How One Can Play with Sums

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Abstract

There are implementations of the celebrated Gosper algorithm (1978) and Zeilberger algorithm (1991) on almost any computer algebra platform. Inspired by Karr's indefinite summation algorithm (1981) I developed an algorithm based on difference fields which can handle indefinite and definite summation problems in a very general setting including Gosper's and Zeilberger's algorithms. The algorithm is available in form of a package developed in the computer algebra system Mathematica. The usage of my summation package `Sigma` will be illustrated by a concrete example.

1 Introduction

Karr developed an algorithm for **indefinite** summation [Kar81, Kar85] based on the theory of difference fields [Coh65]. He introduced so called $\Pi\Sigma$-fields, in which first order linear difference equations can be solved in full generality. This algorithm cannot only deal with series of hypergeometric terms, like Gosper's algorithm [Gos78, PS95a, PS95b], series with q-hypergeometric terms, like [PR97], or holonomic series, like Chyzak's algorithm [CS98], but also with series of terms where for example the harmonic numbers can appear in the denominator. Karr's algorithm is, in a sense, the summation counterpart of Risch's algorithm [Ris69, Ris70] for indefinite integration.

Inspired by this algorithm I developed a summation algorithm based on difference field theory [Sch99, Sch00a, Sch00b, Sch01] in the computer algebra system Mathematica. Based on Bronstein's denominator bounding [Bro00], I was able to streamline Karr's ideas. Moreover, I implemented a user interface that dispenses the user from working explicitly with difference fields. Instead, the user can handle all summation problems in terms of sums and products.

My summation package `Sigma` cannot only handle indefinite summation problems but also can find closed forms of definite multisums. Although Karr's original summation algorithm

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was already capable of carrying out creative telescoping [Zei90], nobody has noticed this possibility until now. With creative telescoping one can compute a recurrence which has a given definite sum as a solution; therefore one can verify automatically a given definite sum identity.
Additionally, I have generalized Karr's algorithm such that linear difference equations of any order can be solved in any given $\Pi\Sigma$-field. Hence one can find solutions of recurrences and thus not only prove definite sum identities, but even discover closed forms of definite sums in a very general setting.
In order to find solutions of a given difference equation, in many cases one has to extend the underlying difference field. Inspired by results in [AP94, HS99], I focus in [Sch01] on problems how one can find appropriate difference field extensions, namely sum extensions and so called d'Alembertian extensions, to find solutions for a recurrence. In this context, my indefinite summation algorithm plays a major role to simplify this solutions further. These aspects will be made clear by proving the following multisum identity:

Theorem 1. For nonnegative integers n we have

$$\sum_{k=0}^{2n}(-1)^k\, k\, \mathrm{H}_k \binom{2n}{k}^3 = \frac{1}{6}\,(1+3n\,\mathrm{H}_n+6n\,\mathrm{H}_{2n}-3n\,\mathrm{H}_{3n})(-1)^n\,\frac{(3n)!}{n!^3} - \frac{1}{6}(-1)^n\frac{n!^3\,(6n)!}{(2n)!^3\,(3n)!}$$

where $\mathrm{H}_k=\sum_{i=1}^{k}\frac{1}{i}$ are the k-th harmonic numbers.

In Section 2 the main goal is to demonstrate the basic strategy [PWZ96] how one can find automatically a closed form of a definite sum. Finally in Section 3 I will not only prove Theorem 1 but even will find the right hand side of the multisum identity.

2 A strategy to handle definite sums

In the following some functions of the Mathematica package `Sigma` are explained by a simple definite summation problem. After loading the summation package

In[1]:= << **Sigma`**

`Sigma - A summation package by Carsten Schneider`

we are able to insert the summation problem:

In[2]:= **mySum = SigmaSum[(i − 1)i^3
SigmaBinomial[2n, i]^3SigmaPower[−1, i]/(2n − i + 1)^3, {i, 0, 2n}]**

$$\text{Out[2]=}\sum_{i=0}^{2n}\left(\frac{(-1+i)\, i^3\, \left(\binom{2n}{i}\right)^3 (-1)^{i}}{(1-i+2n)^3}\right)$$

The functions `SigmaSum` and `SigmaProduct` are used to define sums and products. There are several other functions available, like `SigmaHNumber`, `SigmaBinomial` or `SigmaPower` to define harmonic numbers, binomials or powers. Additional functions are provided to introduce new objects.

Finding a recurrence

For this definite summation problem we are now able to find a closed form. In a first step we generate a recurrence that is satisfied by `mySum`. The idea how to compute a recurrence is based on Zeilberger's creative telescoping method [Zei90].

In[3]:= **rec = GenerateRecurrence[mySum]**

Out[3]= $\{18\ (1+3\ n)\ (2+3\ n)\ (1+6\ n)\ (5+6\ n)\ (13+18\ n+6\ n^2)\ \mathrm{SUM}[n]+12\ (90+814\ n+2543\ n^2+3864\ n^3+3126\ n^4+1296\ n^5+216\ n^6)\ \mathrm{SUM}[1+n]+2\ (1+n)\ (2+n)\ (3+2\ n)^2\ (1+6\ n+6\ n^2)\ \mathrm{SUM}[2+n]==8\ (288+3465\ n+18272\ n^2+55648\ n^3+98548\ n^4+97016\ n^5+48384\ n^6+9408\ n^7)\ (-1)^{2\ n}\}$

Since $(-1)^{2n}=1$ for all integers n, we can replace $(-1)^{2n}$ by 1 in our recurrence.

In[4]:= **rec = rec/.{ $(-1)^{2\ n}\to 1$};**

Solving the recurrence

Next we solve the recurrence in terms of objects given in the recurrence.

In[5]:= **SolveRecurrence[rec[[1]], SUM[n]]**

Out[5]= $\{\{1, 2\ n\}\}$

Unfortunately the algorithm delivers only a particular inhomogeneous solution $2\,n$, which is not a sufficient solution, since it does not have the same initial values as the original summation problem.
Therefore we try to find product extensions which deliver us solutions of the homogeneous version of the recurrence `rec` by calling the function `FindProductExtensions`. This function uses M. Petkovšek's package `Hyper` [Pet92, Pet94, PWZ96] which has to be loaded first.

In[6]:= **<< Hyper`;**

This package is able to find all hypergeometric solutions of a linear recurrence with polynomial coefficients. We obtain the following product extensions:

In[7]:= **FindProductExtensions[rec, SUM[n], Solutions → All]**

```
I use M. Petkovšek's package Hyper to find product extensions.
```

Out[7]= $\left\{\prod_{i=2}^{n}\left(-\frac{3\ (-2+3\ i)\ (-1+3\ i)}{(-1+i)\ i}\right), \prod_{i=1}^{n}\left(-\frac{3\ (-5+6\ i)\ (-1+6\ i)}{(-1+2\ i)^2}\right)\right\}$

Next we solve the recurrence `rec` by using the two product extensions in a rewritten form:

In[8]:= **tower** $=\left\{\left(\frac{(3\ n)!_{.}\ (-1)^{n_{.}}}{((n)!_{.})^3}\right)_{n_{.}}, \left(\frac{((n)!_{.})^3\ (6\ n)!_{.}\ (-1)^{n_{.}}}{((2\ n)!_{.})^3\ (3\ n)!_{.}}\right)_{n_{.}}\right\};$

In[9]:= **recSol = SolveRecurrence[rec[[1]], SUM[n], Tower → tower]**

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11.53 Second
```

Out[9]= $\left\{\left\{0, n\left(\frac{(3\ n)!_{.}\ (-1)^{n_{.}}}{(n)!_{.}^{\,3}}\right)_{n_{.}}\right\}, \left\{0, \left(\frac{(n)!_{.}^{\,3}\ (6\ n)!_{.}\ (-1)^{n_{.}}}{(2\ n)!_{.}^{\,3}\ (3\ n)!_{.}}\right)_{n_{.}}\right\}, \{1, 2\ n\}\right\}$

This has to be interpreted as follows: our algorithm delivers two linear independent solutions of the homogeneous version of the recurrence, namely

$$(-1)^n\, n\, \frac{(3\,n)!}{n!^3} \qquad\qquad (-1)^n \frac{n!^3\,(6\,n)!}{(2\,n)!^3\,(3\,n)!},$$

and one particular solution $2\,n$ of the inhomogeneous recurrence itself.

Finding the linear combination

Finally the closed form of `mySum` is that linear combination of the homogeneous solutions plus the inhomogeneous solution which has exactly the same initial values as `mySum`. This is also computed automatically:

In[10]:= **FindLinearCombination[recSol, mySum, 2]**

Out[10]= $2\ \mathtt{n} - \mathtt{n}\left(\frac{(3\ \mathtt{n})!_{.}\ (-1)^{\mathtt{n}.}}{(\mathtt{n})!_{.}^{\ 3}}\right)_{\mathtt{n}.}$

Summarizing we were able to compute the right hand side of the identity

$$\sum_{i=0}^{2\,n} \frac{(i-1)i^3(-1)^i\binom{2\,n}{i}^3}{(2\,n-i+1)^3} = n(2-(-1)^n\frac{(3\,n)!}{n!^3}). \tag{1}$$

Please note that this example can be solved by just using the hypergeometric packages [PS95a] and [Pet92].

3 A multisum example

In the following we try to find a closed form representation of the multisum in Theorem 1 by using the summation package.

In[11]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{2\ n}} \left(\mathbf{k\ H_k}\ \left(\left(\binom{\mathbf{2\ n}}{\mathbf{k}}\right)^{.\ \mathbf{3}}\ (-\mathbf{1})^{\mathbf{k}.}\right)_{\mathbf{k}.}\right);$

3.1 Finding a recurrence

As already illustrated in the first example, we try to compute a recurrence for the definite sum `mySum`.

Creative telescoping in a difference field setting

Since our algorithm, using Zeilberger's creative telescoping method, is based on a very general difference field setting, we are able to compute a recurrence for this multisum.

In[12]:= **GenerateRecurrence[mySum, RecOrder → 4]**

    2016.48 Second

Out[12]= $\{81\ (1+n)\ (10+117\ n+441\ n^{2}+648\ n^{3}+324\ n^{4})^{2}$
$(579023679111696+$
$6203096595284292\ n+30574972749055508\ n^{2}+$
$92475481987210701\ n^{3}+192864735750636284\ n^{4}+$
$295166120513347017\ n^{5}+344113220933469194\ n^{6}+$
$312890401572444600\ n^{7}+225181229898272112\ n^{8}+$
$129339961859979540\ n^{9}+59474372437202472\ n^{10}+$
$21854565707771808\ n^{11}+6372893337871680\ n^{12}+$
$1455288215784768\ n^{13}+254598040577664\ n^{14}+$
$32934777209856\ n^{15}+2967155877888\ n^{16}+$
$166161051648\ n^{17}+4353564672\ n^{18})\ \mathrm{SUM}[n]+$
$108\ (-8911086594732000+$
$595686855250231800\ n+16380227867435099780\ n^{2}+$
$185672492904312930710\ n^{3}+1271723758536088957353\ n^{4}+$
$6026151073985872712073\ n^{5}+21197749937538020891079\ n^{6}+$
$57793321639546981142298\ n^{7}+125693551925945528389705\ n^{8}+$
$222521457681141044963341\ n^{9}+$
$325368258856450491542511\ n^{10}+$
$397108616509050749048718\ n^{11}+$
$407622807225028518763356\ n^{12}+$
$353729663174629500044400\ n^{13}+$
$260330393614389288503220\ n^{14}+$
$162709980603775713128520\ n^{15}+$
$86335405854765454150272\ n^{16}+38809363531072919958144\ n^{17}+$
$14720133478715210657664\ n^{18}+4681642828665855843072\ n^{19}+$
$1237296059054356451328\ n^{20}+268300933294762027008\ n^{21}+$
$46890597952821408768\ n^{22}+6437495043769780224\ n^{23}+$
$668002856934260736\ n^{24}+49220844925353984\ n^{25}+$
$2293562354761728\ n^{26}+50779978334208\ n^{27})\ \mathrm{SUM}[1+n]+$
$18\ (2+n)\ (3+2\ n)\ (-8228295571986000-29467353203684820\ n+$
$1381518393267116428\ n^{2}+19978139922191293573\ n^{3}+$
$139144387971971638219\ n^{4}+625542630805627460455\ n^{5}+$
$2017285686440215860490\ n^{6}+4933055970124372861135\ n^{7}+$
$9465689765373655917267\ n^{8}+14579998008141370748253\ n^{9}+$
$18312629998410321364656\ n^{10}+18961209332586432771048\ n^{11}+$
$16304139557770332127212\ n^{12}+11695416700671314908740\ n^{13}+$
$7013537868185350191792\ n^{14}+3515617464514069708512\ n^{15}+$
$1469465760759532649280\ n^{16}+509652781805658910464\ n^{17}+$
$145518011266651170048\ n^{18}+33806212169624059392\ n^{19}+$
$6282436535103246336\ n^{20}+910948598145469440\ n^{21}+$
$99231835717287936\ n^{22}+7633845045411840\ n^{23}+$
$369565397876736\ n^{24}+8463329722368\ n^{25})\ \mathrm{SUM}[2+n]+$
$12\ (2+n)\ (3+n)\ (3+2\ n)\ (5+2\ n)$
$(-64001714143920-503422860673228\ n+$
$4002975025720952\ n^{2}+79747990756043705\ n^{3}+$
$565678480977551301\ n^{4}+2447100392628223047\ n^{5}+$
$7404218627394040182\ n^{6}+16709317348234374364\ n^{7}+$
$29191436701822318447\ n^{8}+40425384732611573230\ n^{9}+$
$45074461215631426464\ n^{10}+40878463232569911732\ n^{11}+$
$30338483534960452020\ n^{12}+18477110572629289128\ n^{13}+$
$9232514580951306000\ n^{14}+3772738135947714336\ n^{15}+$
$1252587607610477760\ n^{16}+334329670014178176\ n^{17}+$
$70597472266909440\ n^{18}+11513259270314496\ n^{19}+$
$1397288190984192\ n^{20}+118711550287872\ n^{21}+$
$6295254515712\ n^{22}+156728328192\ n^{23})\ \mathrm{SUM}[3+n]+$
$(2+n)\ (3+n)^{2}\ (4+n)^{2}\ (3+2\ n)\ (5+2\ n)$
$(7+2\ n)^{3}\ (-945554940-7607976456\ n+35254988575\ n^{2}+$
$756814949687\ n^{3}+4816720182041\ n^{4}+17947420546069\ n^{5}+$
$45372683784936\ n^{6}+83005099177032\ n^{7}+$
$113701841575020\ n^{8}+118788006388788\ n^{9}+$
$95405698339488\ n^{10}+58876332512544\ n^{11}+$
$27669385543104\ n^{12}+9716847158592\ n^{13}+$
$2466213765120\ n^{14}+426750114816\ n^{15}+$
$44986834944\ n^{16}+2176782336\ n^{17})\ \mathrm{SUM}[4+n]==$
$0\}$

Here we succeed in finding a recurrence of order 4, but we are not able to find a recurrence with smaller order.

Creative telescoping and sum extensions

Now we could go on, as illustrated in the previous section, to solve the recurrence and finally to combine the computed solutions in order to derive a closed form evaluation. We will not proceed in this way; instead we try to compute again a recurrence but this time we are looking for additional sum extensions such that the recurrence order is smaller. This feature is activated by setting the option `SimplifyByExt→DepthNumber`.

In[13]:= **rec = GenerateRecurrence[mySum, SimplifyByExt → DepthNumber, RecOrder → 2]**

157.97 Second

$$\text{Out[13]}= \{-36\,(1+n)\,(2+n)\,(1+2\,n)\,(3+2\,n)\,(1+3\,n)\,(2+3\,n)\,(1+6\,n)\,(5+6\,n)\,(1+6\,n+6\,n^2)\,(13+18\,n+6\,n^2)\,\text{SUM}[n]-24\,(1+n)\,(2+n)\,(1+2\,n)\,(3+2\,n)\,(1+6\,n+6\,n^2)\,(90+814\,n+2543\,n^2+3864\,n^3+3126\,n^4+1296\,n^5+216\,n^6)\,\text{SUM}[1+n]-4\,(1+n)^2\,(2+n)^2\,(1+2\,n)\,(3+2\,n)^3\,(1+6\,n+6\,n^2)^2\,\text{SUM}[2+n]==$$

$$2\Big((-19512-448728\,n-4422462\,n^2-24996138\,n^3-91349700\,n^4-227427644\,n^5-376226464\,n^6-308925516\,n^7+319086320\,n^8+1617697256\,n^9+3088351728\,n^{10}+3851758512\,n^{11}+3453843392\,n^{12}+2288224320\,n^{13}+1119909888\,n^{14}+396032256\,n^{15}+96095232\,n^{16}+14349312\,n^{17}+995328\,n^{18})\,(-1)^{2\,n}+$$

$$(-14802-376587\,n-3834063\,n^2-21159534\,n^3-71496792\,n^4-157297032\,n^5-232167060\,n^6-231571656\,n^7-153801504\,n^8-65046240\,n^9-15816384\,n^{10}-1679616\,n^{11})$$

$$\sum_{\iota_1=0}^{2\,n}\left(\frac{(-1+\iota_1)\,\iota_1^3\,\left(\left(\binom{2\,n}{\iota_1}\right)^3\,(-1)^{\iota_1}\right)}{(1+2\,n-\iota_1)^3}\right)+$$

$$(-26016-715824\,n-8970272\,n^2-68124912\,n^3-352009200\,n^4-1316397856\,n^5-3697583664\,n^6-7984118976\,n^7-13441452832\,n^8-17772262080\,n^9-18480846528\,n^{10}-15046225664\,n^{11}-9482866944\,n^{12}-4533055488\,n^{13}-1588110336\,n^{14}-384380928\,n^{15}-57397248\,n^{16}-3981312\,n^{17})$$

$$\sum_{\iota_1=0}^{2\,n}\left(\frac{(-1+\iota_1)\,\iota_1^3\,\left(\left(\binom{2\,n}{\iota_1}\right)^3\,(-1)^{\iota_1}\right)}{(1+2\,n-\iota_1)^3\,(2+2\,n-\iota_1)^3\,(3+2\,n-\iota_1)^3}\right)\Big)\}$$

We found two sum extensions *automatically* which allowed us to compute a recurrence of order 2. Please note that the computation time is remarkably smaller, namely 158 seconds, in comparison to the standard approach, namely 2016 seconds.

Here we point out that identity (1) in Section 2 gives us a closed form for the first sum extension. Similar to identity (1) we can find a closed form for the second sum extension and obtain the following identity:

$$\sum_{i=0}^{2n}\frac{(i-1)i^3(-1)^i\binom{2n}{i}^3}{(2n-i+1)^3(2n-i+2)^3(2n-i+3)^3}=(-1)^n\frac{(3n)!}{n!^3}\frac{3\,(1+3\,n)\,(2+3\,n)}{8\,(1+n)^4\,(1+2\,n)^3}+\frac{-3-15\,n-12\,n^2+17\,n^3+38\,n^4+28\,n^5+8\,n^6}{4\,(1+n)^2\,(1+2\,n)^3}.$$

Therefore we are allowed to substitute the two sums by their right hand side evaluations and we obtain the following recurrence:

In[14]:= $\mathbf{rec = 4\ (1+n)\ (2+n)\ (1+2\ n)\ (3+2\ n)\ (1+6\ n+6\ n^2)\ \big(-9\ (1+3\ n)\ (2+3\ n)\ (1+6\ n)\ (5+6\ n)\ (13+18\ n+6\ n^2)\ SUM[n]-6\ (90+814\ n+2543\ n^2+3864\ n^3+3126\ n^4+1296\ n^5+216\ n^6)\ SUM[1+n]-(1+n)\ (2+n)\ (3+2\ n)^2\ (1+6\ n+6\ n^2)\ SUM[2+n]\big) ==}$

$$\mathbf{-\frac{1}{1+n}\Big(6\ (6504+144754\ n+1384851\ n^2+7537254\ n^3+26070977\ n^4+60620448\ n^5+97542252\ n^6+109802520\ n^7+86051628\ n^8+45881424\ n^9+15822864\ n^{10}+3172608\ n^{11}+279936\ n^{12})\Big(\frac{(3\ n)!_{.}\ (-1)^{n}}{((n)!_{.})^3}\Big)_{n}\Big)}$$

3.2 Solving the recurrence

In the following we will solve the recurrence. Similar to Section 2 we can use the function `FindProductExtensions` in order to find the two product extensions

In[15]:= $\mathbf{tower = \Big\{\Big(\frac{(3\ n)!_{.}\ (-1)^{n}}{((n)!_{.})^3}\Big)_{n},\Big(\frac{((n)!_{.})^3\ (6\ n)!_{.}\ (-1)^{n}}{((2\ n)!_{.})^3\ (3\ n)!_{.}}\Big)_{n}\Big\};}$

which deliver us two linear independent solutions of the homogeneous version of the recurrence:

In[16]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], Tower \to tower]}$

Out[16]= $\Big\{\Big\{0,\Big(\frac{(n)!_{.}^{\,3}\ (6\ n)!_{.}\ (-1)^{n}}{(2\ n)!_{.}^{\,3}\ (3\ n)!_{.}}\Big)_{n}\Big\},\Big\{0,n\ \Big(\frac{(3\ n)!_{.}\ (-1)^{n}}{(n)!_{.}^{\,3}}\Big)_{n}\Big\}\Big\}$

Unfortunately we do not find a solution of the inhomogeneous recurrence itself. By setting the option `NestedSumExt`$\to\ \infty$ we find *all* nested sum solutions expressed in terms of objects given in the recurrence and of the two product extensions given in `tower`.

In[17]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], NestedSumExt \to \infty, Tower \to tower, AlgebraicRelationInSumSolutions \to True]}$

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19.78 Second
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Out[17]= $\Big\{\Big\{0,\Big(\frac{(n)!_{.}^{\,3}\ (6\ n)!_{.}\ (-1)^{n}}{(2\ n)!_{.}^{\,3}\ (3\ n)!_{.}}\Big)_{n}\Big\},\Big\{0,n\ \Big(\frac{(3\ n)!_{.}\ (-1)^{n}}{(n)!_{.}^{\,3}}\Big)_{n}\Big\},$

$$\Big\{1,\frac{1}{6}\Big(\frac{(n)!_{.}^{\,3}\ (6\ n)!_{.}\ (-1)^{n}}{(2\ n)!_{.}^{\,3}\ (3\ n)!_{.}}\Big)_{n}\sum_{\iota_1=1}^{n}\Bigg(\Bigg(\iota_1\ (1-6\ \iota_1+6\ \iota_1^2)\ \Big(\frac{(3\ \iota_1)!_{.}\ (-1)^{\iota_1}}{(\iota_1)!_{.}^{\,3}}\Big)_{\iota_1}\sum_{\iota_2=2}^{\iota_1}\big((9360-64710\ \iota_2+63189\ \iota_2^2+413410\ \iota_2^3-1436799\ \iota_2^4+2117172\ \iota_2^5-1737846\ \iota_2^6+826740\ \iota_2^7-213840\ \iota_2^8+23328\ \iota_2^9)/((-1+\iota_2)\ \iota_2\ (-3+2\ \iota_2)\ (-5+3\ \iota_2)\ (-4+3\ \iota_2)\ (13-18\ \iota_2+6\ \iota_2^2)\ (1-6\ \iota_2+6\ \iota_2^2))\big)\Bigg)\Big/\Bigg((-1+2\ \iota_1)^2\ (-2+3\ \iota_1)\ (-1+3\ \iota_1)\ \Big(\frac{(\iota_1)!_{.}^{\,3}\ (6\ \iota_1)!_{.}\ (-1)^{\iota_1}}{(2\ \iota_1)!_{.}^{\,3}\ (3\ \iota_1)!_{.}}\Big)_{\iota_1}\Bigg)\Bigg)\Big\}\Big\}$$

Note that the found sum is indefinite. More generally, these nested sums of indefinite sums are included in the class of d'Alembertian solutions, which are introduced in [AP94]; further results can be found in [Sin91, Sch01].

Solving the recurrence with quantifier elimination

As already mentioned above, the found sum is indefinite and therefore we can apply our **indefinite** summation algorithm in order to simplify that double nested sum further. I consider it as an essential step to eliminate algebraic relations and to simplify nested sums further to simpler nested sums. Therefore, by default, this is done automatically. In the previous attempt I had to force my algorithm not to simplify the found indefinite sum further by setting the option `AlgebraicRelationInSumSolutions→ True`. Omitting this option, we obtain the following inhomogeneous solution which is expressed just by a single nested sum:

In[18]:= $\textbf{recSol} = \textbf{SolveRecurrence}[\textbf{rec}[[1]], \textbf{SUM}[\textbf{n}], \textbf{NestedSumExt} \to \infty, \textbf{Tower} \to \textbf{tower}]$

31.09 Second

Out[18]= $$\left\{\left\{0, \left(\frac{((n)!.)^3\ (6\ n)!.\ (-1)^{n.}}{((2\ n)!.)^3\ (3\ n)!.}\right)_{n.}\right\}, \left\{0, n\left(\frac{(3\ n)!.\ (-1)^{n.}}{((n)!.)^3}\right)_{n.}\right\}, \left\{1, -\left(n\left(\frac{(3\ n)!.\ (-1)^{n.}}{((n)!.)^3}\right)_{n.} \left(60 - 887\ n + 4948\ n^2 - 13599\ n^3 + 19512\ n^4 - 13932\ n^5 + 3888\ n^6 + \left(2 - 25\ n + 117\ n^2 - 258\ n^3 + 270\ n^4 - 108\ n^5\right) \sum_{\iota_1=2}^{n} \left(\left(9360 - 64710\ \iota_1 + 63189\ \iota_1^2 + 413410\ \iota_1^3 - 1436799\ \iota_1^4 + 2117172\ \iota_1^5 - 1737846\ \iota_1^6 + 826740\ \iota_1^7 - 213840\ \iota_1^8 + 23328\ \iota_1^9\right) / \left((-1+\iota_1)\ \iota_1\ (-3+2\ \iota_1)\ (-5+3\ \iota_1)\ (-4+3\ \iota_1)\ \left(13 - 18\ \iota_1 + 6\ \iota_1^2\right)\ \left(1 - 6\ \iota_1 + 6\ \iota_1^2\right)\right)\right)\right)\right) / \left(12\ (-1+2\ n)\ (-2+3\ n)\ (-1+3\ n)\ \left(1 - 6\ n + 6\ n^2\right)\right)\right\}\right\}$$

Solving the recurrence with simple sums

Furthermore, by setting the option `SimpleSumRepresentation→True`, the sum extension is expressed by simpler sums:

In[19]:= $\textbf{recSol} = \textbf{SolveRecurrence}[\textbf{rec}, \textbf{SUM}[\textbf{n}], \textbf{NestedSumExt} \to \infty,$
$\textbf{Tower} \to \textbf{tower}, \textbf{SimpleSumRepresentation} \to \textbf{True}]$

`45.83 Second`

$$\text{Out[19]}= \Big\{\Big\{0, \Big(\frac{(n)!.^3\ (6\ n)!.\ (-1)^{n.}}{(2\ n)!.^3\ (3\ n)!.}\Big)_{n.}\Big\}, \Big\{0, n\ \Big(\frac{(3\ n)!.\ (-1)^{n.}}{(n)!.^3}\Big)_{n.}\Big\},$$
$$\Big\{1, \frac{1}{6\,(-1+2\ n)(-2+3\ n)(-1+3\ n)} \Big(\Big(\frac{(3\ n)!.\ (-1)^{n.}}{(n)!.^3}\Big)_{n.} \Big(-12+91\ n-245\ n^2+261\ n^3-$$
$$90\ n^4+\big(-10\ n+65\ n^2-135\ n^3+90\ n^4\big)\sum_{\iota_1=2}^{n}\Big(\frac{1}{-1+\iota_1}\Big)+$$
$$\big(-12\ n+78\ n^2-162\ n^3+108\ n^4\big)\sum_{\iota_1=2}^{n}\Big(\frac{1}{-3+2\ \iota_1}\Big)+$$
$$\big(6\ n-39\ n^2+81\ n^3-54\ n^4\big)\sum_{\iota_1=2}^{n}\Big(\frac{1}{-5+3\ \iota_1}\Big)+$$
$$\big(6\ n-39\ n^2+81\ n^3-54\ n^4\big)\sum_{\iota_1=2}^{n}\Big(\frac{1}{-4+3\ \iota_1}\Big)\Big)\Big)\Big\}\Big\}$$

Solving the recurrence with standard objects

Finally we notice that these sum extensions can be expressed by the Harmonic numbers. Therefore we can solve the recurrence directly by using the following sum and product extensions:

$$\text{In[20]:=}\ \textbf{tower} = \Big\{\Big(\frac{(\mathbf{3\ n})!.\ (-\mathbf{1})^{\mathbf{n}.}}{((\mathbf{n})!.)^{\mathbf{3}}}\Big)_{\mathbf{n}.}, \Big(\frac{((\mathbf{n})!.)^{\mathbf{3}}\ (\mathbf{6\ n})!.\ (-\mathbf{1})^{\mathbf{n}.}}{((\mathbf{2\ n})!.)^{\mathbf{3}}\ (\mathbf{3\ n})!.}\Big)_{\mathbf{n}.}, \mathbf{H_n}, \mathbf{H_{2\ n}}, \mathbf{H_{3\ n}}\Big\};$$

In[21]:= $\textbf{recSol} = \textbf{SolveRecurrence}[\textbf{rec}, \textbf{SUM}[\textbf{n}], \textbf{NestedSumExt} \to \infty, \textbf{Tower} \to \textbf{tower}]$

`41.52 Second`

$$\text{Out[21]}= \Big\{\Big\{0, \Big(\frac{(n)!.^3\ (6\ n)!.\ (-1)^{n.}}{(2\ n)!.^3\ (3\ n)!.}\Big)_{n.}\Big\}, \Big\{0, n\ \Big(\frac{(3\ n)!.\ (-1)^{n.}}{(n)!.^3}\Big)_{n.}\Big\},$$
$$\Big\{1, \frac{1}{6}\ (1+3\ n\ H_n+6\ n\ H_{2\ n}-3\ n\ H_{3\ n})\ \Big(\frac{(3\ n)!.\ (-1)^{n.}}{(n)!.^3}\Big)_{n.}\Big\}\Big\}$$

3.3 Finding a closed form

In the end, the closed form of `mySum` is that linear combination of the homogeneous solutions plus the inhomogeneous solution which has exactly the same initial values as `mySum`.

In[22]:= $\textbf{result} = \textbf{FindLinearCombination}[\textbf{recSol}, \textbf{mySum}, \mathbf{2}]$

`0.16 Second`

$$\text{Out[22]}= \frac{1}{6}\ (1+3\ n\ H_n+6\ n\ H_{2\ n}-3\ n\ H_{3\ n})\ \Big(\frac{(3\ n)!.\ (-1)^{n.}}{(n)!.^3}\Big)_{n.} - \frac{1}{6}\ \Big(\frac{(n)!.^3\ (6\ n)!.\ (-1)^{n.}}{(2\ n)!.^3\ (3\ n)!.}\Big)_{n.}$$

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