Computation of the Degree of Rational Maps Between Curves *

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ABSTRACT

The degree of a rational map measures how often the map covers the image variety. In particular, when the rational map is a parametrization, the degree measures how often the parametrization traces the image. We show how the degree of rational maps between algebraic curves can be determined efficiently. In the process, we also give a complete proof of Sederberg's approach for making a parametrization proper.

1. INTRODUCTION

We are dealing with rational maps between algebraic varieties, in particular algebraic curves. All statements about such a rational map are meant to hold for almost all points in the corresponding variety, i.e. for a non-empty (Zariski) open subset of the variety.

The degree of a rational map ϕ between two irreducible varieties W_1 and W_2 measures how often the map covers the image variety, i.e. the cardinality of the generic fibre $\phi^{-1}(P)$, for $P \in W_2$. If the degree is 1, this means that the rational map is rationally invertible, i.e. a birationality.

A rational parametrization of an algebraic variety is a special rational map, from a whole affine space onto the variety. Such a rational parametrization is proper, i.e. 1-1, if the degree of the rational map is 1. In this case the parametrization traces the variety exactly once.

In general, determining the degree of a rational map can be achieved by elimination theoretic methods. In the case of curves, we show that the degree can be computed by a few gcd computations. In the process, we also give a complete proof of Sederberg's approach for making a parametrization proper (see [?]).

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Related questions of proper reparametrization of curves are treated in [?]. The relation of the degree of a parametrization to the implicitization of curves is also analyzed in [?] and [?].

2. THE DEGREE OF A RATIONAL MAP

In this section we briefly recall some of the basic properties of the degree of a rational map defined between irreducible varieties of the same dimension; for further details we refer to [?] and [?].

Throughout this paper let $\mathbb{K}$ be an algebraically closed field of characteristic 0. Whenever we consider an arbitrary field $\mathbb{L}$, then by $\overline{\mathbb{L}}$ we denote the algebraic closure of $\mathbb{L}$.

Let W_1 and W_2 be irreducible varieties over $\mathbb{K}$, such that $\dim(W_1) = \dim(W_2)$. Let $\phi : W_1 \rightarrow W_2$ be a rational map such that $\phi(W_1) \subset W_2$ is dense, i.e. ϕ is dominant. Now, we consider the monomorphism $\phi^* : \mathbb{K}(W_2) \rightarrow \mathbb{K}(W_1)$ induced by ϕ over the fields of rational functions, and the field extensions

$$\mathbb{K} \subset \phi^*(\mathbb{K}(W_2)) \subset \mathbb{K}(W_1).$$

Then, since the transcendence degree of field extensions is additive, taking into account that $\dim(W_1) = \dim(W_2)$ and that ϕ is dominant, one has that the transcendence degree of $\mathbb{K}(W_1)$ over $\phi^*(\mathbb{K}(W_2))$ is zero, and hence the extension is algebraic. Moreover, since $\mathbb{K}(W_1)$ can be obtained by adjoining to $\phi^*(\mathbb{K}(W_2))$ the variables of W_1, we see that $\phi^*(\mathbb{K}(W_2)) \subset \mathbb{K}(W_1)$ is finite.

Definition 1. The *degree* of the dominant rational map ϕ from W_1 to W_2 is the degree of the finite algebraic field extension $\phi^*(\mathbb{K}(W_2)$ over $\mathbb{K}(W_1)$, that is

$$\operatorname{degree}(\phi) = [\mathbb{K}(W_1) : \phi^*(\mathbb{K}(W_2))].$$

Observe that the notion of degree can be used to characterize the birationality of rational maps as follows.

Lemma 1. *A dominant rational map $\phi : W_1 \rightarrow W_2$ between irreducible varieties of the same dimension is birational if and only if* $\operatorname{degree}(\phi) = 1$.

Also, taking into account the definition of degree of a rational map and that the degree of algebraic field extensions is multiplicative, one deduces the following lemma.

Lemma 2. *Let $\phi_1 : W_1 \to W_2$ and $\phi_2 : W_2 \to W_3$ be dominant rational maps between irreducible varieties of the same dimension. Then*

$$\text{degree}(\phi_2 \circ \phi_1) = \text{degree}(\phi_1) \cdot \text{degree}(\phi_2).$$

From the computational point of view, one may approach the determination of the degree of a rational map by computing the degree of the algebraic field extension. However, the problem can be approached using that the degree of the map is the cardinality of a generic fibre. Those points where the cardinality of the fibre does not equal the degree of the map are called *ramification points* of the rational map. More precisely, one may apply the following result (see Proposition 7.16 in [?]).

Theorem 1. *Let $\phi : W_1 \to W_2$ be a dominant rational map between irreducible varieties of the same dimension. There exists a non–empty open subset U of W_2 such that for every $P \in U$ the cardinality of the fibre $\phi^{-1}(P)$ is equal to* degree(ϕ).

Thus, a direct application of this result, combined with elimination techniques, provides a method for computing the degree. More precisely, let $W_1 \subset \mathbb{K}^r$ and $W_2 \subset \mathbb{K}^s$ be irreducible varieties of the same dimension defined over $\mathbb{K}$ by $\{F_1(\bar{x}), \ldots, F_n(\bar{x})\} \subset \mathbb{K}[\bar{x}]$, and $\{G_1(\bar{y}), \ldots, G_m(\bar{y})\} \subset \mathbb{K}[\bar{y}]$, respectively, where $\bar{x} = (x_1, \ldots, x_r), \bar{y} = (y_1, \ldots, y_s)$. Let

$$\phi = \left(\frac{\phi_1}{\phi_{s+1,1}}, \ldots, \frac{\phi_s}{\phi_{s+1,s}}\right) : W_1 \to W_2$$

be a dominant rational map, where $\phi_i, \phi_{s+1,i}$ are polynomials over $\mathbb{K}$, and $\gcd(\phi_i, \phi_{s+1,i}) = 1$.

Then, the following corollary follows from Theorem 1.

Corollary. *Let $\phi : W_1 \to W_2$ be a dominant rational map between irreducible varieties of the same dimension.*

(a) *Let $\overline{b} = (b_1, \ldots, b_s)$ be a generic element of W_2. Then the degree of ϕ is equal to the cardinality of the finite set*

$$\left\{ a \in W_1 \;\middle|\; \phi(a) = b, \prod_{i=1}^{s} \phi_{s+1,i}(a) \neq 0 \right\}.$$

(b) *Let $\overline{a} = (a_1, \ldots, a_r)$ be a generic element of W_1. Then the degree of ϕ is equal to the cardinality of the finite set*

$$\left\{ a' \in W_1 \;\middle|\; \phi(a') = \phi(a), \prod_{i=1}^{s} \phi_{s+1,i}(a') \neq 0 \right\}.$$

3. CASE OF RATIONAL PARAMETRIZATIONS

A rational parametrization of an algebraic variety W is a tuple of rational functions defining a dominant rational map from an affine space to W. In this section we consider the particular but important case of rational maps induced by such rational parametrizations. First we analyze the problem for general unirational varieties. Afterwards we show how these results can be improved for the case of rational curves.

Let W be a unirational variety defined over $\mathbb{K}$ by the rational parametrization

$$\mathcal{P}(t_1, \ldots, t_n) = \left(\frac{\chi_1}{\chi_{s+1,1}}, \ldots, \frac{\chi_s}{\chi_{s+1,s}}\right)$$

in reduced form; i.e. $\gcd(\chi_i, \chi_{s+1,i}) = 1$ for $i = 1, \ldots, s$. Associated with the parametrization $\mathcal{P}(t)$, we consider the following rational map

$$\begin{array}{cccc} \phi_{\mathcal{P}} : & \mathbb{K}^n & \longrightarrow & W \\ & (a_1, \ldots, a_n) & \longmapsto & \mathcal{P}(a_1, \ldots, a_n) \end{array}.$$

The degree of the parametrization $\mathcal{P}$ is the maximum degree of the rational function components of the parametrization; that is

$$\deg_t(\mathcal{P}) = \max\left\{ \deg_t\left(\frac{\chi_1}{\chi_{s+1,1}}\right), \ldots, \deg_t\left(\frac{\chi_s}{\chi_{s+1,s}}\right) \right\}.$$

Note that in general $\deg_t(\mathcal{P}) \neq \text{degree}(\phi_{\mathcal{P}})$.

Lemma 1 implies that $\mathcal{P}$ is proper if and only if degree$(\phi_{\mathcal{P}}) = 1$. In addition, degree$(\phi_{\mathcal{P}})$ measures, intuitively speaking, the number of times that $\mathcal{P}(\bar{t})$ traces W when $\bar{t}$ takes values in $\mathbb{K}^n$. This is specially clear when W is a curve. In [?], the tracing index of plane curve parametrizations (i.e. the degree of the induced rational map) is analyzed.

Now, for $i = 1, \ldots, s$ we introduce the polynomials

$$G_i(\bar{t}, \bar{h}) = \chi_i(\bar{t})\chi_{s+1,i}(\bar{h}) - \chi_i(\bar{h})\chi_{s+1,i}(\bar{t}) \;\; i = 1, \ldots, s$$

where $\bar{t} = (t_1, \ldots, t_n)$ and $\bar{h} = (h_1, \ldots, h_n)$. From the corollary to Theorem 1 we immediately get the following theorem.

Theorem 2. *Let $\overline{b}$ be a generic element of $\mathbb{K}^n$. Then*

$$\text{degree}(\phi_{\mathcal{P}}) = \text{Card}\left(\left\{ \bar{a} \in \mathbb{K}^n \;\middle|\; \begin{array}{l} G_1(\bar{a}, \bar{b}) = 0, \\ \vdots \\ G_s(\bar{a}, \bar{b}) = 0, \\ \prod_{i=1}^{s} \chi_{s+1,i}(\bar{a}) \neq 0 \end{array} \right\}\right).$$

Remark. Lemma 1 and Theorem 2 provide an algorithmic criterion for deciding the properness of a rational parametrization.

Example. We consider the rational parametrization

$$\mathcal{P}(t_1, t_2) = \left(\frac{\left(t_2^4 + t_1\right)^2}{t_2^2 + t_2^4 + t_1}, \frac{\left(t_2^4 + t_1\right) t_2^2}{t_2^2 + t_2^4 + t_1}, \frac{\left(t_2^4 + t_1\right)^3}{\left(t_2^2 + t_2^4 + t_1\right)^2} \right).$$

In order to compute the degree of the rational map $\phi_{\mathcal{P}}$, first we get

$$\{\bar{a} \in \overline{\mathbb{K}(b_1, b_2)}^{\,2} \mid G_i(\bar{a}, \bar{b}) = 0, i = 1, 2, 3\} =$$

$$= \{(0,0), (b_1, b_2), (b_1, -b_2)\}$$

Therefore, since the denominator of the rational map vanishes at $(0,0)$, we deduce that $\text{degree}(\phi_{\mathcal{P}}) = 2$. □

Space Curves

In this subsection we deal with the problem of computing the degree in the case of space curve parametrizations. In general, the application of Theorem 2 involves elimination techniques such as Gröbner bases. We will see in this section that, in the case of curves, Theorem 2 can be improved and the degree can be determined by gcd computations.

For this purpose, in the sequel, let $\mathcal{D}$ be the rational space curve defined by the rational parametrization in reduced form

$$\mathcal{P}(t) = \left(\frac{\chi_1(t)}{\chi_{s+1,1}(t)}, \ldots, \frac{\chi_s(t)}{\chi_{s+1,s}(t)}\right),$$

let $\phi_{\mathcal{P}}$ be the rational map induced by $\mathcal{P}$, and for $i = 1, \ldots, s$ let

$$G_i(t,h) = \chi_i(t)\chi_{s+1,i}(h) - \chi_i(h)\chi_{s+1,i}(t).$$

We start by showing that Theorem 2 can be improved for the case of curves.

Theorem 3. *Let b be a generic element of $\mathbb{K}$. Then*

$$\text{degree}(\phi_{\mathcal{P}}) = \text{Card}\left(\left\{ a \in \mathbb{K} \;\middle|\; \begin{array}{c} G_1(a,b) = 0, \\ \vdots \\ G_s(a,b) = 0 \end{array} \right\}\right).$$

Proof: Let $V_1 = \{a \in \mathbb{K} \mid G_i(a,b) = 0,\ i = 1, \ldots, s\}$, and $V_2 = \{a \in \mathbb{K} \mid \prod_{i=1}^{s} \chi_{s+1,i}(a) = 0\}$. We prove that $V_1 \cap V_2 = \emptyset$. Indeed: let $a_0 \in V_1 \cap V_2$, then there exists $i \in \{1, \ldots, s\}$ such that $\chi_{s+1,i}(a_0) = 0$. Hence, $\chi_i(b)\chi_{s+1,i}(a_0) = 0$. Thus, since $\chi_i(b)$ is not identically 0, because it is a denominator of $\mathcal{P}$ and b is generic, one gets that $\chi_{s+1,i}(a_0) = 0$. But this is impossible since $\gcd(\chi_i, \chi_{s+1,i}) = 1$.

Finally, since $V_1 \cap V_2 = \emptyset$, an application of Theorem 2 concludes the proof. □

Now, we see that the cardinality involved in Theorem 3 can be computed as the degree of a gcd. For this purpose we use the following technical lemma, which is proved in [?].

Lemma 3. *Let $p(t), q(t) \in \mathbb{K}[t]^\star$ be relatively prime such that at least one of them is non-constant, and let $R(h)$ be the resultant*

$$R(h) = \text{Res}_t(p(t) - hq(t), p'(t) - hq'(t)).$$

Then, $E = \{b \in \mathbb{K} \mid R(b) \neq 0\}$ is non–empty, and for all $b \in E$ the polynomial $p(t) - bq(t)$ is squarefree.

Theorem 4. *Let $\mathcal{P}$ be a rational parametrization of the space curve $\mathcal{D}$, and let the polynomials G_i be defined as above. Then*

$$\text{degree}(\phi_{\mathcal{P}}) = \deg_t(\gcd(G_1(t,h), \ldots, G_s(t,h))).$$

Proof. In order to prove Theorem 4 one just has to see that the primitive part $L(t,h)$ of $\gcd(G_1(t,h), \ldots, G_s(t,h))$ w.r.t. t is square–free. Let us assume that $L(t,h)$ is not square–free, and let us suppose w.l.o.g. that the first component of $\mathcal{P}$ is non–constant. Then $G_1, \ldots, G_s$ are not square–free, either. In particular, there exist non-constant polynomials $M, N \in \mathbb{K}[t,h]$, and $\ell \in \mathbb{N}, \ell \geq 2$, such that $G_1 = M^\ell N$. Note that, since L is primitive as a polynomial in $\mathbb{K}[h][t]$, one has that M depends on t. Now, consider the set $\Omega \subset \mathbb{K}$ consisting in all values of h such that $\chi_{s+1,1}(h) \neq 0$ and such that $M(t,h)$ is not a constant polynomial. Observe that $\chi_{s+1,1}(h)$ cannot be identically zero because it is a denominator, and that M does depend on t. Therefore, Ω is a non–empty open subset of $\mathbb{K}$. We consider the polynomial $G_1^*(t,h) = \chi_1(t) - \frac{\chi_1(h)}{\chi_{s+1,1}(h)}\chi_{s+1,1}(t)$ in $\mathbb{K}(h)[t]$. Note that the image of the first component of $\mathcal{P}$ is dense in $\mathbb{K}$. For every $h_0 \in \Omega$, since $\chi_{s+1,1}(h_0) \neq 0$, $G_1^*(t,h_0)$ is defined. Moreover, for every $h_0 \in \Omega$, $G_1^*(t,h_0)$ is not square–free, because $M(t,h_0)$ is non–constant, and the polynomials $\chi_1(t), \chi_{s,1}(t)$ are not simultaneosly constant (by assumption the first component of $\mathcal{P}$ is not constant) and coprime. But this is impossible because of Lemma 3. □

Applying Theorem 4 and Lemma 1, one gets the following characterization of the properness of a parametrization that fits perfectly with Sederberg's criterion (see [?]).

Corollary. *The parametrization $\mathcal{P}(t)$ is proper if and only if $\deg_t(\gcd(G_1(t,h), \ldots, G_s(t,h))) = 1$.*

Example. Let $\mathcal{D}$ be parametrized by

$$\mathcal{P}(t) = \left(\frac{t^2+1+t}{t^2+1}, \frac{t^3+3+t}{t^2+2}, \frac{t^5+1}{t^2+3}, \frac{t^2+t^4}{t^2}\right)$$

The polynomials $G_i(t,h)$ are

$$\begin{array}{rl} G_1(t,h) = & th^2 + t - ht^2 - h, \\ G_2(t,h) = & t^3h^2 + 2\,t^3 + 3\,h^2 + th^2 + 2\,t - h^3t^2 - 2\,h^3 \\ & -3\,t^2 - ht^2 - 2\,h, \\ G_3(t,h) = & t^5h^2 + 3\,t^5 + h^2 - h^5t^2 - 3\,h^5 - t^2, \\ G_4(t,h) = & t^2h^2\left(t^2 - h^2\right). \end{array}$$

Applying Theorem 4 one deduces that

$$\text{degree}(\phi_{\mathcal{P}}) = \deg_t(\gcd(G_1, G_2, G_3, G_4)) = \deg_t(t-h) = 1.$$

Therefore, $\mathcal{P}$ is proper. □

In addition to the previous analysis, we recall how the degree of $\phi_{\mathcal{P}}$ behaves under reparametrizations and how it is related to the degree of the curve.

Lemma 4. *Let $R(t) \in \mathbb{K}(t)$ be a non-constant rational* *function, and let $\phi_R : \mathbb{K} \to \mathbb{K}$ be the rational map induced by $R(t)$. Then* $\text{degree}(\phi_R) = \deg_t(R(t))$.

Proof: Let $R(t) = \frac{p(t)}{q(t)}$ be in reduced form, and let $G(t,h) = p(t)q(h) - p(h)q(t)$. A similar reasoning as in the proof of Theorem 3 shows that

$$\text{degree}(\phi_R) = \text{Card}(\{t \in \mathbb{K} \mid G(t,h) = 0\}).$$

Now, reasoning as in the proof of Theorem 4, one deduces that the primitive part w.r.t h of $G(t,h)$ is square–free. Moreover, $\deg_t(R) = \deg_t(G)$. Therefore we have

$$\text{Card}(\{t \in \mathbb{K} \mid G(t,h) = 0\}) = \deg_t(R).$$

$\square$

Theorem 5. *Let $R(t) \in \mathbb{K}(t) \setminus \mathbb{K}$. Then*

$$\text{degree}(\phi_{\mathcal{P}(R(t))}) = \deg_t(R(t)) \cdot \text{degree}(\phi_{\mathcal{P}(t)}).$$

Proof: This follows from Lemma 2 and Lemma 4. $\square$

Plane Curves

In this section, we treat the special case of plane curves. More precisely, we show how the degree of the map and the degree of the curve are related.

For this purpose, in the sequel, let $\mathcal{C}$ be the rational curve defined by the rational parametrization in reduced form

$$\mathcal{P}(t) = \left(\frac{\chi_1(t)}{\chi_{3,1}(t)}, \frac{\chi_2(t)}{\chi_{3,2}(t)} \right),$$

let $\phi_{\mathcal{P}}$ the rational map induced by $\mathcal{P}$, and let $G_1(t,h)$ and $G_2(t,h)$ be as above.

Clearly, the results presented for space curves can also be stated for plane curves. We start this analysis by summarizing these results.

Theorem 6. *Let $\mathcal{P}$ be a rational parametrization of the plane curve $\mathcal{C}$, and let the polynomials G_1, G_2 be defined as above. Then*

(1) $\text{degree}(\phi_{\mathcal{P}}) = \text{Card}\left(\{t \in \mathbb{K} \mid G_1(t,h) = G_2(t,h) = 0\}\right)$.

(2) $\text{degree}(\phi_{\mathcal{P}}) = \deg_t(\gcd(G_1(t,h), G_2(t,h)))$.

(3) The parametrization $\mathcal{P}(t)$ is proper if and only if $\deg_t(\gcd(G_1(t,h), G_2(t,h))) = 1$

(4) Let $R(t)$ be a non–constant rational function over $\mathbb{K}$. Then $\text{degree}(\phi_{\mathcal{P}(R(t))}) = \deg_t(R(t)) \cdot \text{degree}(\phi_{\mathcal{P}(t)})$.

In Theorem 8 we prove how the degree of a proper parametrization and the degree of the curve are related. In the next theorem we characterize the properness of the parametrization by means of the degree.

Theorem 7. *Let $f(x,y) \in \mathbb{K}[x,y]$ be the defining polynomial of the plane curve $\mathcal{C}$. Then $\mathcal{P}(t)$ is proper if and only if*

$$\deg(\mathcal{P}(t)) = \max\{\deg_x(f), \deg_y(f)\}.$$

Furthermore, if $\mathcal{P}(t)$ is proper, then $\deg(\frac{\chi_1}{\chi_{3,1}}) = \deg_y(f)$, and $\deg(\frac{\chi_2}{\chi_{3,2}}) = \deg_x(f)$.

Proof: We first prove the result for the special case of parametrizations having a constant component; i.e. for lines parallel to the axes. Afterwards, we consider the general case. Let $\mathcal{P}(t)$ be a parametrization such that one of its two components is constant, say $\mathcal{P}(t) = (\chi_1(t), \lambda)$ where $\lambda \in \mathbb{K}$. Then the curve $\mathcal{C}$ is the line defined by $y = \lambda$. Hence, (t, λ) is a proper parametrization of $\mathcal{C}$. So, every proper parametrization of $\mathcal{C}$ is of the form $(\frac{at+b}{ct+d}, \lambda)$, where $a, b, c, d, \in \mathbb{K}$ and $ad - bc \neq 0$. Therefore, $\deg(\chi_1) = 1$, and the theorem clearly holds.

In order to prove the general case, let $\mathcal{P}(t)$ be proper, in reduced form, and such that none of its components is constant. Then we prove that

$$\max\{\deg(\chi_2), \deg(\chi_{3,2})\} = \deg_x(f),$$

and analogously one can prove that $\max\{\deg(\chi_1), \deg(\chi_{3,1})\} = \deg_y(f)$. From these relations, we immediately get that $\deg(\mathcal{P}(t)) = \max\{\deg_x(f), \deg_y(f)\}$. For this purpose, we define $\mathcal{S}$ as the subset of $\mathbb{K}$ containing: (a) all the second coordinates of those points on $\mathcal{C}$ that are not generated by $\mathcal{P}(t)$; (b) those $b \in \mathbb{K}$ such that the polynomial $\chi_2(t) - b\chi_{3,2}(t)$ has multiple roots; (c) $lc(\chi_2)/lc(\chi_{3,2})$, where " lc" denotes the leading coefficient; (d) those $b \in \mathbb{K}$ such that the polynomial $f(x,b)$ has multiple roots; (e) the roots of the leading coefficient, with respect to x, of $f(x,y)$.

We claim that $\mathcal{S}$ is finite. Indeed: $\mathcal{P}(t)$ is a parametrization, so only finitely many points on the curve are not generated by $\mathcal{P}(t)$, and therefore only finitely many field elements satisfy (a). According to Lemma 3 there are only finitely many field elements satisfying (b). The argument for (c) is trivial. An element $b \in \mathbb{K}$ satisfies (d) iff b is the second coordinate of a singular point of $\mathcal{C}$ or the line $y = b$ is tangent to the curve at some simple point. Since $\mathcal{C}$ is irreducible, it has only finitely many singular points. Moreover, $y = b$ is tangent to $\mathcal{C}$ at some point (a,b) if (a,b) is a solution of the system $\{f = 0, \frac{\partial f}{\partial x} = 0\}$. However, by Bézout's Theorem, this system has only finitely many solutions; note that f is not a line. So only finitely many field elements satisfy (d). Since the leading coefficient, with respect to x, of $f(x,y)$ is a non-zero univariate polynomial (note that, since $\mathcal{C}$ is not a line, f is a non-linear irreducible bivariate polynomial), only finitely many field elements satisfy (e). Therefore, $\mathcal{S}$ is finite.

Now we take an element $b \in \mathbb{K} \setminus \mathcal{S}$ and we consider the intersection of $\mathcal{C}$ and the line of equation $y = b$. Since $b \notin \mathcal{S}$, by (e), one has that the degree of $f(x,b)$ is exactly $\deg_x(f(x,y))$, say $m := \deg_x(f(x,y))$. Furthermore, by (d), $f(x,b)$ has m different roots, say $\{r_1, \ldots, r_m\}$. So, there are m different points on $\mathcal{C}$ having b as a second coordinate (i.e. $\{(r_i, b)\}_{i=1,\ldots,m}$), and they can be generated by $\mathcal{P}(t)$, because of (a).

On the other hand, we consider the polynomial $M(t) = \chi_2(t) - b\chi_{3,2}(t)$. We note that, since every point (r_i, b) is generated by some value of the parameter t, $\deg_t(M) \geq m$. But, since $\mathcal{P}(t)$ is proper, and since M cannot have multiple roots, we get that $\deg_t(M) = m = \deg_x(f(x,y))$. Now, since b is not the quotient of the leading coefficients of χ_2 and $\chi_{3,2}$, one has that $\deg_x(f(x,y)) = \deg(M) = \max\{\deg(\chi_2), \deg(\chi_{3,2})\}$.

Conversely, let $\mathcal{P}(t)$ be a parametrization of $\mathcal{C}$ such that $\deg(\mathcal{P}(t)) = \max\{\deg_x(f), \deg_y(f)\}$. By Lüroth's theorem there is a proper parametrization of $\mathcal{C}$. Let $\mathcal{Q}(t)$ be any proper parametrization of $\mathcal{C}$. Then, there exists $R(t) \in \mathbb{K}(t)$ such that $\mathcal{Q}(R(t)) = \mathcal{P}(t)$. Now, since $\mathcal{Q}(t)$ is proper, one deduces that $\deg(\mathcal{Q}(t)) = \max\{\deg_x(f), \deg_y(f)\} = \deg(\mathcal{P}(t))$. Therefore, since the degree is multiplicative with respect to composition, one deduces that $R(t)$ is of degree one, and hence invertible. Thus, $\mathcal{P}(t)$ is proper. □

Theorem 8. *Let $f(x,y) \in \mathbb{K}[x,y]$ be the defining polynomial of $\mathcal{C}$, and let $n = \max\{\deg_x(f), \deg_y(f)\}$. Then,*

$$\text{degree}(\phi_{\mathcal{P}}) = \frac{\deg(\mathcal{P}(t))}{n}.$$

Proof: By Lüroth's theorem it follows that there exists a proper parametrization $\mathcal{P}'(t)$ of $\mathcal{C}$, and there exists $R(t) \in \mathbb{K}(t) \setminus \mathbb{K}$ such that $\mathcal{P}(t) = \mathcal{P}'(R(t))$. Applying Theorem 4 and using the fact that $\mathcal{P}'(t)$ is proper, we can derive that

$$\text{degree}(\phi_{\mathcal{P}}) = \deg_t(R(t)) \cdot \text{degree}(\phi_{\mathcal{P}'}) = \deg_t(R(t)).$$

Furthermore, we also have

$$\deg(\mathcal{P}(t)) = \deg_t(R(t)) \cdot \deg(\mathcal{P}'(t)).$$

Thus,

$$\text{degree}(\phi_{\mathcal{P}}) = \frac{\deg(\mathcal{P}(t))}{\deg(\mathcal{P}'(t))}.$$

Moreover, taking into account that $\mathcal{P}'(t)$ is proper, by Theorem 5 one has that $\deg(\mathcal{P}'(t)) = n$. Therefore the theorem holds. □

Finally we can combine these results to show how the degree of $\phi_{\mathcal{P}}$ appears in the implicitization problem. Proofs of this relation appear in [**?**] and also in [**?**] and [**?**].

Theorem 9. *Let $f(x,y)$ be the defining polynomial of $\mathcal{C}$, let $\mathcal{P}$ be a parametrization of $\mathcal{C}$, and let $m = \text{degree}(\phi_{\mathcal{P}})$. Then for some constant $\lambda \in \mathbb{K}$ we have*

$$\text{Res}_t(\chi_{3,1}(t)x - \chi_1(t), \chi_{3,2}(t)y - \chi_2(t)) = \lambda \cdot (f(x,y))^m.$$

4. RATIONAL MAPS BETWEEN RATIONAL CURVES

In this section we deal with the problem of computing the degree of a dominant rational map between two rational curves. For this purpose, let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two rational space curves, let

$$\mathcal{P}(t) = \left(\frac{\chi_1(t)}{\chi_{s+1,1}(t)}, \ldots, \frac{\chi_s(t)}{\chi_{s+1,s}(t)}\right),$$

be a rational parametrization of $\mathcal{D}_1$ in reduced form, let $\phi_{\mathcal{P}}$ be the rational map induced by $\mathcal{P}$, and let

$$\phi : \mathcal{D}_1 \to \mathcal{D}_2$$

be a dominant rational map between $\mathcal{D}_1$ and $\mathcal{D}_2$. In this situation $\phi(\mathcal{P}(t))$ is a rational parametrization of $\mathcal{D}_2$. Let us say that $\phi(\mathcal{P}(t))$ is written as

$$\phi(\mathcal{P}(t)) = \left(\frac{\psi_1(t)}{\psi_{r+1,1}(t)}, \ldots, \frac{\psi_r(t)}{\psi_{r+1,r}(t)}\right).$$

Also we consider, for $i = 1, \ldots, s$, the polynomials:

$$G_i(t,h) = \chi_i(t)\chi_{s+1,i}(h) - \chi_i(h)\chi_{s+1,i}(t),$$

and, for $i = 1, \ldots, r$, the polynomials

$$\tilde{G}_i(t,h) = \psi_i(t)\psi_{r+1,i}(h) - \psi_i(h)\psi_{r+1,i}(t).$$

In this situation, the following theorem holds.

Theorem 10.

$$\text{degree}(\phi) = \frac{\deg_t(\gcd(\tilde{G}_1, \ldots, \tilde{G}_r))}{\deg_t(\gcd(G_1, \ldots, G_s))}.$$

Proof: Let $\mathcal{Q}(t) = \phi(\mathcal{P}(t))$ and $\phi_{\mathcal{Q}}$ be the map induced by $\mathcal{Q}$. Then

$$\phi_{\mathcal{Q}} : \mathbb{K} \xrightarrow{\phi_{\mathcal{P}}} \mathcal{D}_1 \xrightarrow{\phi} \mathcal{D}_2.$$

Applying Lemma 2 one has that $\text{degree}(\phi_{\mathcal{Q}}) = \text{degree}(\phi_{\mathcal{P}}) \cdot \text{degree}(\phi)$. Moreover, from Theorem 4 we get that

$$\text{degree}(\phi_{\mathcal{Q}}) = \deg_t(\gcd(\tilde{G}_1, \ldots, \tilde{G}_r)),$$
$$\text{degree}(\phi_{\mathcal{P}}) = \deg_t(\gcd(G_1, \ldots, G_s)).$$

Therefore, the theorem holds. □

Corollary. *If $\mathcal{P}(t)$ is proper then*

$$\text{degree}(\phi) = \deg_t(\gcd(\tilde{G}_1, \ldots, \tilde{G}_r)).$$

Remark. Note that Theorem 10 provides an algorithmic criterion for deciding whether ϕ is a birational map.

In addition, a result similar to Theorem 8 can also be established for plane curves.

Theorem 11. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be rational plane curves with defining polynomials $f(x,y)$ and $g(x,y)$, respectively. If $n = \max\{\deg_x(f), \deg_y(f)\}$ and $m = \max\{\deg_x(g), \deg_y(g)\}$, then,*

$$\frac{m}{n} = \frac{\deg(\phi(\mathcal{P}(t))}{\deg(\mathcal{P}(t)) \cdot \text{degree}(\phi)}.$$

Proof: Let $\mathcal{Q}(t) = \phi(\mathcal{P}(t)$ and $\phi_{\mathcal{Q}}$ be the map induced by $\mathcal{Q}$. By Theorem 8 one has

$$\text{degree}(\phi_{\mathcal{P}}) = \frac{\deg(\mathcal{P}(t))}{n}, \ \text{degree}(\phi_{\mathcal{Q}}) = \frac{\deg(\mathcal{Q}(t))}{m}.$$

Furthermore, Lemma 2 implies that

$$\text{degree}(\phi_{\mathcal{Q}}) = \text{degree}(\phi_{\mathcal{P}}) \cdot \text{degree}(\phi).$$

Therefore, the theorem holds. □

Example. Let $\mathcal{D}_1$ be the plane curve of implicit equation

$$f(x,y) = 3\,x^3y^2 + 6\,x^2y + x^2y^2 - 2\,xy - y + 8\,x^3y + 8\,x^3,$$

and $\mathcal{D}_2$ be the plane curve of implicit equation

$g(x,y) = 85 - 26\,x - 144\,y + 55\,xy - 18\,x^2 + 149\,y^2 + 3\,x^4 + 9\,x^3y - 3\,x^4y + 26\,x^2y^2 - 8\,x^3y^2 - 74\,xy^2 - x^3 - 8\,x^2y^3 - 144\,y^3 + 64\,y^4 + 48\,xy^3$

We want to analyze whether the rational map

$$\begin{array}{cccc} \phi: & \mathcal{D}_1 & \longrightarrow & \mathcal{D}_2 \\ & (x,y) & \longmapsto & \left(\frac{x^2+1}{x}, \frac{y+x}{y}\right) \end{array}$$

is birational. For this purpose, we consider a proper rational parametrization of $\mathcal{D}_1$, namely

$$\mathcal{P}(t) = \left(\frac{t}{t^2+2}, \frac{t^3}{t+1}\right).$$

Now, we compute

$$\phi(\mathcal{P}(t)) = \left(\frac{5\,t^2 + t^4 + 4}{t\,(t^2+2)}, \frac{t^4 + 2\,t^2 + t + 1}{t^2\,(t^2+2)}\right),$$

and the polynomials

$\tilde{G}_1(t,h) = 5\,t^2h^3 + 10\,t^2h + h^3t^4 + 2\,ht^4 + 4\,h^3 + 8\,h - 5\,h^2t^3 - 10\,h^2t - t^3h^4 - 2\,th^4 - 4\,t^3 - 8\,t$

$\tilde{G}_2(t,h) = th^4 + 2\,h^2t + h^4 + 2\,h^2 - ht^4 - 2\,t^2h - t^4 - 2\,t^2.$

So we see that

$$\gcd(\tilde{G}_1, \tilde{G}_2)) = t - h.$$

Therefore, applying the corollary to Theorem 10, we conclude that ϕ is a birational map.

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