Diophantine equations of nonlinear physics. Part 1: Nonlinear evolution PDEs*

Elena Kartashova

RISC, J.Kepler University, Linz, Austria

e-mail: lena@risc.uni-linz.ac.at

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1 Introduction

Theoretical physics could be regarded as an intermediate party between experimental physics and pure mathematics and its subject is a creation of new methods of solving physical problems which are not yet solvable by the methods of pure mathematics. These unusual and often not strict methods led to astonishing results a good feeling of which one can get from [1] where the author constructs correspondence between n-wave PDE of nonlinear physics and n-orthogonal coordinate systems in differential geometry, represents theory of surfaces as a chapter of soliton theory and formulates some other surprising statements. The main idea of him is to show mathematicians how to use physical methods and ideas in order to handle nonlinear PDEs.

Our purpose here is a bit different - we would like to show that many problems in nonlinear physics could be reformulated and solved as algebraic or number-theoretical ones. This text which is supposed to consist of four parts is written for mathematicians specialized in algebra, number theory and symbolic computations. The whole way will be presented here - from physical problem setting till the method of finding some solutions of nonlinear PDEs using solutions of some algebraic systems till the presentation of a few most interesting from physical and mathematical point of view problems to solve. In order to make this text self-sufficient we begin our **Part 1** with formulation of basic definitions and results concerning partial differential equations (PDEs) which were twenty years ago the necessary part of any mathematical education but not anymore. "Courses in this subject have even disappeared from the obligatory program of many universities (for example, in Paris). Moreover such remarkable textbooks as the classical three-volume work of Goursat have been removed as superfluous from the library" [2]. Basic facts about PDEs are studied now mainly in some physical courses, for instance in courses of electrodynamics or meteorology.

On the other hand, our main subject here is to show how to construct some algebraic systems from a given PDE and how to find some solutions of PDE using solutions of these systems. The whole ideology of this procedure is based on the profound knowledge of a few general facts about PDEs and that is our reason of including *Section 2: Mathematical basics on PDEs*, in this text. General definitions of PDE's types (linearity, nonlinearity, order, etc) and classes of second order PDEs (elliptic, parabolic, hyperbolic) with different sorts of initial/boundary conditions are given as well as many illustrative examples.

In *Section 3: Dispersive waves*, another classification of PDEs - into dispersive and non-dispersive equations - is presented which is successfully used in physics. Mostly dispersive PDEs do not have hyperbolic type but not necessarily, an example of dispersive PDE is given which is hyperbolic. Besides that this classification deals with PDEs of an arbitrary order. Therefore it *can not* be reduced to the known mathematical one. The physical background of the concepts of a dispersive wave, dispersion relation, wave system and connected questions is briefly discussed in order to demonstrate why the concept of dispersive PDE turned out to be such a powerful tool for finding solutions of nonlinear PDEs though it is practically not known by pure mathematicians. The only mentioning of it by mathematician we have found in the book of V. I. Arnold [2] who writes about important physical principles and concepts such as energy, variational principle, the Lagragians, dispersion relations, the Hamiltonian formalism, etc. which gave a rise for the development of large areas in mathematics (theory of Fourier series and integrals, functional analysis, algebraic geometry and many others). But even V.I.Arnold could not find place for it in the consequent mathematical presentation of the theory of PDEs and the words "dispersion relation" appear only in the introduction.

In the last three *Sections* of **Part 1** concepts of wave resonance, resonantly interacting waves, resonance conditions, etc. are introduced as well as some simple variant of perturbation technique which are used for construction of a so-called kinetic equation corresponding to a given dispersive PDE. As an intermediate step some system of algebraic equations (the resonance conditions) and system of ODEs for amplitudes of resonantly interacting waves are constructed. Kinetic equation is to some extent equivalent to the initial PDE provided that some *intuitive additional conditions* about wave system are hold. Solutions of kinetic equation are then solutions of initial PDE. As conclusion, the question of kinetic equation validity is arisen which will be treated in the **Part 2**.

2 Mathematical basics on PDEs

There are many phenomena in nature, which, even though occurring over finite regions of space and time, can be described in terms of properties that prevail at each point of space and time separately. This description originated with Newton, who with the aid of differential calculus showed us how to grasp a global phenomenon, for example, elliptic orbit of a planet, by means of a locally applied law, for instance, $F = ma$ where F, m and a

mean force, mass and acceleration correspondingly. This manner of making nature comprehensible has been extended from the motion of single point particles to the behavior of other forms of matter and energy, be it in the form of gasses, fluids, light, heat, electricity, signals travelling along optical fibers and neurons, gravitation, etc. This extension consists of formulating or stating a partial differential equation governing the phenomenon, and then solving that differential equation for the purpose of predicting measurable properties of the phenomenon.

Simply put, a differential equation is an equation that contains derivatives. To solve an equation of this type is to find a function that satisfies the differential equation. The difference between solving an algebraic equation and solving a differential equation is that with the former you are looking typically for a number and with the later you are looking for a function.

Ordinary differential equations (ODEs) are equations of one variable and their theory is well developed: general theorems about the existence of a solution are proven and solutions in general form (Wronskian, Green function, etc.) are written out for many classes of them. In contrast to ordinary differential equations, there is no unified theory of partial differential equations (PDEs). Some equations are known to have exact solutions. For the others a lot of results are obtained of a type "some specific properties of a boundary-value problem for ..." Concerning many specific PDEs there is no theoretical results at all. The only mathematical classification of PDEs which does exist concerns PDEs of second order.

As our first step here let us introduce the main definitions and formulate important mathematical results concerning PDEs taking for simplicity PDE of a second order with two variables. The general form of such an equation could be written as

$$a\frac{\partial^2\psi}{\partial x^2} + b\frac{\partial^2\psi}{\partial x\partial t} + c\frac{\partial^2\psi}{\partial t^2} + d\frac{\partial\psi}{\partial x} + e\frac{\partial\psi}{\partial t} + f\psi = g$$

where x and t are two variables, ψ is an unknown function of x and t and a, b, c, d, e, f may depend on x, t and even ψ.

The **order** of the equation is given by the order of its highest derivative.

Thus, if one of the coefficients a, b or c is non-zero, then the PDE is of the **second order**.

If $a = b = c = 0$ but d and e are non-zero, then the PDE is of the **first order**.

If the coefficients a, b, c, d, e and f do not depend on ψ, then PDE is **linear** otherwise it is **nonlinear**.

If $g = 0$, then PDE is **homogeneous** otherwise it is **inhomogeneous**.

An equation which is linear with respect to the derivatives, is called **quasilinear**. It means that coefficients a, b, c, d, e and f might be functions of t, x, ψ but do not depend on ψ_x or ψ_t. These equations are called **linearized** in physical texts and often posses analytical solutions.

Some examples.

- first order linear PDE (translation equation):

$$\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial t} = 0$$

- first order nonlinear PDE (simple wave equation):

$$\psi \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial t} = 0$$

- first order linear inhomogeneous PDE:

$$e^x \frac{\partial \psi}{\partial x} + 4 \frac{\partial \psi}{\partial t} = t$$

- second order linear PDE:

$$\frac{\partial^2 \psi}{\partial x^2} + 4 \frac{\partial \psi}{\partial t} = 0$$

- second order linear inhomogeneous PDE:

$$\frac{1}{x} \frac{\partial \psi}{\partial x} - \frac{\partial^2 \psi}{\partial t^2} = x^2$$

- second order nonlinear PDE:

$$\frac{\partial^2 \psi}{\partial x^2} - 2e^x \frac{\partial^2 \psi}{\partial t^2} = \psi^3$$

General solution of a PDE (named also a general integral) includes as a rule some arbitrary functions and it means that in order to get a unique solution of PDE we need some additional information which could be given in the form of some initial or boundary conditions. Let us say that a problem described by PDE is well-posed if

- the problem has exactly one solution for given initial/boundary conditions (uniqueness),
- a small change in initial/boundary conditions produces a small change in the solution (stability).

If a problem is not well-posed, it is useless for applications. If the problem has no solution at all, or more that one solution, it gives us no predictive power whatsoever. If the solution exists and is unique but small changes in data result lead to big changes in the solution, then the solution is still useless because, in practice, all boundary data, and so on, come from measurements which have small errors in them.

Solving PDE then involves evaluating a function $\psi(x,t)$ in some region of the (x,t)-plane defined normally by some additional conditions like values of ψ or its derivative on the boundary of the region we consider.

There are several common types of boundary/initial conditions:

- Dirichlet conditions: the function ψ is given on the boundary.
- Neumann conditions: when we specify the normal derivative $(\nabla\psi)_n = \frac{\partial\psi}{\partial x}$.
- Robin (mixed) conditions: a combination of ψ and $(\nabla\psi)_n$ are given.
- Cauchy (initial) conditions: ψ and $\frac{\partial\psi}{\partial t}$ are given at some initial value of t.

Example.

Let us consider equation

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2}$$

with Dirichlet type boundary conditions

$$\psi(0,t) = 0, \quad \psi(\pi,t) = 0,$$

and

$$\psi(x,0) = 0, \quad \psi(x,\pi) = 0.$$

Looking for solutions of the form $\psi(x,t) = X(x)T(t)$ we find that any function of the form

$$\psi(x,t) = A\sin(nx)\sin(nt)$$

with integer n gives a solution. Thus there are infinitely many solutions to the problem! It is ill-posed.

By the following classification method we can decide what sort of data should be applied to a given PDE in order to get a well-posed problem. For simplicity we do this only for the case of a PDE in two variables, x and t (classification could be generalized for the case of n variables).

Consider the (possibly nonlinear) PDE

$$a\frac{\partial^2\psi}{\partial x^2} + b\frac{\partial^2\psi}{\partial x \partial t} + c\frac{\partial^2\psi}{\partial t^2} = f(x,t,\psi,\frac{\partial\psi}{\partial x},\frac{\partial\psi}{\partial t}) \tag{1}$$

Suppose we prescribe ψ, $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial t}$ along some curve in the (x,t)-plane (see Figure below) and suppose this curve is given in a parameterized form as $(x(s),t(s))$ where s denotes arc length along this curve. Let us specify function ψ along the curve, i.e.

$$\psi(x(s),t(s)) = \Psi(s),$$

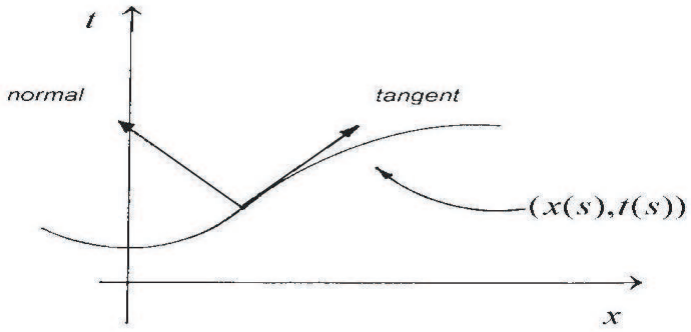
then we have also implicitly specified the tangential derivative of Ψ along this curve, since, differentiating with respect to s we find

$$\frac{\boldsymbol{d}}{\boldsymbol{d}s}\psi(x(s),t(s)) = \frac{\partial\psi}{\partial x}\frac{\boldsymbol{d}x}{\boldsymbol{d}s} + \frac{\partial\psi}{\partial t}\frac{\boldsymbol{d}t}{\boldsymbol{d}s} = \Psi'(s)$$

Now notice that vector $(\frac{\boldsymbol{d}x}{\boldsymbol{d}s},\frac{\boldsymbol{d}t}{\boldsymbol{d}s})$ is the unit vector tangent to the curve $(x(s),t(s))$ and hence

$$\frac{\partial\psi}{\partial x}\frac{\boldsymbol{d}x}{\boldsymbol{d}s} + \frac{\partial\psi}{\partial t}\frac{\boldsymbol{d}t}{\boldsymbol{d}s} = L(s)\nabla\psi = \frac{\partial\psi}{\partial L}.$$

That is, the tangential derivative of ψ along the curve s is already known if the value of ψ is known along the curve.



Once we specify, say, $\frac{\partial \psi}{\partial x}$ along the curve then $\frac{\partial \psi}{\partial t}$ is determined there by the initial equation.

The usual thing to do is to specify the normal derivative of ψ along the curve, then

$$(\frac{\boldsymbol{d}x}{\boldsymbol{d}s}, \frac{\boldsymbol{d}t}{\boldsymbol{d}s})$$

is the unit tangent and

$$(-\frac{\boldsymbol{d}t}{\boldsymbol{d}s}, \frac{\boldsymbol{d}x}{\boldsymbol{d}s})$$

is the unit normal. Thus

$$\frac{\partial \psi}{\partial n} = n \nabla \psi = -\frac{\partial \psi}{\partial x}\frac{\boldsymbol{d}t}{\boldsymbol{d}s} + \frac{\partial \psi}{\partial t}\frac{\boldsymbol{d}x}{\boldsymbol{d}s}.$$

and hence, if we prescribe $\frac{\partial \psi}{\partial n} = \Theta(s)$ along the curve , as well as $\psi = \Psi(s)$, we have two equations for the unknowns $\frac{\partial \psi}{\partial x}$ and $\frac{\partial \psi}{\partial t}$ along the curve which could be written in matrix form as following:

$$\begin{pmatrix} \frac{\boldsymbol{d}x}{\boldsymbol{d}s} & \frac{\boldsymbol{d}t}{\boldsymbol{d}s} \\ -\frac{\boldsymbol{d}t}{\boldsymbol{d}s} & \frac{\boldsymbol{d}x}{\boldsymbol{d}s} \end{pmatrix} \begin{pmatrix} \frac{\partial \psi}{\partial x} \\ \frac{\partial \psi}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\boldsymbol{d}\Psi}{\boldsymbol{d}s} \\ \Theta \end{pmatrix}$$

Recalling that $(\frac{\boldsymbol{d}x}{\boldsymbol{d}s})^2 + (\frac{\boldsymbol{d}t}{\boldsymbol{d}s})^2 = 1$, we find that

$$\frac{\partial \psi}{\partial x} = \frac{\boldsymbol{d}x}{\boldsymbol{d}s}\Psi'(s) - \frac{\boldsymbol{d}t}{\boldsymbol{d}s}\Theta(s)$$
$$\frac{\partial \psi}{\partial t} = \frac{\boldsymbol{d}t}{\boldsymbol{d}s}\Psi'(s) + \frac{\boldsymbol{d}x}{\boldsymbol{d}s}\Theta(s)$$

Thus, by prescribing ψ and $\frac{\partial\psi}{\partial n}$ along the curve $(x(s), t(s))$, we know all of ψ, $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial t}$ along this curve. Now assume that a, b, c and f are functions only of x, t, ψ, $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial t}$, this means that all of these too are known along the curve. That is, we have one equation for the three unknowns $\frac{\partial^2\psi}{\partial x^2}$, $\frac{\partial^2\psi}{\partial x\partial t}$ and $\frac{\partial^2\psi}{\partial t^2}$ along the curve. But we can find two other equations by differentiating expressions for the first partial derivatives, namely

$$\frac{\boldsymbol{d}}{\boldsymbol{d}s}\frac{\partial\psi}{\partial x} = \frac{\boldsymbol{d}x}{\boldsymbol{d}s}\frac{\partial^2\psi}{\partial x^2} + \frac{\boldsymbol{d}t}{\boldsymbol{d}s}\frac{\partial^2\psi}{\partial x\partial t},$$
$$\frac{\boldsymbol{d}}{\boldsymbol{d}s}\frac{\partial\psi}{\partial t} = \frac{\boldsymbol{d}x}{\boldsymbol{d}s}\frac{\partial^2\psi}{\partial x\partial t} + \frac{\boldsymbol{d}t}{\boldsymbol{d}s}\frac{\partial^2\psi}{\partial t^2}$$

This gives us the system of equations

$$\begin{pmatrix} \frac{\boldsymbol{d}x}{\boldsymbol{d}s} & \frac{\boldsymbol{d}t}{\boldsymbol{d}s} & 0 \\ 0 & \frac{\boldsymbol{d}x}{\boldsymbol{d}s} & \frac{\boldsymbol{d}t}{\boldsymbol{d}s} \\ a & b & c \end{pmatrix} \begin{pmatrix} \frac{\partial^2\psi}{\partial x^2} \\ \frac{\partial^2\psi}{\partial x\partial t} \\ \frac{\partial^2\psi}{\partial t^2} \end{pmatrix} = \begin{pmatrix} \frac{\boldsymbol{d}}{\boldsymbol{d}s}(\frac{\boldsymbol{d}x}{\boldsymbol{d}s}\Psi' - \frac{\boldsymbol{d}t}{\boldsymbol{d}s}\Theta) \\ \frac{\boldsymbol{d}}{\boldsymbol{d}s}(\frac{\boldsymbol{d}x}{\boldsymbol{d}s}\Psi' + \frac{\boldsymbol{d}t}{\boldsymbol{d}s}\Theta) \\ f \end{pmatrix}$$

which provides solution if its determinant is non-zero:

$$\begin{vmatrix} \frac{\boldsymbol{d}x}{\boldsymbol{d}s} & \frac{\boldsymbol{d}t}{\boldsymbol{d}s} & 0 \\ 0 & \frac{\boldsymbol{d}x}{\boldsymbol{d}s} & \frac{\boldsymbol{d}t}{\boldsymbol{d}s} \\ a & b & c \end{vmatrix} \neq 0$$

i.e. if

$$a(\frac{\boldsymbol{d}t}{\boldsymbol{d}s})^2 - b\frac{\boldsymbol{d}x}{\boldsymbol{d}s}\frac{\boldsymbol{d}t}{\boldsymbol{d}s} + c(\frac{\boldsymbol{d}x}{\boldsymbol{d}s})^2 \neq 0.$$

If however

$$a(\frac{\boldsymbol{d}t}{\boldsymbol{d}s})^2 - b\frac{\boldsymbol{d}x}{\boldsymbol{d}s}\frac{\boldsymbol{d}t}{\boldsymbol{d}s} + c(\frac{\boldsymbol{d}x}{\boldsymbol{d}s})^2 = 0,$$

then in general the system of equations above will be inconsistent and will have no solutions. That means that the second partial derivatives *can not exist* along this curve and that means that PDE *can not have a solution*. If we eliminate the parameter s, we may rewrite the condition of the existence of solution in the form

$$\frac{\boldsymbol{d}x}{\boldsymbol{d}t} = \frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}.$$

Any curve which satisfy these this condition is called *characteristic curve* or *characteristic*. Thus, Cauchy data, for instance, determine the solution of the problem *only if* the curve is nowhere parallel to a characteristic. Obviously, there are three different cases to be regarded:

- $b^2 < 4ac$, the characteristics are pure imaginary,
- $b^2 = 4ac$, a single family of real characteristics,
- $b^2 > 4ac$, a double family of real characteristics.

According to these conditions, all second order PDEs are divided into three classes which are named *elliptic, parabolic* and *hyperbolic* by appealing analogy with conical sections so that these names remind us about geometrical reasoning which led to this partition. To each class some specific boundary/initial conditions have to be prescribed in order to provide a well-posed problem.

Class 1: $b^2 < 4ac$, elliptic equations.

Examples:

- Laplace equation:
$$\bigtriangledown^2\psi = 0,$$
(note that operator $\bigtriangledown^2\psi$ is commonly written as $\triangle$ in pure mathematical texts),
- Helmholtz equation:
$$\bigtriangledown^2\psi + \alpha^2\psi = 0,$$
- Poisson's equation:
$$\bigtriangledown^2\psi = -4\pi\rho$$

Class 2: $b^2 = 4ac$, parabolic equations.

Examples:

- heat conduction:
$$\triangle\psi = \frac{1}{\alpha}\frac{\partial\psi}{\partial t},$$
and the proportionality factor is called heat conductivity. The same equation is used also for description of diffusion (with ψ as concentration of the dissolved substance in the solvent and α - diffusivity) and also is a form of the Navier-Stokes equation for a laminar fluid (with ψ as internal friction of an incompressible fluid and α - kinematic viscosity).

- Schrödinger equation:
$$\triangle\psi = \frac{2m}{i\hbar}\frac{\partial\psi}{\partial t},$$
which describes a vibration process (here m is mass of the particle, $\hbar$ is Plank's constant and $i^2 = -1$).

Class 3: $b^2 > 4ac$, hyperbolic equations.

Examples:

- wave equation:
$$\triangle\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2},$$
which is used in particular in acoustics (c is sound speed), electrodynamics of varying fields (c is light speed), in optics, in the theory of water waves, etc.

- four dimensional potential equation:
$$\sum_k \frac{\partial^2\psi}{\partial x_k^2} = 0,$$
which is used in special theory of relativity and is written out for three space variables x_1, x_2, x_3 and one time variable $x_4 = ict$.

Now that we have specified three main classes of second order PDEs it is possible to describe for each class what sort of boundary/initial conditions will provide us a well-posed problem:

- parabolic - One initial (Cauchy)+ some boundary condition,
- elliptic - Dirichlet/Neumann/Robin,
- hyperbolic - One initial (Cauchy) + some boundary condition.

In this way we have specified a well-posed problem for a PDE of second order. But it does not mean that we are able to solve it! Even in case of first order the task of finding solutions is not solved completely. For some specific types of them the general solution is found, for instance, in case when x, t, ψ are not included in the first order PDE explicitly, i.e. it has form
$$P(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial t}) = 0,$$

then it has solution

$$\psi = \lambda_1 x + \lambda_2 t + \mu,$$

where $P(\lambda_1, \lambda_2) = 0$

or in case of generalized Clero equation

$$\psi = x\frac{\partial \psi}{\partial x} + t\frac{\partial \psi}{\partial t} + f(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial t})$$

with general solution

$$\psi = \lambda x + \mu t + f(\lambda, \mu).$$

As to general methods, methods of characteristics, Lax pair, Green functions, canonical transformations, etc. produce often solutions for some specific PDE or groups of them. A great number of interesting results are obtained by the method of inverse scattering transform which allows to construct a PDE with given solution. In this way solutions of some famous PDEs have been obtained. Simply put, the inverse scattering transform is a nonlinear analog of the Fourier transform used for linear problems. Its value lies in the fact that it allows certain nonlinear problems to be treated by what are essentially linear methods. For some important PDEs a lot of specific results are obtained, for instance for PDE of only two variables x and t:

$$\begin{aligned} \frac{\partial \psi}{\partial t} + 6\psi\frac{\partial \psi}{\partial x} + \frac{\partial^3 \psi}{\partial x^3} = 0, \\ \psi(x, 0) = \psi_i(x), \\ \psi(0, t) = \psi_b(t), \end{aligned}$$

with initial condition $\psi_i(x)$ and boundary condition $\psi_b(t)$. This equation is called linearized Korteweg-de Vries (KdV) equation and is generic equation for the study of weak turbulence among long waves because it describes many physical phenomena such as plasma waves, Rossby waves, magma flow, surface water waves, etc. Hundreds of investigators dealt with this problem but no complete mathematical theory of this equation had yet been obtained. Of course, there are numerous number of results like, for instance, soliton solutions on the infinite line, five different types of solution depending on the relation between the initial and boundary values, solutions for positive quarter-plane problem (i.e. $x > 0$ and $t > 0$), solutions for negative quarter-plane problem (i.e. $x < 0$ and $t > 0$), solutions for time-dependent boundary conditions, numerical solutions, etc.

Sometimes by simple transformation a nonlinear PDE could become linear, for instance Thomas equation

$$\psi_{xy} + \alpha\psi_x + \beta\psi_y + \psi_x\psi_y = 0$$

could be made linear with a change of variables: $\psi = log\theta$ for some positively defined function θ:

$$\psi_x = (log\theta)_x = \frac{\theta_x}{\theta}, \quad \psi_y = (log\theta)_y = \frac{\theta_y}{\theta},$$
$$\psi_{xy} = (\frac{\theta_x}{\theta})_y = \frac{\theta_{xy}\theta - \theta_y\theta_x}{\theta^2}, \quad \psi_x\psi_y = \frac{\theta_x\theta_y}{\theta^2}$$

and substituting this into Thomas equation we get finally linear PDE

$$\theta_{xy} + \alpha\theta_x + \beta\theta_y = 0.$$

General solution for particular case $\beta = 0$ can be obtained as a result of one once more change of variables: $\theta = \phi e^{k_1 y}$ which leads to

$$\theta_x = \phi_x e^{k_1 y}, \quad \theta_{xy} = \phi_{xy} e^{k_1 y} + k_1\phi_x e^{k_1 y} \quad \text{and} \quad \phi_{xy} + (k_1 + \alpha)\phi_x = 0,$$

and finally

$$\partial_x(\phi_y + (k_1 + \alpha)\phi) = 0.$$

It gives

$$\phi_y - k_2\phi = f(y) \quad \text{with} \quad k_2 = -(k_1 + \alpha)$$

and arbitrary function $f(y)$. Now general solution can be obtained by the method of variation of a constant. As a first step let us solve homogeneous part of this equation, i.e. $\phi_y - k_2\phi = 0$ and $\phi(x, y) = g(x)e^{k_2 y}$ with arbitrary $g(x)$. As a second step, suppose that $g(x)$ is function on x, y, i.e. $g(x, y)$, then initial equation takes form

$$(g(x, y)e^{k_2 y})_y - k_2 g(x, y)e^{k_2 y} = f(y),$$

$$g(x, y)_y e^{k_2 y} + k_2 g(x, y)e^{k_2 y} - k_2 g(x, y)e^{k_2 y} = f(y),$$

$$g(x, y)_y e^{k_2 y} = f(y), \quad g(x, y) = \int f(y)e^{-k_2 y}dy + h(x),$$

and finally the general solution of Thomas equation with $\beta = 0$ can be written out as

$$\phi(x, y) = g(x, y)e^{k_2 y} = e^{k_2 y}(\int f(y)e^{-k_2 y}dy + h(x)) = \hat{f}(y) + e^{k_2 y}h(x)$$

with two arbitrary functions $\hat{f}(y)$ and $h(x)$. But generally speaking each new nonlinear PDE could turn out to be unsolvable!

Exercises:

1.1 Define the type and (if possible) class of following PDEs of two variables x and t:

- $$\psi_{tt} - \alpha^2 \bigtriangledown^2 \psi + \beta^2 \psi + \psi\psi_x = 0,$$
- $$\psi_{tt} - \psi_{xx} + \psi = 0,$$
- $$\psi_{tt} - \alpha^2 \bigtriangledown^2 \psi - \beta^2 \bigtriangledown^2 \psi_{tt} + 3e^x J^2(\psi, t) = 0,$$
- $$\psi_t + \alpha\psi_x + \beta\psi_{xxx} + \psi\psi_t + \psi^2 = 0.$$

1.2 Prove that Burger's equation

$$\psi_t + \psi\psi_x = \alpha\psi_{xx}$$

could be transformed to linear heat $\theta_t - \theta_{xx} = 0$ equation by substitution

$$\psi = -2(log\theta)_x.$$

How does this substitution change the class of the initial equation? Why is it important?

1.3 Is Thomas's equation quasi-linear?

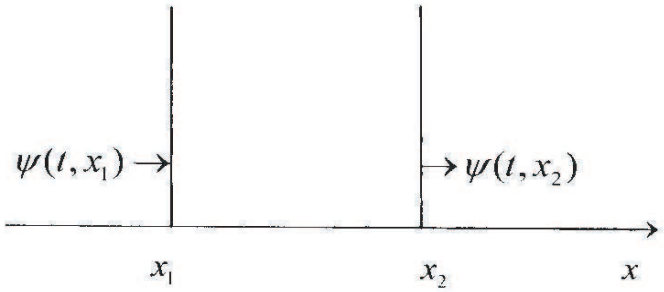
1.4 Is Burger's equation quasi-linear?

3 Dispersive waves

Now, that all the PDEs of the second order are divided into three non-intersecting classes, some additional conditions provide for each class a well-posed problem which might be solvable or at least some important information about its solutions could be obtained. The only problem is that physicists are not satisfied - they would like to know how to solve nonlinear PDEs. In order to show that nonlinear equations in physics are not exotics of a sort let us now describe a class of problems which leads immediately to some nonlinear PDE. In many physical problems we are trying to find some relation between two quantities say unit density ρ of some physical thing and its unit flow ψ so that the velocity of a flow could be defined as ψ/ρ. Both ρ and ψ are functions of space and time variables, say, $\rho = \rho(x,t)$ and $\psi = \psi(x,t)$. Most physical systems have some conservation law which could be written out as

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho dx + \psi(x_2,t) - \psi(x_1,t) = 0$$

in any fixed interval x_1, x_2 (see Fig. below)



and if ρ and ψ are smooth then in the limit $x_1 \to x_2$ then the conservation law takes a form

$$\psi_t + \psi_x = 0.$$

Very often there exist some intuitive or empirical considerations allowing us to regard flow as a function of density, that is, $\psi = F(\rho)$, which leads immediately to the nonlinear PDE

$$\psi_t + c(\psi)\psi_x = 0$$

with $c(\psi) = F_x(\rho)$. This PDE describes a lot of different physical or technical problems such as flood waves in rivers (wave's height h plays the role of density and $c(h)$ is the flow velocity), transport flow (ρ now is number of cars at the unit length of a highway, ψ is number of cars passing a line x in the unit of time and highway has no outlets), erosion in mountains, chemical exchange processes, absorption, etc.

It means that most of physical problems are described by nonlinear PDEs which are very often of order more than 2 and mathematics provides no systematic methods to solve them. Nevertheless, a very interesting way has been found by physicists in order to get some general information about physically important PDEs. In order to present here these results as a first step we restrict ourselves in this section to linear PDEs.

Actually the main idea is very simple. If we are not able to classify equations due to their *form* - let us try to classify them due to form of their *solutions*. The simplest possible solution whatsoever would be some nice smooth periodical function like *sine* or *cosine*. Let us see, whether linear PDEs of arbitrary finite order which admit these type of solutions have also some other properties in common. Taking into account that the minimal number of variables in any PDE is equal two and without loose of generality let us regard following function of two variables:

$$\psi(x,t) = a\sin(\omega t - kx)$$

with

$$\omega = 2\pi f = \frac{2\pi}{T}; \quad k = \frac{2\pi}{\lambda};$$

where ω is wave frequency in radians and per second, f is wave frequency in Hertz, k is wave number, T is wave period, λ is wave length. The constant a is called wave amplitude. The wave period is the time it takes two successive wave crests. The wave length λ is the distance two successive wave crests (see Fig. 1.1).

The basic feature of the wave as defined above is that the whole pattern *moves along the x-axis* as the time changes. Consider for simplicity the point $x = 0, t = 0$, where $\psi = 0$. If now t starts to increase, the points $x_0(t)$ defined by $x_0(t)/\lambda = t/T$ will have the property that $\psi(x_0(t), t) = 0$ for all t. The point x_0, where $\psi = 0$, thus moves with the velocity λ/T along the x-axis.

An additional angle α in the expression $\psi = a\sin(\omega t - kx + \alpha)$ is called a *phase term*. The whole argument of a *sine* is called a *phase*. Since $\sin(2\pi n +$

$\alpha) = \sin(2\pi n + \alpha)$ for any $n \in Z \cup \{0\}$, phase difference of any multiple of 2π do not matter at all. The phase of a point(x_1, t_1) will be equal to the phase of the point (x_2, t_2) if

$$\omega t_1 + k x_1 = \omega t_2 + k x_2,$$

that is,

$$\frac{x_2 - x_1}{t_2 - t_1} = \frac{\omega}{k} = \frac{\lambda}{T}$$

or

$$x_2 = x_1 + \frac{\omega}{k}(t_2 - t_1).$$

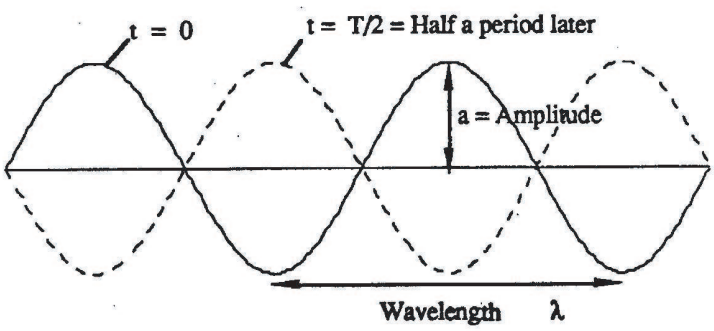


Figure 1: A linear wave.

Thus point x_2 on the x-axis which moves with velocity ω/k will have the same phase for all times. Therefore, the velocity $c_{ph} = \omega/k = \lambda/T$ is called **phase velocity** associated with the wave and this wave is called a **travelling wave**. Another important notion is a **standing wave** which does not travel but remains in a constant position and occurs in nature whenever a travelling wave reflects perfectly from a boundary.

In this case reasoning similar to those concerning phase velocity lead to the following definition of **group velocity** which indeed is generalization of phase velocity for the case of travelling wave:

$$c_{gr} = \frac{\boldsymbol{d}\omega}{\boldsymbol{d}k},$$

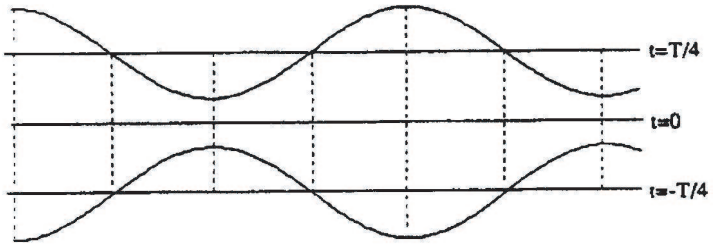


Figure 2: The standing wave.

it is, so to say, "phase velocity" of the wave's envelope (see Figure below).

Difference between phase and group velocities is now obvious:

- Phase velocity $c_{ph} = \omega/k$ is velocity of points which belong to a one wave and have the same phase.
- Group velocity $c_{gr} = \boldsymbol{d}\omega/\boldsymbol{d}k$ is velocity of points which belong to a group of waves and have the same amplitude.

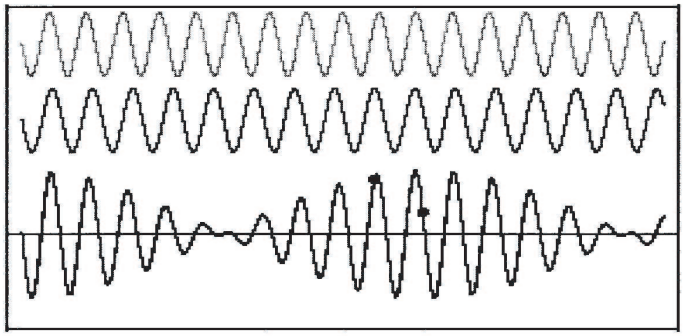


Figure 3: The wave packet.

In general these two velocities *do not coincide* (see **Exercises** below) and group velocity plays more important role then phase velocity because it is the velocity at which energy propagates.

Now notice that in physics it is usual to regard a wave in a form of *complex exponent* using Euler formula

$$\exp ix = \cos x + i \sin x$$

because the operations like differentiation, multiplication, etc. become more compact. Therefore from now on we will use also the complex representation for a linear wave

$$\psi(x) = A \exp i(kx - \omega t)$$

keeping in mind that when necessary one takes the real part of it, namely

$$\Re(\psi(x)) = |A| \cos(kx - \omega t + \alpha), \quad \alpha = argA.$$

Let us now write out explicitly a linear operator corresponding to 2-dimensional PDE with constant coefficients in a form

$$P(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}),$$

where P is a polynomial which degree coincides with the order of PDE. After substituting a wave solution in it we will get

$$P(-i\omega, ik) = 0.$$

It means that in order to satisfy initial PDE wave number k and frequency ω have to be connected in some way, i.e. there exist some function f such, that

$$f(\omega, k) = 0.$$

This connection is called **dispersion relation**. A solution of the dispersion relation is called **dispersion function**, $\omega = \omega(k)$. In general case, there could be a few solutions, each of them is called **mode** of the dispersion function. Since ω/k is phase velocity of a wave, ω is supposed to be a real-value function. Another important point is that the very notion of **dispersive wave systems** (that is, dispersive PDE) has been introduced in order to point out the difference between phase velocities of different waves providing solutions of the same PDE. But the case of $\omega(k) = ck$ gives us *the same phase velocity* for all waves and therefore ideologically does not fall into the class of dispersive waves. That is the reason why the following condition on the ω has to be satisfied:

$$\frac{\partial^2 \omega}{\partial k^2} \neq 0$$

Actually this condition excludes also case of $\omega(k) = c_1 k + c_2$ due to some physical reasons which we are not going to explain here (details see in [3],

[4]). The very important fact is that the initial PDE could be reconstructed by substituting $-i\omega$ by $\partial/\partial t$ and ix by $\partial/\partial x$.

Example

Let us regard a linear PDE

$$\psi_{tt} + \alpha^2 \phi_{xxxx} = 0.$$

and find explicit form of the dispersion function for it. First of all we need some preliminary calculations:

$$\psi_t = \frac{\partial}{\partial t}\psi = \omega(-i)A\exp i(kx - \omega t),$$
$$\psi_{tt} = \frac{\partial}{\partial t}(\frac{\partial}{\partial t}\psi) = (\omega(-i))^2 A\exp i(kx - \omega t) = -\omega^2 A\exp i(kx - \omega t),$$
$$\psi_x = \frac{\partial}{\partial x}\psi = kiA\exp i(kx - \omega t),$$
$$\text{.....},$$
$$\psi_{xxxx} = \frac{\partial^4}{\partial x^4}\psi = (ki)^4 A\exp i(kx - \omega t) = k^4 A\exp i(kx - \omega t).$$

and substituting these results into initial PDE one gets:

$$0 = \psi_{tt} + \alpha^2\psi_{xxxx} =$$
$$= -\omega^2 A\exp i(kx - \omega t) + \alpha^2 k^4 A\exp i(kx - \omega t),$$

which leads to the dispersion function

$$\omega(k) = \pm\alpha k^2$$

with two modes: $\omega(k) = \alpha k^2$ and $\omega(k) = -\alpha k^2$.

All definitions and procedure above could be easily reformulated for a case of more space variables, namely $x_1, x_2, ..., x_n$. In this case linear operator generated by initial PDE takes form

$$P(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}),$$

and correspondingly dispersion function could be computed from

$$P(-i\omega, ik_1, ..., ik_n) = 0.$$

In this case we will have not a wave number k but a *wave vector* $\vec{k} = (k_1, ..., k_n)$ and the condition of non-zero second derivative of the dispersion function takes a matrix form:

$$|\frac{\partial^2 \omega}{\partial k_i \partial k_j}| \neq 0$$

Since ideally behavior of each physical system has to be described in time and space, it is normal in physics to use different notations for these variables, for instance of a form

$$P(t, x_1, ..., x_n) = 0.$$

and the same in PDEs. In case when a PDE does not have variable t, it is called *stationary PDE* as, for instance, Laplace equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0.$$

In the opposite case PDE is called *evolution PDE*. The importance of the different nature of these variables is recognized also in mathematics where the variables are called correspondingly *time-like* and *space-like* variables. Obviously the notion of dispersion is only important for evolution PDEs.

The division of the variables into these two classes originated from the special relativity theory where time and three-dimensional space are treated together as a single four-dimensional manifold called *space-time* or *Minkovski space*. In Minkovski space a metrics allowing to compute an interval s along a curve between two events is defined analogously to distance in Euclidean space:

$$\boldsymbol{d}s^2 = \boldsymbol{d}x^2 + \boldsymbol{d}y^2 + \boldsymbol{d}z^2 - c^2\boldsymbol{d}t^2$$

where c is speed of light (sometimes the opposite choice of signs in the right part is chosen). However, note that whereas Euclidean distances are always positive, Minkovski's intervals may be positive, zero or negative. Events with space-time interval zero are separated by the propagation of a light signal. Events with a positive space-time interval are in each other's future or past, and the value of the interval defines the proper time measured by an observer travelling between them.

Note that in fact mathematical classification of PDEs into hyperbolic, parabolic or elliptic classes is not only too restrictive (strict definitions are only possible for PDEs of second order) but also too abstract - it has no memory whatsoever about the difference between space and time variables and all

the reasoning could be carried out for the case of any variables $x_1, x_2, ..., x_s$. On the other hand, physical classification is *based* on the existence of two types of variables and it helps to construct physical interpretation of the solutions of PDE. What these two classifications have in common is following worrying fact - both of them give us some ways to construct some solutions of some PDEs in some special cases but not one of them provides either a method to solve any specific PDE or at least a possibility to establish that any specific PDE has a solution. Big advantage of physical classification is due to the possibility to treat PDE of an arbitrary order.

Let us summarize the results obtained:

- We have constructed a one-to-one correspondence between linear (evolution) PDE $L(\psi) = 0$ of arbitrary order allowing a wave solution $\psi(\vec{x}) = A \exp i(\vec{k}\vec{x} - \omega t)$ and some polynomial P which defines dispersion function $\omega = \omega(\vec{k})$.
- The number of variables of dispersion function ω coincides with the number of space variables of the equation $L(\psi) = 0$.
- Given dispersion function allows us to re-construct the corresponding linear PDE.

In this way the partitioning of all linear evolution PDEs into two classes - dispersive and non-dispersive - has been obtained.

This partition is not complementary to a standard mathematical one.

For instance, though hyperbolic PDEs normally do not have dispersive wave solutions, the hyperbolic equation $\psi_{tt} - \alpha^2 \bigtriangledown^2 \psi + \beta^2 \psi = 0$ has them. Another case is the equation $\psi_{tt} + \alpha^2 \phi_{xxxx} = 0$ which could not be classified as hyperbolic, parabolic or elliptic but belongs to the class of dispersive PDEs.

Obviously in this way PDEs are able to generate only *polynomial* dispersion relations. In fact, if some specific PDE is regarded together with special initial/boundary conditions, it may generate a transcendental dispersion function such like

$$\omega(k) = k \tanh \alpha k$$

and in general it is possible just to choose *arbitrary* dispersion function and to construct corresponding (perhaps integro-differential) equation in partial variables.

Exercises:

2.1. Show that two waves $\sin(\omega t - kx)$ and $\sin(\omega t + kx)$ are moving in opposite directions.

2.2. Show that if ω and k are allowed to be negative and arbitrary phase terms could be included, then all functions

$$\begin{aligned} a\sin(kx - \omega t), \\ a\sin(kx + \omega t + \frac{\pi}{4}), \\ a\cos(\omega t - kx) + b\sin(\omega t - kx) \end{aligned}$$

may be written as $\tilde{a}\sin(\tilde{\omega}t - \tilde{k}x + \tilde{\alpha})$ for appropriate choice of $\tilde{a}, \tilde{\omega}, \tilde{k}, \tilde{\alpha}$.

2.3. Compute dispersion relations for following PDEs of two variables x and t:

- $$\psi_{tt} - \alpha^2 \bigtriangledown^2 \psi + \beta^2 \psi + \psi\psi_x = 0,$$
- $$\psi_{tt} - \psi_{xx} + \psi = 0,$$
- $$\psi_{tt} - \alpha^2 \bigtriangledown^2 \psi - \beta^2 \bigtriangledown^2 \psi_{tt} + 3e^x J^2(\psi, t) = 0,$$
- $$\psi_t + \alpha\psi_x + \beta\psi_{xxx} + \psi\psi_t + \psi = 0.$$

2.4. The same in case of three space variables, i.e. $\vec{x} = (x_1, x_2, x_3)$.

2.5. Compute phase and group velocities for each PDE from 2.3.

4 Wave resonances and perturbation techniques

Now that we know what a linear dispersive wave is, we are going to introduce the notion of the wave resonance which is the mile-stone for the whole theory of nonlinear dispersive waves, that is for the theory of nonlinear dispersive PDEs. We begin with a well-known example illustrating the general idea of the resonance in some oscillating system.

Let us consider a linear oscillator driven by a small force

$$x_{tt} + p^2 x = \varepsilon e^{i\Omega t}.$$

Here p is eigenfrequency of the system, Ω is frequency of the driving force and $\varepsilon > 0$ is a small parameter. Deviation of this system from equilibrium is small (of order ε), if there is no resonance between the frequency of the driving force $\varepsilon e^{i\Omega t}$ and an eigenfrequency of the system. If these frequencies coincide then the amplitude of resonator grows linearly with the time and this situation is called **resonance** in physics. Mathematically it means **existence of unbounded solutions**.

Let us now regard a (weakly) nonlinear PDE of the form

$$L(\psi) = \varepsilon N(\psi)$$

where L is an arbitrary linear dispersive operator and N is an arbitrary nonlinear operator of *second order* which makes the formulae below more simple. Two linear waves providing solutions of $L(\psi) = 0$ could be written out as

$$A_1 \exp i[\vec{k}_1\vec{x} - \omega(\vec{k}_1)t],$$
$$A_2 \exp i[\vec{k}_2\vec{x} - \omega(\vec{k}_2)t],$$

and their amplitudes A_1, A_2 are constant. Intuitively natural to expect that *solutions of weakly nonlinear PDE will have the same form as linear waves but perhaps with amplitudes depending on time.* Taking into account that nonlinearity is small, each amplitude is regarded as *a slow-varying function of time*, that is $A_j = A_j(t/\varepsilon)$. Standard notation is $A_j = A_j(T)$ where $T = t/\varepsilon$ is called **slow time**.

Since wave energy is by definition proportional to amplitude's square A_j^2 it means that in case of nonlinear PDE *waves exchange their energy.* This

effect can also be described as "waves are interacting with each other" or "there exists energy transfer through the wave spectrum" or similar. Unlike linear waves for which their linear combination was also solution of $L(\psi) = 0$, it is not the case for nonlinear waves!

Substitution of two linear waves into the operator $\epsilon N(\psi)$ generates terms like

$$\exp i[(\vec{k}_1 \pm \vec{k}_2)\vec{x} - [\omega(\vec{k}_1) \pm \omega(\vec{k}_2)]t$$

which play the role of a small driving force for the linear wave system (same as in case of linear oscillator above). This driving force gives a small effect on a wave system till resonance occurs. i.e. till the wave number and the wave frequency of the driving force does not coincide with some wave number and some frequency of eigen-wave:

$$\omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) = 0,$$
$$\vec{k}_1 \pm \vec{k}_2 \pm \vec{k}_3 = 0.$$

This system is called **resonance conditions** and it describes **resonance curves** or **resonance surfaces** depending on the number of space variables. Second equation is written out in vector form, that is we have one linear resonance condition if $dim(\vec{k}) = 1$ and few linear conditions if $1 \leq dim(\vec{k})$. In fact, it is not one system but a few different systems according to different nontrivial choices of signs "+" and "-". Each specific combination generates a system to be solved and therefore resonance conditions describe not one but *a few different algebraic systems*. The dispersion ω is known algebraic function of space variables and its specific form could be computed from $L(\psi) = 0$. Therefore resonant conditions represent *an algebraic system* (strictly speaking, we would have to say a system of algebraic systems but for brevity we will use words "algebraic system" for resonance conditions.)

The number of unknowns is equal to the number of space variables.

The number of equations depends on the number of space variables *and* on the number of possible nontrivial combinations of signs "+" and "-" in it.

The solutions of this algebraic system(s), if any, provide *coordinates of wave vectors* of three resonantly interacting waves

$$A_1(T)\exp i[\vec{k}_1\vec{x} - \omega(\vec{k}_1)t],$$
$$A_2(T)\exp i[\vec{k}_2\vec{x} - \omega(\vec{k}_2)t],$$
$$A_3(T)\exp i[\vec{k}_3\vec{x} - \omega(\vec{k}_3)t],$$

and one can see immediately that one main purpose - to find solutions of the initial PDE - is not solved yet. We have found all $\vec{k}_i$ but functions $A_1(T), A_2(T), A_3(T)$ are also to be found. These could be done by one of the perturbation methods (they are also called asymptotic methods) which are much in use in physics and are dealing with equations having some small parameter $\varepsilon > 0$ (see [6]). The main idea of such a method is very simple - an unknown solution, depending on ε, is written out in a form of infinite series on different powers of ε and coefficients before any power of ε are computed consequently.

In order to demonstrate how it works let us take a simple algebraic equation

$$x^2 - (3 - 2\varepsilon)x + 2 + \varepsilon = 0$$

and try to find its asymptotic solutions. In case $\varepsilon = 0$ we get

$$x^2 - 3x + 2 = 0$$

with roots $x = 1$ and $x = 2$. The first equation is called *perturbed* and the second - *unperturbed.* Natural suggestion is that the solutions of perturbed equation differ only a little bit from the solutions of unperturbed one.

First step. Look for the perturbed solutions in a form

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + ...$$

with x_0 as a solution of unperturbed equation which in our case means $x_0 = 1$ or $x_0 = 2$.

Second step. Substitute this infinite series into the given equation and rewrite it in such a way that all the terms with same degree of ε are collected together. In our case it leads to

$$\varepsilon^0 : x_0^2 - 3x_0 + 2,$$
$$\varepsilon^1 : 2x_0x_1 - 3x_1 - 2x_0 + 1,$$
$$\varepsilon^2 : 2x_0x_2 + x_1^2 - 3x_2 - 2x_1,$$
$$...$$

Third step. Suppose that all coefficients corresponding to the consequent powers of ε are zero and solve the system (notice that first equation here is unperturbed initial equation whose solutions are already known):

$$\begin{aligned} x_0^2 - 3x_0 + 2 = 0, \\ 2x_0x_1 - 3x_1 - 2x_0 + 1 = 0, \\ 2x_0x_2 + x_1^2 - 3x_2 - 2x_1 = 0, \\ ... \end{aligned}$$

Then the resulting solutions of perturbed equation are

$$\begin{aligned} x_0 = 1, \quad x_1 = -1, \quad x_2 = 3, ... \\ x = 1 - \varepsilon + 3\varepsilon^2 + ... \end{aligned}$$

and

$$\begin{aligned} x_0 = 2, \quad x_1 = 3, \quad x_2 = -3, ... \\ x = 2 + 3\varepsilon - 3\varepsilon^2 + ... \end{aligned}$$

Now we see that perturbation technique generates a system of equations which being solved consequently gives us the coefficients in front of any desirable degree of ϵ in the general solution of the perturbed equation. Equation above was chosen as example because it's exact solutions are known and one would be able to compare them with the results obtained by perturbation method. Indeed, exact solutions of perturbed equation are

$$x = \frac{1}{2}[3 + 2\varepsilon \pm \sqrt{1 + 8\varepsilon + 4\varepsilon^2}]$$

and substituting binomial representation for the expression under the square root

$$\begin{aligned} (1 + 8\varepsilon + 4\varepsilon^2)^{\frac{1}{2}} = 1 + (8 \ + 4\varepsilon^2) + \frac{\frac{1}{2}(\frac{-1}{2})}{2!}(8\varepsilon + 4\varepsilon^2)^2 + ... = \\ = 1 + 4\varepsilon + 2\varepsilon^2 - \frac{1}{8}(64\varepsilon^2 + ...) = \\ = 1 + 4\varepsilon - 6\varepsilon^2 + ... \end{aligned}$$

into the exact solution one gets finally

$$\begin{aligned} x = \frac{1}{2}(3 + 2\varepsilon - 1 - 4\varepsilon + 6\varepsilon^2 + ...) = \\ = 1 - \varepsilon + 3\varepsilon^2 + ... \\ x = \frac{1}{2}(3 + 2\varepsilon + 1 + 4\varepsilon - 6\varepsilon^2 + ...) = \\ = 2 + 3\varepsilon - 3\varepsilon^2 + ... \end{aligned}$$

as before.

This check gives mathematician a feeling that "there is something in perturbation methods" but not much more - no strict mathematical proof of them being always correct and/or providing always good approximate (physicists call them *asymptotic*) solutions or something of this kind. On the other hand, perturbation methods are very successfully used in physics and engineering for the problems having no known exact solutions.

We demonstrate below how to use perturbation method for barotropic vorticity equation (BVE) is also known as Obukhov-Charney or Hasegawa-Mima equation. The number of names given to it is explained by its importance in many physical applications - from geophysics to astrophysics to plasma physics: the equation was again and again re-discovered by specialists in very different branches of physics. In particular, taken in spherical form it describes planetary (or Rossby) waves in an ocean or in the Earth atmosphere which appear due to the rotation of the Earth.

Being regarded on a sphere, BVE takes following form

$$\frac{\partial \triangle \psi}{\partial t} + 2\frac{\partial \psi}{\partial \lambda} + \varepsilon J(\psi, \triangle \psi) = 0. \tag{2}$$

Here ψ is the stream-function; variables t, ϕ and λ physically mean the time, the latitude ($-\pi/2 \leq \phi \leq \pi/2$) and the longitude ($0 \leq \lambda \leq 2\pi$) respectively; $0 < \varepsilon << 1$ is small parameter. The spherical Laplacian and Jacobian are given by formulae

$$\triangle \psi = \frac{\partial^2 \psi}{\partial \phi^2} + \frac{1}{\cos^2 \phi}\frac{\partial^2 \psi}{\partial \lambda^2} - \tan \phi \frac{\partial \psi}{\partial \phi}$$

and

$$J(a, b) = \frac{1}{cos\phi}(\frac{\partial a}{\partial \lambda}\frac{\partial b}{\partial \phi} - \frac{\partial a}{\partial \phi}\frac{\partial b}{\partial \lambda})$$

respectively. The linear part of spherical BVE has wave solutions in the form

$$AP_n^m(\mu) \exp i[m\lambda + \frac{2m}{n(n+1)}t],$$

where $\vec{k} = (m, n)$, $\mu = \sin\phi$, $\omega = -2m/[n(n+1)]$, $P_n^m(x)$ is the associated Legendre function of degree n and order m and A is constant wave amplitude.

In this place mathematician gets a shock - till now linear wave was supposed to have much more simple form, namely, $A\exp i[\vec{k}\vec{x} - \omega(\vec{k})t]$ without any additional multiplier of a functional form. A physicist says that it is intuitively clear that if multiplier is some *oscillatory function of only space variables* then we will still have a wave of a sort "but it would be difficult to include it in an overall definition. We seem to be left at present with the looser idea that whenever oscillations in space are coupled with oscillation in time through a dispersion relation, we expect the typical effects of dispersive waves" [3]. By the way, most physically important dispersive equations have this form of waves.

Now let us keep in mind that a wave is something more complicated then just a *cosine* but still smooth and periodical let us look where perturbation method will lead us. An approximate solution has a form

$$\psi = \psi_0(\lambda, \phi, t, T) + \varepsilon\psi_1(\lambda, \phi, t, T) + \varepsilon^2\psi_2(\lambda, \phi, t, T) + ...$$

where $T = t/\varepsilon$ is the "slow" time and the zero approximation ψ_0 is given as a sum of three linear waves:

$$\psi_0(\lambda, \phi, t, T) = \sum_{k=1}^{3} A_k(T)P^{(k)}\exp(m_k\lambda - \omega_k t)$$

with notations $P^{(k)} = P_{n_k}^{m_k}$ and $\omega_k = \omega(m_k, n_k)$. Then

$$\varepsilon^0: \quad \frac{\partial\triangle\psi_0}{\partial t} + 2\frac{\partial\psi_0}{\partial\lambda} = 0,$$
$$\varepsilon^1: \quad \frac{\partial\triangle\psi_1}{\partial t} + 2\frac{\partial\psi_1}{\partial\lambda} = -J(\psi_0, \triangle\psi_0) - \frac{\partial\triangle\psi_0}{\partial T},$$
$$\varepsilon^2: .....$$

To calculate the right part of last equation some preliminary calculations are necessary (notation $\theta_k = \exp(m_k\lambda - \omega_k t)$):

$$\frac{\partial\psi_0}{\partial\lambda} = i\sum_{k=1}^{3} P^{(k)} m_k [A_k \exp(i\theta_k) - A_k^* \exp(-i\theta_k)];$$

$$\frac{\partial^2\psi_0}{\partial\lambda^2} = -\sum_{k=1}^{3} P^{(k)} m_k^2 [A_k \exp(i\theta_k) + A_k^* \exp(-i\theta_k)];$$

$$\frac{\partial\psi_0}{\partial\phi} = \sum_{k=1}^{3} \frac{d}{d\phi} P^{(k)} \cos\phi [A_k \exp(i\theta_k) + A_k^* \exp(-i\theta_k)];$$

$$\frac{\partial^2\psi_0}{\partial\phi^2} = \sum_{k=1}^{3} [\frac{d^2}{d\phi^2} P^{(k)} \cos^2\phi - \frac{d}{d\phi} P^{(k)} \sin\phi][A_k \exp(i\theta_k) + A_k^* \exp(-i\theta_k)];$$

$$\triangle\psi_0 = -\sum_{k=1}^{3} P^{(k)} N_k [A_k \exp(i\theta_k) + A_k^* \exp(-i\theta_k)];$$

(where notation $N_k = n_k(n_k + 1)$ and the definition of Legendre function as a solution of the following equation:

$$(1 - z^2)\frac{d^2P}{dz^2} - 2z\frac{dP}{dz} + [n(n+1) - \frac{m^2}{1 - z^2}]P = 0$$

has been used in order to obtain the last equation);

$$\frac{\partial\triangle\psi_0}{\partial\lambda} = i\sum_{k=1}^{3} P^{(k)} m_k N_k [A_k \exp(i\theta_k) - A_k^* \exp(-i\theta_k)];$$

$$\frac{\partial\triangle\psi_0}{\partial\phi} = -\sum_{k=1}^{3} N_k \frac{d}{d\phi} P^{(k)} \cos\phi [A_k \exp(i\theta_k) + A_k^* \exp(-i\theta_k)];$$

$$J(\psi_0, \triangle\psi_0) =$$

$$-i\sum_{j,k=1}^{3} N_k m_j P^{(j)} \frac{d}{d\phi} P^{(k)} [A_j \exp(i\theta_j) - A_j^* \exp(-i\theta_j)][A_k \exp(i\theta_k) + A_k^* \exp(-i\theta_k)] +$$

$$+i\sum_{j,k=1}^{3} N_k m_k P^{(k)} \frac{d}{d\phi} P^{(j)} [A_j \exp(i\theta_j) + A_j^* \exp(-i\theta_j)][A_k \exp(i\theta_k) - A_k^* \exp(-i\theta_k)];$$

$$\frac{\partial\triangle\psi_0}{\partial T} = \sum_{k=1}^{3} P^{(k)} N_k [\frac{dA_k}{dT} \exp(i\theta_k) + \frac{dA_k^*}{dT} \exp(-i\theta_k)].$$

The condition of unbounded growth of the left hand can be written out in the form

$$J(\psi_0, \triangle\psi_0) = \frac{\partial\triangle\psi_0}{\partial T}$$

and therefore the resonance conditions take form of

$$\theta_j + \theta_k = \theta_i \quad \forall j, k, i = 1, 2, 3.$$

Let us fix some specific resonance condition, say,

$$\theta_1 + \theta_2 = \theta_3,$$

which means

$$\omega_1 + \omega_2 = \omega_3, \quad \& \quad m_1 + m_2 = m_3, \tag{3}$$

then

$$\frac{\partial\triangle\psi_0}{\partial T} \cong -N_3 P^{(3)}[\frac{dA_3}{dT}\exp(i\theta_3) + \frac{dA_3^*}{dT}\exp(-i\theta_3)],$$
$$J(\psi_0, \triangle\psi_0) \cong -i(N_1 - N_2)(m_2 P^{(2)}\frac{d}{d\phi}P^{(1)}) - m_1 P(1)\frac{d}{d\phi}P(2)\cdot$$
$$A_1 A_2 \exp[i(\theta_1 + \theta_2)] - A_1^* A_2^* \exp[-i(\theta_1 + \theta_2)],$$

where notation $\cong$ means that only terms which can generate given resonance are written out. Let us substitute these expressions into the coefficient by ε^1, i.e. into the equation

$$\frac{\partial\triangle\psi_1}{\partial t} + 2\frac{\partial\psi_1}{\partial\lambda} = -J(\psi_0, \triangle\psi_0) - \frac{\partial\triangle\psi_0}{\partial T},$$

multiply both parts of it by

$$P^{(3)}\sin\phi[A_3\exp(i\theta_3) + A_3^*\exp(-i\theta_3)]$$

and integrate all over the sphere with $t \to \infty$. As a result two following equations can be obtained:

$$N_3\frac{dA_3}{dT} = 2iZ(N_2 - N_1)A_1 A_2,$$
$$N_3\frac{dA_3^*}{dT} = -2iZ(N_2 - N_1)A_1^* A_2^*,$$

where

$$Z = \int_{-\pi/2}^{\pi/2} [m_2 P^{(2)} \frac{d}{d\phi} P^{(1)} - m_1 P^{(1)} \frac{d}{d\phi} P^{(2)}] \frac{d}{d\phi} P^{(3)} d\phi.$$

The same procedure obviously provides the analogous equations for A_2 and A_3 while fixing corresponding resonance conditions:

$$\begin{aligned} N_1 \frac{dA_1}{dT} &= -2iZ(N_2 - N_3) A_3 A_2^*, \\ N_1 \frac{dA_1^*}{dT} &= 2iZ(N_2 - N_3) A_3^* A_2^*, \\ N_2 \frac{dA_2}{dT} &= -2iZ(N_3 - N_1) A_1^* A_3, \\ N_2 \frac{dA_2^*}{dT} &= 2iZ(N_3 - N_1) A_1 A_3^*. \end{aligned}$$

This system of 6 ODEs on A_i, A_i^* is normally written in the form

$$\begin{cases} N_1 \dot{A}_1 = Z(N_2 - N_3) A_2 A_3, \\ N_2 \dot{A}_2 = Z(N_3 - N_1) A_1 A_3, \\ N_3 \dot{A}_3 = Z(N_1 - N_2) A_1 A_2, \end{cases} \tag{4}$$

for real-valued amplitudes A_1, A_2, A_3 meaning that two analogous systems of ODEs have to be solved - one for real parts of amplitudes and one for their imaginary parts.

Obviously only the case when $Z \neq 0$ has to be regraded while otherwise $\dot{A}_i = 0$ and it is linear case. It could be shown that $Z \neq 0$ if the following conditions keep true:

$$\begin{cases} |n_1 - n_2| \leq n_3 \leq n_1 + n_2, \\ n_1 + n_2 + n_3 \quad \text{is odd}, \\ m_i \leq n_i \quad \forall i = 1, 2, 3, \end{cases}$$

Coefficient of Z is called *interaction coefficient* and plays important physical role - it characterizes the velocity of energy exchange between the waves of the triad. Explicit form of the interaction coefficients depends on the form of dispersion function (that is, on the form of initial PDE) and coordinates of wave vectors. Interaction coefficients must not be identically equal to zero (otherwise amplitudes would not be changing and one would be back in linear case). Therefore *some additional restrictions are sometimes added*

to the general resonance conditions, namely in our case resonant conditions take form:

$$\begin{cases} \omega_1 + \omega_2 = \omega_3, \\ m_1 + m_2 = m_3, \\ m_i \leq n_i \quad \forall i = 1, 2, 3, \\ |n_1 - n_2| \leq n_3 \leq n_1 + n_2, \\ n_1 + n_2 + n_3 \quad \text{is odd}, \\ n_i \neq n_j \quad \forall i \neq j. \end{cases} \tag{5}$$

Notice that Sys.(4) can be solved explicitly in terms of Jacobian elliptic functions [5]. Here we are not going into the full details of its derivation but just give the main idea. First of all, we note that the system has two conserved integrals. The first one is called *energy conservation law* and could be obtained by multiplying the i-th equation by A_i, $i = 1, 2, 3$ and adding all three of them:

$$N_1 A_1^2 + N_2 A_2^2 + N_3 A_3^2 = c_1.$$

The second conserved integral is called *enstrophy conservation law* and is obtained by multiplying the i-th equation by $N_i A_i$, $i = 1, 2, 3$ and adding all three of them:

$$N_1^2 A_1^2 + N_2^2 A_2^2 + N_3^2 A_3^2 = c_2.$$

The constants c_1 and c_2 depend, of course, on the initial values of $A_{i0}, i = 1, 2, 3$. Using these two conservation laws one can easily obtain expressions for A_2 and A_3 in terms of A_1. Substitution of these expressions into the first equation of our ODE system gives us differential equation of the first order which explicit solution is one of Jacobian elliptic functions. In similar way we will get expressions for the amplitudes A_2 and A_3 as well, so that the general solution has form:

$$\begin{aligned} A_1 &= b_1 cn(t/t_0 - \lambda), \\ A_2 &= b_2 dn(t/t_0 - \lambda), \\ A_3 &= b_3 sn(t/t_0 - \lambda), \end{aligned}$$

where

$$\begin{aligned} b_1^2 &= A_{10}^2 + N_3(N_2 - N_3)/N_1(N_2 - N_1)A_{30}^2, \\ b_2^2 &= A_{20}^2 + N_3(N_3 - N_1)/N_2(N_2 - N_1)A_{30}^2, \\ b_3^2 &= A_{30}^2 + N_1(N_2 - N_1)/N_3(N_2 - N_3)A_{10}^2, \end{aligned}$$

the constant t_0 depends on wave numbers and initial values of amplitudes as follows:

$$t_0 = \frac{1}{Z}[\frac{N_2 - N_3}{N_1}(\frac{N_3 - N_1}{N_2}A_{30}^2 + \frac{N_2 - N_1}{N_3}A_{20}^2)]^{1/2}$$

and the constant λ is obtained from the initial conditions.

The Jacobian elliptic functions cn, sn, dn can be defined in the following way. Let

$$u = \int_0^{\phi} \frac{d\theta}{(1 - \mu \sin^2 \theta)^{1/2}}$$

and let define three following functions:

$$\begin{aligned} snu &= \sin \phi, \\ cnu &= \cos \phi, \\ dnu &= (1 - \mu \sin^2 \phi)^{1/2}. \end{aligned}$$

These functions are periodic in u. The periods of snu and cnu are equal to $4K$ and the periods of dnu is equal to $2K$ where

$$K = \int_0^{\pi/2} \frac{d\theta}{(1 - \mu sin^2 \theta)^{1/2}}$$

is complete elliptic integral and

$$\mu = \frac{A_{10}^2 \frac{N_1}{N_3 - N_2} + A_{20}^2 \frac{N_2}{N_3 - N_1}}{A_{30}^2 \frac{N_3}{N_2 - N_1} + A_{20}^2 \frac{N_2}{N_3 - N_1}}$$

is modulus of the elliptic functions. The period of energy exchange within the modes of a resonant triad could be calculated as

$$\tau = 4pt_0K, \quad p = 1, 2, 3, ...$$

Summing up, we described in all details the energetic behavior of *three arbitrary resonantly interacting waves*. As a result system of ODEs for slowly changing waves amplitudes is obtained

$$\begin{cases} \dot{A}_1 = \mathfrak{C}_1 A_2 A_3, \\ \dot{A}_2 = \mathfrak{C}_2 A_1 A_3, \\ \dot{A}_3 = \mathfrak{C}_3 A_1 A_2, \end{cases}$$

where A_i are functions of one variable T and of a few parameters which are the wave numbers giving solutions of resonance conditions (three parameters for the case $dim(\vec{k}) = 1$, six parameters for the case $dim(\vec{k}) = 2$ and so on). The only difference with the results obtained for perturbed polynomial is the following: perturbation technique applied for a polynomial generates system of polynomials to be solved while applied for PDE it generates a system of ODEs.

The only problem left is that the space of wave numbers is infinite and it means that in fact we have to solve a much more complicated system of ODEs, namely

$$\begin{cases} \dot{A}_1 = \mathfrak{C}_{1,3}A_2A_3 + \mathfrak{C}_{1,4}A_2A_4 + ... + \mathfrak{C}_{1,s}A_2A_s + ..., \\ \dot{A}_2 = \mathfrak{C}_{2,3}A_1A_3 + \mathfrak{C}_{2,4}A_1A_4 + ... + \mathfrak{C}_{2,s}A_1A_s + ..., \\ ..... \end{cases}$$

for the case of quadratic nonlinearity. This system connects all existing waves and they are infinitely many which makes it difficult to solve these ODEs, whether analytically or numerically. What are the possibilities we will see in the next Section.

5 Classification of dispersive PDEs

Let us notice fist that all the considerations above could be carried out for the nonlinearity of arbitrary order n in which case resonance conditions

$$\begin{cases} \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm ... \pm \omega(\vec{k}_{n+1}) = 0, \\ \vec{k}_1 \pm \vec{k}_2 \pm ... \pm \vec{k}_{n+1} = 0. \end{cases} \tag{6}$$

describe $(n+1)$-wave interactions so that cubic nonlinearity generates four-wave interactions, nonlinearity of fourth order generates five-wave interactions and so on. Corresponding system of ODEs on the slowly-changing amplitudes will have terms like $A_1A_2...A_{n+1}$ on the right hand:

$$\begin{cases} \dot{A}_1 = \mathfrak{C}_1A_2A_3A_4...A_{n+1} + ..., \\ \dot{A}_2 = \mathfrak{C}_2A_1A_3A_4...A_{n+1} + ..., \\ ... \\ \dot{A}_{n+1} = \mathfrak{C}_rA_1A_2A_3...A_n + ..., \\ ... \end{cases} \tag{7}$$

Let us sum up what we got till now:

Beginning with PDE $L(\psi) = \epsilon N(\psi)$ some algebraic system (resonance conditions) and some system of ODEs have been constructed. When solved together they may provide solution(s) of the initial PDE.

To solve this system of ODEs a special method
of **wave kinetic equation** has been developed in 60-th (see, for instance, [14], [10]) and applied for many different types evolution PDEs. Kinetic equation is approximately equivalent to the initial nonlinear PDE but has more simple form allowing direct numerical computations of each wave amplitudes in a given domain of wave spectrum. Wave kinetic equation is an averaged equation imposed on a certain set of correlation functions and it is in fact one limiting case of the quantum Bose-Einstein equation while the Boltzman kinetic equation is its other limit. Some statistical assumptions have been used in order to obtain kinetic equations and limit of its applicability then is a very complicated problem which should be solved separately for each specific equation [13].

Simply formulated, conditions used for obtaining of kinetic equation can be described as following:

- each wave takes part in resonant interactions,
- each wave interacts with all other waves simultaneously,
- all wave amplitudes are of the same order.

As a result, the Sys.(7) has been reduced to an equation of a form

$$\frac{d}{dT}A_i = \int \mathcal{G}(\vec{k}_i)\delta(\sum \vec{k}_i)d\vec{k}_i$$

which is solved in respect to each separate wave amplitude A_i. Here function $\mathcal{G}$ depends on the form of initial nonlinear PDE and notation $\delta(\sum \vec{k}_i)$ is used for delta-function which is equal to zero on the solutions of Sys.(6).1) and is non-zero otherwise. Wave kinetic equation is often written out in the form

$$\frac{d}{dT}A_i^2 = \int \mathcal{G}(\vec{k}_i)\delta(\sum \vec{k}_i)d\vec{k}_i$$

because amplitude square is proportional to the wave energy and this form allows to treat the results produced by kinetic equation in terms of energy

exchange between the interacting waves.

Thus, the existence of resonances, i.e. solutions of Sys.(6), in a $(n+1)$-wave system for some specific PDE allows physicist to replace the PDE by kinetic equation which govern energy transfer through the spectrum. A very interesting fact is that even some classification of dispersive PDEs due to their integrability properties was constructed basing on solutions of Sys.(6). We present it below very briefly and for more details see [7].

First of all, it was proven that in case when Eq.(6.1) does not have any solutions, initial nonlinear PDE could be transformed into a *linear* PDE by *canonic transformation*. If, on the other hand, Eq.(6.1) does have some solutions, then initial PDE will still have some nonlinearity after canonic transformation and it could be written out as

$$\Sigma_i \frac{T_i \delta(\vec{k}_1 \pm \vec{k}_2 \pm ... \pm \vec{k}_i)}{\omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm ... \pm \omega(\vec{k}_i)}$$

where δ is a delta-function and so-called here *vertex coefficient* T_i and in fact T_{i_0} for some specific i_0 is just some generalization of *interaction coefficient* from the system for slowly changing amplitudes in case of i_0-waves interactions. In case of zero vertex coefficient, $T_i = 0$ we fall into the class of equations having *soliton* solutions and in case of non-zero vertex coefficients, as it was shown before, the kinetic equation is constructed which *is, to some extent, equivalent* to the initial PDE. The words "to some extent" are here in order to remind us that some additional physically natural assumptions were used in order to construct kinetic equation.

Now it looks like all dispersive PDEs are classified and the work of physicist is finished while the only task left to mathematician is to prove convergence of perturbation methods. But *it's not true!* Kinetic equations have been written for infinite interaction domains, i.e. for an infinite plane or an infinite channel and were used successfully for about 25 years for description of many types of waves, mostly in cases for 3- and 4-waves interactions. The results of laboratory experiments showed that "reasonable agreement is normally obtained with theoretical results for the infinite case" [8], if the wavelength are small enough in comparison to the size of the experimental basin. The cases when wavelengths are comparable with the characteristic sizes of the experimental basin remain, as a rule, unexplained [9] and have been named "effects of finite lengths" [8]. Attempts to put some additional

"physically relevant" terms into the kinetic equations in order to make them applicable to long-wavelength systems have failed. The author of the pioneering work in this field, O. Phillips, who obtained the first kinetic equation in [10] wrote in [11] that "new physics, new mathematics and new intuition is required" in order to understand energetic behavior of long-wavelength systems (they are also called *large-scale systems*).

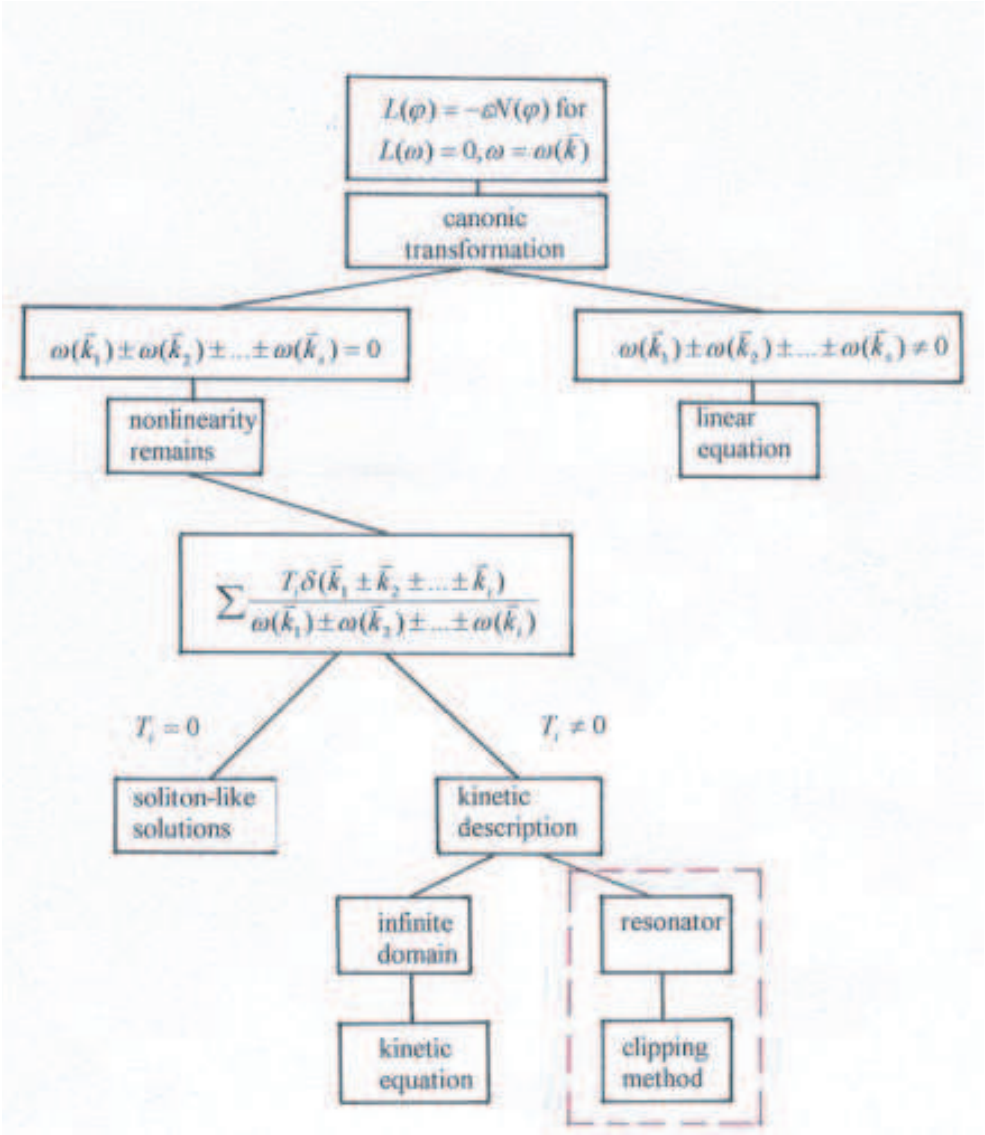
From physical point of view large-scale wave system is a system where wave *do notice the boundaries*. Mathematically it means that initial PDE has to be regarded with zero or periodical boundary conditions. These wave systems are called *resonators* and are very important in numerous number of physical applications [12]. From mathematical point of view, difference between infinite interaction domain and resonator could formulated very easily:

- In case of infinite interaction domain Sys.(6) have to be solved in real numbers.
- In resonators Sys.(6) have to be solved in integers.

It means that in resonators Sys.(6) describe a system of Diophantine or algebraic equations in integers. When a specialist in number theory or in algebra looks at assumptions used for obtaining of kinetic equations together with discrete Sys.(6), he starts to worry. Why do any pair of integers (m, n) has to be part of some solution? Or, even more, to be part of infinitely many solutions? Is it really true that this algebraic system *always* has infinitely many solutions? Rather not. At least, one has to prove it.

In the next Part, **Part 2: Properties of discrete resonant systems**, it will be proven that all these physical assumptions do not hold in resonators, and this is the reason why method of kinetic equation does not work there. It means that this class of wave systems can not be described in terms of kinetic equation *principally* and has to be studied separately. In Figure below we show where this "missing term" in the classification of dispersive PDEs has to be inserted.

In Part 2 a new approach will be presented to study discrete resonance systems based mainly on number theory methods. Some general properties of these systems are described analytically and corresponding results for many specific physical problems examples are presented.


$L(\varphi) = -\varepsilon N(\varphi)$ for
$L(\omega) = 0, \omega = \omega(\bar{k})$
canonic transformation
$\omega(\bar{k}_1) \pm \omega(\bar{k}_2) \pm \ldots \pm \omega(\bar{k}_s) = 0$
$\omega(\bar{k}_1) \pm \omega(\bar{k}_2) \pm \ldots \pm \omega(\bar{k}_s) \neq 0$
nonlinearity remains
linear equation
$\sum \frac{T_i \delta(\bar{k}_1 \pm \bar{k}_2 \pm \ldots \pm \bar{k}_i)}{\omega(\bar{k}_1) \pm \omega(\bar{k}_2) \pm \ldots \pm \omega(\bar{k}_i)}$
$T_i = 0$
$T_i \neq 0$
soliton-like solutions
kinetic description
infinite domain
resonator
kinetic equation
clipping method

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