(\cos x \sin \frac{1}{xy} + \sin \frac{1}{xy} \sin y \right) + \partial_x - \partial_y \right]$$

with $|l_2(\tilde{B})| \sim 5 \cdot 10^{-4}$. On the other hand, in the domain $0.001 \leq x, y \leq 1$ qualitatively different approximation is needed while in this domain $|l_2(\tilde{B})| \sim 10^2$ (see Fig. 3).
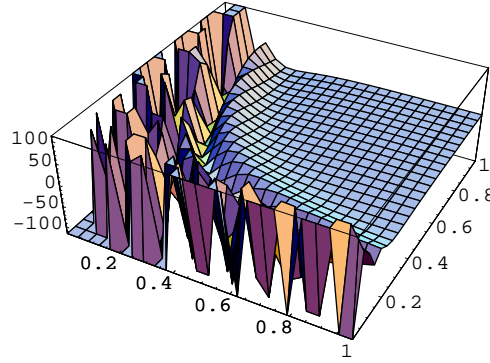

Figure 3: Invariant $l_2(\tilde{B})$ in the domain $0.0001 \leq x, y \leq 1$

Obviously, if we get enough approximate factorizations of the given LPDE with different solvable first-order factors we can write out explicitly general solution of the initial LPDE. Otherwise, one gets a chain of the linear first-order equations

$$A_{i0,n}\psi_0 = 0, \ A_{i0,n-1}\psi_1 = \psi_0, \ ....$$

to be solved numerically which is a great numerical simplification, of course, specially for higher order LPDEs. On the other hand, while performing

numerical simulations, one has to take into account a lot of other factors, first of all, initial and boundary conditions. It would be a nontrivial task to include them into the exact formulae given by BK-factorization. In order to estimate usefulness of this approach from numerical point of view we still have to answer all the questions concerning computation time, stability, computation error, etc. For instance, coming back to the example of approximate factorization given in the previous section, one have to estimate what is numerically more reasonable for a given set on initial and boundary conditions - to solve numerically the system of equations

$$\begin{cases} [\frac{1}{2}(\cos x \sin \frac{1}{xy} + \sin \frac{1}{xy} \sin y) + \partial_x - \partial_y] \circ \psi_0 = 0 \\ [\frac{1}{2}(-\cos x \sin \frac{1}{xy} + \sin \frac{1}{xy} \sin y) + \partial_x + \partial_y] \circ \psi_1 = \psi_0 \end{cases}$$

or one equation $B \circ \psi = 0$. Some answers can be given by the method presented in [9] where a symbolic approach is used to generate automatically finite difference schemes for LPDEs and to check their von Neumann stability. Some preliminary steps to be taken in this direction might be following: (1) to take a non-factorizable but solvable operator, for instance, $A_1 = \partial_x \partial_y + x \partial_x + 2$, then LPDE $A_1(\psi) = 0$, has general solution

$$\psi = -\partial_x \Big(X(x)e^{-xy} + \int e^{x(y'-y)} Y(y') dy' \Big)$$

with two arbitrary functions $X(x)$ and $Y(y)$; (2) to construct its approximate factorization $\widetilde{A}_1 = L_1 \circ L_2$; (3) to get computational schemes using [9] - for A_1 and $\widetilde{A}_1$; (4) compute both numerically; (5) to compare results for A_1 and $\widetilde{A}_1$ with the general solution for some classes of initial data and for a fixed choice of computational scheme.

6   Summary

We presented here invariant formulation of BK-factorization and formulated some ideas about using it for approximate factorization of LPDOs, i.e. non-commutative polynomials. The great number of results is known

on factorization of commutative polynomials (see, for instance, [10] and others) where measure of the difference between factorizable and non-factorizable polynomials can be introduced as a function of the coefficients of a polynomial. It is not the case for non-commutative polynomials the reason being that infinitesimal changes in the coefficients can change drastically the solution of the corresponding LPDE, as well as changes of the initial and boundary conditions. The problem of defining a reasonable measure for non-commutative polynomials is under the study.

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