Degree Bounds to Find Polynomial Solutions of Parameterized Linear Difference Equations in ΠΣ-Fields*

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Abstract An important application of solving parameterized linear difference equations in ΠΣ-fields, a very general class of difference fields, is simplifying and proving of nested multisum expressions and identities. Together with other reduction techniques described elsewhere, the algorithms considered in this article can be used to search for all solutions of such difference equations. More precisely, within a typical reduction step one often is faced with subproblems to find all solutions of linear difference equations where the solutions live in a polynomial ring. The algorithms under consideration deliver degree bounds for these polynomial solutions.

Key words Linear Difference Equations – ΠΣ-Fields – Degree Bounds

1 Introduction

M. Karr defined in [11,12] a very general class of difference fields, so called ΠΣ-fields, under two aspects. First, ΠΣ-fields allow us to describe indefinite nested multi-sums in a formal way, and second one is capable of solving first order linear difference equations in ΠΣ-fields; this amounts to simplify indefinite nested multi-sums by elimination of sum quantifiers. In [18,19] we streamlined Karr's ideas based on [6] to a compact algorithm and generalized the underlying reduction techniques which enables one to search for all solutions of parameterized linear difference equations of arbitrary order

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in $\Pi\Sigma$-fields; see problem *PLDE*. With this algorithm one is not only able to deal with indefinite summation, but one can also prove various definite multi-sum identities by applying Zeilberger's creative telescoping trick [26]. Moreover, by using our algorithm one can solve recurrences, obtained by creative telescoping, and hence one can discover definite multi-sum identities. These algorithms are available in form of a package called Sigma [17, 23, 21] in the computer algebra system Mathematica.

In order to solve parameterized linear difference equations in $\Pi\Sigma$-fields, one generates a reduction process that is introduced in [19]. In this reduction one is faced with subproblems to find all solutions of parameterized linear difference equations where the solutions live in a polynomial ring. Then one can apply further reduction techniques given in [19, 20] to solve this difference equation if one knows a degree bound of the solutions in that polynomial ring. As illustrated in Section 2 these reduction techniques are well known in one of the simplest cases of $\Pi\Sigma$-fields. In particular in [1, 14, 4, 16] one computes these degree bounds of the polynomial solutions.

In this article we consider degree bounds of linear difference equations in the more general setting of $\Pi\Sigma$-fields. Namely, based on [11] we develop algorithms to compute degree bounds for first order linear difference equations. Whereas in Karr's work theoretical and computational aspects are mixed, we try to separate his results in several parts to achieve more transparency. Furthermore, all proof steps are carefully carried out, whereas in Karr's work the essential proofs are omitted.
Similarly to the first order case, one needs degree bounds for linear difference equations of higher order. As it turns out, it is much harder to find such degree bounds. In this article we generalize Karr's degree bounds to the higher order case which enables one to treat at least some special cases of linear difference equations. In this sense the degree bounds under consideration contribute to important developments to solve linear difference equations in $\Pi\Sigma$-fields.

First the degree bound problem is introduced in the context of symbolic summation. After defining $\Pi\Sigma$-fields in Section 3, some basic strategies for finding degree bounds are given in Section 4. In Sections 5 and 6 methods are developed that find various degree bounds. Especially in Section 7 this leads us to an algorithm that solves the degree bound problem for first order linear difference equations. Furthermore, we show some important properties of this algorithm which are the key step in order to obtain refined summation algorithms in [22]. Further degree bounds are given in Section 8.

2 Symbolic Summation and the Degree Bound Problem

In [2, 1, 3] S. Abromov is concerned in finding all solutions $g(k)$ in the field of rational functions $\mathbb{K}(k)$ with characteristic 0 that fulfill linear difference equations of the type

$$a_m(k)\,g(k+m)+\cdots+a_0(k)\,g(k)=f(k) \tag{1}$$

where $a_i(k)$ and $f(k)$ are polynomials in $\mathbb{K}[k]$. Looking closer at this problem, one immediately sees that this problem can be formalized in the difference field $(\mathbb{K}(k), \sigma)$ with the field automorphism $\sigma : \mathbb{K}(k) \to \mathbb{K}(k)$ defined by $\sigma(k) = k + 1$.

More generally, a *difference field* is a field $\mathbb{F}$ together with a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$, which we denote by $(\mathbb{F}, \sigma)$. Moreover, the subset $\mathbb{K} := \{k \in \mathbb{F} \,|\, \sigma(k) = k\}$ is called the *constant field* of the difference field $(\mathbb{F}, \sigma)$. It is easy to see that the constant field $\mathbb{K}$ of a difference field $(\mathbb{F}, \sigma)$ is a subfield of $\mathbb{F}$. Subsequently, we assume that **all** fields are of characteristic 0. Since for any automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ and any $q \in \mathbb{Q}$ we have $\sigma(q) = q$, it follows that $\mathbb{Q}$ is a subfield of $\mathbb{K}$.

Problem (1) is included in the following problem.

PLDE: Solving **P**arameterized **L**inear **D**ifference **E**quations.

- **Given** $(a_1 \dots, a_m) \in \mathbb{F}^m$ with $a_1\, a_m \neq 0$ and $(f_1, \dots, f_n) \in \mathbb{F}^n$.
- **Find all** $g \in \mathbb{F}$ and $(c_1, \dots, c_n) \in \mathbb{K}^n$ with

$$a_1\, \sigma^{m-1}(g) + \dots + a_m\, g = c_1\, f_1 + \dots + c_n\, f_n. \tag{2}$$

Note that in any difference field $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, the field $\mathbb{F}$ can be interpreted as a vector space over $\mathbb{K}$. Hence problem *PLDE* can be described by the following set called solution space.

Definition 1 *Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$ and let $\mathbb{V}$ be a subspace of $\mathbb{F}$ over $\mathbb{K}$. Let $\mathbf{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}^m$ and $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. We define the* solution space *for $\boldsymbol{a}, \boldsymbol{f}$ in $\mathbb{V}$ by*

$$\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V}) = \{(c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{V} : \ (2)\ \textit{holds}\}.$$

It is easy to see that $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$ is a vector space over $\mathbb{K}$. Moreover, in [19] based on [8, Thm. XII (page 272)] it is proven that this vector space has finite dimension.

Proposition 1 *Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$ and assume $\boldsymbol{f} \in \mathbb{F}^n$ and $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^m$. Let $\mathbb{V}$ be a subspace of $\mathbb{F}$ over $\mathbb{K}$. Then $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$ is a vector space over $\mathbb{K}$ with maximal dimension $m + n - 1$.*

Problem *PLDE* plays an important role in symbolic summation. For instance, take $\mathbb{F} = \mathbb{K}(k)$ and consider the case $m = 2$ with $a_1 = 1$ and $a_2 = -1$. Then for $n = 1$ one obtains telescoping and therefore indefinite summation for a rational function $f_1 = f'(k) \in \mathbb{K}(k)$. Moreover, specializing to $\mathbb{K} = \mathbb{K}'(\nu)$ and $f_i = f'(\nu + i - 1, k) \in \mathbb{K}'(\nu)(k)$ for $1 \leq i \leq n$, one formulates the creative telescoping problem [26] of order $n - 1$ for definite rational sums. Furthermore, setting $n = 1$ in problem *PLDE* is nothing else than (1). Finally, solving the general problem *PLDE* allows us to sum over ∂-finite summand expressions as described in [7].

Slight variations of the algorithms in [2,3,1] allow us to solve problem *PLDE* for the difference field $(\mathbb{K}(k), \sigma)$ from above. More generally, with results from this article and from [19,20], algorithms have been developed that can attack problem *PLDE* in ΠΣ-fields, introduced in [11,12].

Loosely spoken, these are difference fields $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ where $\mathbb{F}:=\mathbb{K}(t_1)\dots,(t_e)$ is a rational function field and the application of σ on the t_i's is recursively defined over $1\leq i\leq e$ with $\sigma(t_i)=\alpha_i\,t_i+\beta_i$ for $\alpha_i,\beta_i\in\mathbb{K}(t_1)\dots(t_{i-1})$; we omitted some technical conditions given in Section 3. This means, as worked out in [23,21], that one can discover and prove a huge class of indefinite and definite multi-sum identities by applying the summation principles of *telescoping*, *creative telescoping* and *solving recurrences* in the $\Pi\Sigma$-field setting; see for instance [13,9,10,5]. Subsequently, we will illustrate our algorithms with the following summation examples.

Example 1 **Given** the k-th harmonic numbers $H_k:=\sum_{i=1}^{k}$. **Show** that there does not exist an $r(k)\in\mathbb{Q}(k)$ with $r(k)=H_k$. Suppose such an $r(k)$ exists. Then we have $r(k+1)-r(k)=\frac{1}{k+1}$. In Example 7 we show (algorithmically) that such an $r(k)$ does not exist in the difference field $(\mathbb{Q}(k),\sigma)$ with $\sigma(k)=k+1$, a contradiction.

Example 2 **Given** the summand $f(k)=H_k(kH_k-1)/k^2$. **Find** a closed form for the sum $\sum_{k=1}^{n}f(k)$ in terms of H_k and the harmonic numbers of 3rd kind $H_k^{(3)}=\sum_{i=1}^{k}\frac{1}{i^3}$. In order to accomplish this task, we first construct the $\Pi\Sigma$-field $(\mathbb{Q}(k)(h_3)(h),\sigma)$ where σ is defined by $\sigma(k)=k+1$, $\sigma(h_3)=h_3+\frac{1}{(k+1)^3}$ and $\sigma(h)=h+\frac{1}{k+1}$. Note that the k-shifts $S_k\,H_k=H_k+\frac{1}{k+1}$ and $S_k\,H_k^{(3)}=H_k^{(3)}+\frac{1}{(k+1)^3}$ are reflected by the action of σ on h and h_3. Then with our algorithms, see Example 8, we obtain for the difference equation

$$\sigma(g)-g=f \tag{3}$$

with $f=h(h\,k-1)/k^2$ the solution $g=\frac{1}{3k^2}\big[3h-3k\,h^2+k^2h^3-k^2h_3\big]$ with the shifted version $\sigma(g)=\frac{1}{3}\big[h^3-h_3\big]$. In other words, we obtain for the *telescoping equation*

$$g(k+1)-g(k)=f(k) \tag{4}$$

the solution $g(k+1)=\frac{1}{3}\big[H_k^3-H_k^{(3)}\big]$ and $g(k)=\frac{1}{3k^2}\big[3H_k-3k\,H_k^2+k^2H_k^3-k^2H_k^{(3)}\big]$. By telescoping, we get the closed form

$$\sum_{k=1}^{n}\frac{H_k(kH_k-1)}{k^2}=\frac{1}{3}\Big[H_n^3-H_n^{(3)}\Big].$$

Example 3 **Given** the summand $f(k)=\frac{H_k(3k(k+1)H_k+3k-1)}{k^2(kH_k-3)}\prod_{i=3}^{k}\frac{iH_i-3}{iH_i}$. **Find** a closed form evaluation of the sum $\sum_{k=3}^{n}f(k)$ in terms of the sums and products given in $f(k)$. First we construct the $\Pi\Sigma$-field $(\mathbb{Q}(k)(h)(p),\sigma)$ where σ is defined by $\sigma(k)=k+1$, $\sigma(h)=h+\frac{1}{k+1}$ and $\sigma(p)=\frac{h(k+1)-2}{h(k+1)+1}p$. Note that the k-shift $S_k\prod_{i=3}^{k}\frac{iH_i-3}{iH_i}=\frac{H_k(k+1)-2}{H_k(k+1)+1}\prod_{i=3}^{k}\frac{iH_i-3}{iH_i}$ is reflected by the action of σ on p. Then with our algorithms, see Example 9, we obtain for (3) with $f=\frac{h(3k(k+1)h+3k-1)}{k^2(k\,h-3)}p$ the solution $g=-\frac{h(hk-1)^2(k(h+3)-1)}{k^2(hk-3)}p$ with the shifted version $\sigma(g)=-h^2(h+3)p$. In other words, we obtain for (4) the solutions $g(k+1) = -H_k^2(H_k+3)\prod_{i=3}^{k}\frac{iH_i-3}{iH_i}$ and $g(k) = -\frac{H_k(H_k k-1)^2(k(H_k+3)-1)}{k^2(H_k k-3)}\prod_{i=3}^{k}\frac{iH_i-3}{iH_i}$. By telescoping, we arrive at

$$\sum_{k=3}^{n}\frac{H_k\big(3k(k+1)H_k+3k-1\big)\prod_{i=3}^{k}\frac{iH_i-3}{iH_i}}{k^2(kH_k-3)} = \frac{81}{8} - \big(H_n^3+3H_n^2\big)\prod_{i=3}^{n}\frac{iH_i-3}{iH_i}. \tag{5}$$

Example 4 Similarly, one is able to derive the identity

$$\prod_{k=3}^{n}\frac{(kH_k-3)((k+1)H_k-2)^3}{kH_k((k+1)H_k+1)^3} = \frac{1331}{125}\frac{((n+1)H_n-2)^3}{((n+1)H_n+1)^3}\left(\prod_{i=3}^{n}\frac{iH_i-3}{iH_i}\right)^4. \tag{6}$$

Namely, solving

$$\sigma(g) - \frac{(k\,h-3)((k+1)h-2)^3}{k\,h((k+1)h+1)^3}g = 0 \tag{7}$$

in the ΠΣ-field $(\mathbb{Q}(k)(h)(p),\sigma)$, as in Example 3, gives the solution $g = \frac{k\,h\,p^4}{k\,h-3}$ and its shifted version $\sigma(g) = \frac{((k+1)h-2)^3}{((k+1)h+1)^3}p^4$; see Example 10. Therefore we obtain the solution $g(k+1) = \frac{((k+1)H_k-2)^3}{((k+1)H_k+1)^3}\left(\prod_{i=3}^{k}\frac{iH_i-3}{iH_i}\right)^4$ for $g(k+1) - \frac{(kH_k-3)((k+1)H_k-2)^3}{kH_k((k+1)H_k+1)^3}g(k) = 0$. This gives identity (6).

Example 5 Prove that

$$\sum_{k=0}^{n} H_k\,H_{n-k} = 2n - 2nH_n + (1+n)H_n^2 - (n+1)H_n^{(2)}, \quad n \geq 0 \tag{8}$$

where $H_n^{(2)} = \sum_{i=1}^{n}\frac{1}{i^2}$. Denote $f(n,k) := H_k\,H_{n-k}$. Then we try to solve the following *creative telescoping problem* [26]: find constants $c_1(n)$ and $c_2(n)$, not all zero, and $g(n,k)$ such that

$$g(n,k+1) - g(n,k) = c_1(n)f(n,k) + c_2(n)f(n+1,k). \tag{9}$$

We accomplish this task in the ΠΣ-field $(\mathbb{Q}(n)(k)(h)(h'),\sigma)$ with constant field $\mathbb{Q}(n)$ defined by $\sigma(k) = k+1$, $\sigma(h) = h + \frac{1}{k+1}$ and $\sigma(h') = h' - \frac{1}{n-k}$. Note that the k-shift $S_k\,H_{n-k} = H_{n-k} - \frac{1}{n-k}$ is reflected by the action of σ on h'. Moreover, observe that $S_n H_{n-k} = H_{n-k} + \frac{1}{n-k+1}$. Hence, (9) can be rephrased as follows: find $c_i \in \mathbb{Q}(n)$, not all zero, and $g \in \mathbb{Q}(n)(k)(h)(h')$ with

$$\sigma(g) - g = c_1\,h\,h' + c_2\,h(h' + \frac{1}{n-k+1}). \tag{10}$$

Now we apply our algorithms, see Example 11, and we obtain the solution

$$\begin{aligned} c_1 =& n+2, \quad c_2 = -(n+1),\\ \sigma(g) =& k(2+h(-1+h')-h') + h(-1+h') + (1+n)h',\\ g =& (1+n)^2h' + k(2+n(-2+h)(-1+h') + (-2+h)h')\\ &+ k^2(-2+h+h'-hh')/(1+n-k); \end{aligned} \tag{11}$$

we get the solution for (9) by replacing h and h' with H_k and H_{n-k}. Denote the left hand side of (8) by $S(n)$. Now summing the creative telescoping equation (9) over k from 0 to n and using the relation $S(n+1) = \sum_{k=0}^{n} f(n+1,k) + f(n+1,n+1)$, one obtains the recurrence relation

$$(n+1)S(n+1) - (n+2)S(n) = 2(n+1)H_n - 2n.$$

Finally, it is an easy exercise to check that also the right hand side of (8) is a solution of this recurrence. Checking that both sides of (8) are equal for $n = 0$ proves the identity. For further details we refer to [21].

Example 6 Express the sum $S(n) := \sum_{k=1}^{n} \frac{(-1)^k}{k^3}\binom{n}{k}$ in terms of H_n, $H_n^{(2)}$ and $H_n^{(3)}$. In order to accomplish this task, we first derive for our sum the recurrence relation

$$\begin{aligned}(n+3)^3 S(n+3) &- (n+3)(3n^2+15n+19)S(n+2)\\ &+ 3(n+2)^2(n+3)S(n+1) - (n+1)(n+2)(n+3)S(n) = -1 \quad (12)\end{aligned}$$

by creative telescoping; one can either use our difference field algorithms, like in Example 5, or one can directly apply Zeilberger's algorithm [26]. Second, we solve this recurrence in terms of H_n, $H_n^{(2)}$ and $H_n^{(3)}$. Namely, we take the $\Pi\Sigma$-field $\mathbb{Q}(n)(h_2)(h_3)(h)$ with $\sigma(n) = n+1$, $\sigma(h_2) = h_2 + \frac{1}{(n+1)^2}$, $\sigma(h_3) = h_3 + \frac{1}{(n+1)^3}$ and $\sigma(h) = h + \frac{1}{n+1}$, and we compute the basis

$$\{(0,1), (0,h), (0,h^2+h_2), (1, -\frac{1}{6}(h^3 + 3hh_2 + 2h_3))\} \quad (13)$$

over $\mathbb{Q}$ for the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{Q}(n)(h_2)(h_3)(h))$ with $\boldsymbol{a} = \big((n+3)^3, -(n+3)(3n^2+15n+19), 3(n+2)^2(n+3), -(n+1)(n+2)(n+3)\big)$ and $\boldsymbol{f} = (-1)$; see Example 12. This means that we obtain the solutions 1, H_n, $H_n^2 + H_n^{(2)}$ of the homogeneous version of (12) and the particular solution $-\frac{1}{6}\big(H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}\big)$. Finally, we obtain the identity

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^3}\binom{n}{k} = -\frac{1}{6}\Big(H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}\Big) \quad (14)$$

by composing the particular linear combination of the homogeneous solutions plus the inhomogeneous solution that matches the first three initial values of $S(n)$ for $n = 0, 1, 2$.

All these summation problems can be solved automatically with the summation package Sigma [18]. More precisely, one solves problem *PLDE* for a $\Pi\Sigma$-field $(\mathbb{F}(t), \sigma)$ with constant field $\mathbb{K}$, i.e., computes a basis of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$. In order to accomplish this task, the following reduction is applied.

(I) In a first step we look for a *universal denominator bound*, i.e., a polynomial $d \in \mathbb{F}[t]^*$ such that for all $(c_1, \dots, c_n, g) \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ we have $d\,g \in \mathbb{F}[t]$. Then given such a $d \in \mathbb{F}[t]^*$, (2) holds if and only if

$$\frac{a_1}{\sigma^{m-1}(d)}\,\sigma^{m-1}(g') + \dots + \frac{a_{m-1}}{\sigma(d)}\,\sigma(g') + \frac{a_m}{d}\,g' = c_1\,f_1 + \dots + c_n\,f_n \quad (15)$$

holds with $g' = g\,d \in \mathbb{F}[t]$. Hence, if we can compute a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{F}[t])$ for $\boldsymbol{a'} := (\frac{a_1}{\sigma^{m-1}(d)}, \dots, \frac{a_{m-1}}{\sigma(d)}, \frac{a_m}{d}) \in \mathbb{F}(t)^m$, say $\{(c_{i1}, \dots, c_{in}, g_i)\}_{1 \leq i \leq r}$, one gets the basis $\{(c_{i1}, \dots, c_{in}, \frac{g_i}{d})\}_{1 \leq i \leq r}$ for $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$.

Remark. This strategy popped up the first time in [2,3] for the case $\mathbb{F} = \mathbb{K}$ and $\sigma(t) = t+1$; see also [25]. In [20], based on [11,6], these ideas are carried over to general ΠΣ-fields.

Summarizing, it suffices to find a basis of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$ for a given $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$.

(II) Next we restrict the solution range $\mathbb{F}[t]$ to a finite dimensional subspace

$$\mathbb{F}[t]_b := \{f \in \mathbb{F}[t] \mid \deg(f) \leq b\}$$

of $\mathbb{F}[t]$ over $\mathbb{F}$ for some $b \in \mathbb{N}_0 \cup \{-1\}$, i.e., we consider the following problem.

DegB: **Deg**ree **B**ounding

- **Given** a ΠΣ-field $(\mathbb{F}(t), \sigma)$, $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$.
- **Find** a *degree bound* $b \in \mathbb{N}_0 \cup \{-1\}$ such that $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]) = \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_b)$.

Remark. In [1,14,4,16] several algorithms are introduced that determine a degree bound of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{K}[t])$; as it turns out in [15], all these algorithms are equivalent and compute the same degree bounds.

(III) Finally, one has to find a basis for $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_b)$. For the rational case $\mathbb{F} = \mathbb{K}$ this problem can be reduced to solving a linear system of equations over $\mathbb{K}$. For a general ΠΣ-field $(\mathbb{F}(t), \sigma)$ this task can be accomplished by solving various *PLDE*-problems in the smaller ΠΣ-field $(\mathbb{F}, \sigma)$; see [19].

This article delivers one of the key steps in order to solve problem *PLDE* in ΠΣ-fields. More precisely, we will explain how one can solve problem *DegB* in ΠΣ-fields for the first order case, i.e., $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^2$. Moreover, these ideas are then generalized to the higher order case for some special cases.

Example 7 (Cont. Exp. 1) Given $(\mathbb{Q}(k), \sigma)$ with $\sigma(k) = k+1$, we show that there does not exist a $g \in \mathbb{Q}(k)$ with

$$\sigma(g) - g = \frac{1}{k+1}. \quad (16)$$

In order to accomplish this task, we apply our reduction techniques form above. Namely, by the algorithms given in [20] we obtain the denominator bound $d = 1$. This means that it suffices to look for a g in $\mathbb{Q}[k]$. Then we compute the degree bound $b = 0$; see Example 17. Hence we have to look for a $g \in \mathbb{Q}[k]_0 = \mathbb{Q}$ with (16). Obviously such a solution cannot exist.

Example 8 (Cont. Exp. 2) In order to find a $g \in \mathbb{Q}(k)(h_3)(h)$ for (3) with $f = h(h\,k-1)/k^2$, we compute (by using the algorithms in [20]) the denominator bound $d = 1$, i.e., $g \in \mathbb{Q}(k)(h_3)[h]$. Next, by our algorithms, see Example 16, we derive the degree bound $b = 3$. This means that any solution from $\mathbb{Q}(k)(h_3)[h]$ lies already in $\mathbb{Q}(k)(h_3)[h]_3$. Finally, by reduction techniques in [19] we obtain the solution $g = \frac{1}{3k^2}\big[3h - 3k\,h^2 + k^2h^3 - k^2h_3\big]$.

Example 9 (Cont. Exp. 3) In order to find a solution $g \in \mathbb{Q}(k)(h)(p)$ for (3) with $f = \frac{h(3k(k+1)h+3k-1)}{k^2(k\,h-3)}p$, we proceed as follows. First we compute the denominator bound $d = 1$; afterwards we obtain the degree bound $b = 1$, i.e., $g \in \mathbb{Q}(k)(h)[p]_1$; see Example 15. Finally, by the reduction techniques in [19] it follows that $g = g'\,p$ for some $g' \in \mathbb{Q}(k)(h)$, i.e., $\frac{h(k-1)-2}{h(k+1)+1}\sigma(g') - g' = \frac{h(3k(k+1)h+3k-1)}{k^2(kh-3)}$. Now we repeat our reduction techniques to find g'. First we compute a denominator bound $d = hk - 3$, i.e., $g' = \frac{g''}{hk-3}$ for some $g'' \in \mathbb{Q}(k)[h]$. Hence g'' must fulfill

$$\frac{h(k-1)-2}{(h(k+1)+1)\sigma(d)}\sigma(g'') - \frac{1}{d}g'' = \frac{h\big(3k(k+1)h+3k-1\big)}{k^2(kh-3)}. \qquad (17)$$

Finally, by our algorithm, see Exp. 19, we get the degree bound $b = 4$, i.e., $g'' \in \mathbb{Q}(k)[h]_4$. To this end, we find the solution $g'' = -\frac{h(hk-1)^2(k(h+3)-1)}{k^2}$ by using the algorithms given in [19].

Example 10 (Cont. Exp. 4) In order to find a $g \in \mathbb{Q}(k)(h)(p)$ with (7), we get the denominator bound $d = 1$. Next, we obtain the degree bound $b = 4$, i.e., $g \in \mathbb{Q}(k)(h)[p]_4$; see Example 14. Finally, we compute $g = \frac{k\,h\,p^4}{k\,h-3}$; see [19].

Example 11 (Cont. Exp. 5) We find the solution (11) for (10) as follows. We compute the denominator bound $d = 1$ and degree bound $b = 2$, i.e., $g \in \mathbb{Q}(k)(h)[h']_2$; see Exp. 18. Finally, by the algorithms in [19] we get (11).

Example 12 (Cont. Exp. 6) We find the basis (13) as follows. We compute the denominator bound $d = 1$ and the degree bound $b = 3$; see Example 21. Finally, by the algorithms in [19] we obtain (13).

3 A Constructive Theory of ΠΣ-Extensions and ΠΣ-Fields

We shall introduce ΠΣ-extensions and ΠΣ-fields. In particular we shall work out that a huge class of multi-sum expressions can be represented in ΠΣ-fields in a completely automatic fashion. Then, given such a ΠΣ-field, we can proceed as in the Examples 1–5 in order to attack various symbolic summation problems.

3.1 The Definition of ΠΣ-Extensions and ΠΣ-Fields

ΠΣ-extensions, introduced first in [11,12] and further analyzed in [6,18, 24], are defined in terms of difference field extensions. Namely, a difference field $(\mathbb{E},\sigma')$ is a *difference field extension* of $(\mathbb{F},\sigma)$ if $\mathbb{F}$ is a subfield of $\mathbb{E}$ and $\sigma'(g)=\sigma(g)$ for all $g\in\mathbb{F}$; note that from now on σ and σ' are not distinguished anymore.

Definition 2 *Let $(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$. We call this is a Σ^*-extension (resp. Π-extension) if $\mathbb{F}(t)$ is a rational function field, $\sigma(t)=t+a$ (resp. $\sigma(t)=a\,t$) for some $a\in\mathbb{F}^*$, and $\mathrm{const}_\sigma\mathbb{F}(t)=\mathrm{const}_\sigma\mathbb{F}$.*

According to [11] we introduce for a difference field $(\mathbb{F},\sigma)$ the set

$$\mathrm{H}_{(\mathbb{F},\sigma)}:=\{\sigma(g)/g\,|\,g\in\mathbb{F}^*\}.$$

Note that $\mathrm{H}_{(\mathbb{F},\sigma)}$ forms a multiplicative group which we call also *homogeneous group*. With this definition we can formulate equivalent descriptions of Π-extensions, see [12, Thm. 2.2], and of Σ^*-extensions, see [18, Cor. 2.2.4].

Theorem 1 *Let $(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$.* **(1)** *Then this is a Π-extension iff $\sigma(t)=a\,t$, $t\neq 0$, $a\in\mathbb{F}^*$ and there is no $n>0$ with $a^n\in\mathrm{H}_{(\mathbb{F},\sigma)}$.* **(2)** *Then this is a Σ^*-extension iff $\sigma(t)=t+a$, $t\notin\mathbb{F}$, $a\in\mathbb{F}^*$, and there is no $g\in\mathbb{F}$ with $\sigma(g)-g=a$.*

Observe that this result states that the elimination of sums (resp. products) and building up Σ^*-extensions (resp. Π-extensions) are closely related. Namely, given $(\mathbb{F},\sigma)$ and $a\in\mathbb{F}^*$ there are the following situations.

- If one fails to find a $g\in\mathbb{F}$ with $\sigma(g)-g=a$, i.e., if one cannot solve the telescoping problem in $\mathbb{F}$, one can adjoin the solution t with $\sigma(t)-t=a$ to $\mathbb{F}$ in form of the Σ^*-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$.
- In a similar fashion one tries to handle the product case. Namely, one first looks for a $g\in\mathbb{F}$ with $\sigma(g)=a\,g$; see Exp. 4. If one fails to find such a g, and even more, if one fails to find a $g\in\mathbb{F}$ and $n>0$ with $\sigma(g)=a^n\,g$, i.e., if there is no $n>0$ with $a^n\notin\mathrm{H}_{(\mathbb{F},\sigma)}$, one can adjoin[1] t with $\sigma(t)=a\,t$ to $\mathbb{F}$ in form of the Π-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$.

Before we consider further constructive aspects in Subsection 3.2, we complete our definition of ΠΣ-extensions and ΠΣ-fields.

Definition 3 *We call $(\mathbb{F}(t),\sigma)$ a* Σ-extension *of $(\mathbb{F},\sigma)$ if* **(1)** *$\sigma(t)=\alpha\,t+\beta$ with $\alpha,\beta\in\mathbb{F}^*$ and $t\notin\mathbb{F}$,* **(2)** *there does not exist a $g\in\mathbb{F}(t)\setminus\mathbb{F}$ with $\frac{\sigma(g)}{g}\in\mathbb{F}$, and* **(3)** *for all $n\in\mathbb{Z}^*$ we have that $\alpha^n\in\mathrm{H}_{(\mathbb{F},\sigma)}\Rightarrow\alpha\in\mathrm{H}_{(\mathbb{F},\sigma)}$.*

[1] The only problematic case occurs if there exists an $n>1$ with $a^n\in\mathrm{H}_{(\mathbb{F},\sigma)}$ but $a\notin\mathrm{H}_{(\mathbb{F},\sigma)}$. For instance, the object $(-1)^k$ cannot be adjoined in form of a Π-extension since $(-1)^2=1\in\mathrm{H}_{(\mathbb{F},\sigma)}$; such objects can be only treated in form of a difference ring extension, like $\mathbb{F}[(-1)^n]$, where zero-divisors occur: $(1-(-1)^n)(1+(-1)^n)=0$. For further details we refer to [24].

By [18, Thm. 2.2.3][2] it follows for any Σ-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ that $\mathbb{F}(t)$ is a rational function field and $\text{const}_\sigma\mathbb{F}(t)=\text{const}_\sigma\mathbb{F}$. Furthermore, an alternative description of Σ-extensions can be given; see [18, Cor. 2.2.3].

Theorem 2 *Let $(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$ where $\alpha,\beta\in\mathbb{F}^*$. Then $(\mathbb{F}(t),\sigma)$ is a* Σ-extension *of $(\mathbb{F},\sigma)$ if and only if there is no $g\in\mathbb{F}$ with $\sigma(g)-\alpha\,g=\beta$ and property (3) from Def. 3 holds.*

Therefore the class of Σ-extensions with $\alpha=1$ coincide with the class of Σ^*-extensions. Later we also need the special case of simple Σ-extensions.

Definition 4 *A Σ-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$ is called* simple Σ-extension *if for all $n>0$ we have $\alpha^n\notin \mathrm{H}_{(\mathbb{F},\sigma)}$.*

Finally, we introduce (nested) $\Pi\Sigma$-extensions and $\Pi\Sigma$-fields.

Definition 5 *A difference field extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ is called* (nested) $\Pi\Sigma$-extension *if $(\mathbb{F}(t_1,\dots,t_{i-1})(t_i),\sigma)$ is a Π- or Σ-extension of $(\mathbb{F}(t_1,\dots,t_{i-1}),\sigma)$ for all $1\le i\le e$. (If $i=0$, $(\mathbb{F}(t_1),\sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$.) A difference field $(\mathbb{F},\sigma)$ is a* $\Pi\Sigma$-field over $\mathbb{K}$ *if $\mathbb{F}=\mathbb{K}(t_1)\dots(t_e)$, $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{K},\sigma)$ and* $\text{const}_\sigma\mathbb{K}=\mathbb{K}$.

3.2 The Construction of $\Pi\Sigma$-Fields

Given an indefinite nested product-sum expression $f(k)$, as in Examples 2–5, one can construct completely automatically the underlying $\Pi\Sigma$-field in most instances. More precisely, suppose that we are given a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ over $\mathbb{K}$ and $\alpha,\beta\in\mathbb{F}$. In particular, suppose that $\mathbb{K}$ is σ-computable[3].

Definition 6 *A field $\mathbb{K}$ is called* σ-computable *if* **(1)** *for any $k\in\mathbb{K}$ one can decide if $k\in\mathbb{Z}$,* **(2)** *there is an algorithm that can factorize multivariate polynomials in $\mathbb{K}[t_1,\dots,t_n]$, and* **(3)** *there is an algorithm that can compute a basis of the submodule $\{(n_1,\dots,n_k)\in\mathbb{Z}^k\,|\,c_1^{n_1}\cdots c_k^{n_k}=1\}$ of $\mathbb{Z}^k$ over $\mathbb{Z}$ for any $(c_1,\dots,c_k)\in\mathbb{K}^k$.*

Then one can decide algorithmically if one can construct a $\Pi\Sigma$-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$. Namely, due to [11, Theorem 9] there is the following result.

Theorem 3 *Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma$-field over a σ-computable constant field $\mathbb{K}$. Then there is an algorithm that solves problem HG.*

HG: **H***omogeneous* **G***roup.*

- **Given** $(\mathbb{F},\sigma)$ *and $c,\alpha\in\mathbb{F}^*$.*
- **Decide** *if there exists a $d\in\mathbb{Z}$ with $c\,\alpha^d\in\mathrm{H}_{(\mathbb{F},\sigma)}$. If yes,* **compute** *such a d.*

---

[2] This is a corrected version of [11, Thm. 3] or [12, Thm. 2.3].

[3] For instance, any rational function field $\mathbb{K}=\mathbb{A}(x_1,\dots,x_r)$ over an algebraic number field $\mathbb{A}$ is σ-computable; see [24].

Moreover, using Karr's algorithm [11] or our streamlined algorithm [18,19], together with the degree bound algorithms given in Sections 5 and 6, we obtain the following result; see Remark 1.

Theorem 4 *Let $(\mathbb{F}, \sigma)$ be a ΠΣ-field over a σ-computable constant field $\mathbb{K}$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^2$ and $\boldsymbol{f} \in \mathbb{F}^n$. Then there exists an algorithm that computes a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$, i.e., one can solve problem PLDE with $m = 2$.*

Summarizing, these algorithms allow us to construct a ΠΣ-field for a given summation problem, like for instance in Examples 2–5.

Example 13 For Example 3 we proceed as follows. In order to describe the summand $f(k)$ on the left hand side of (5), we first represent the objects k, H_k and $\prod_{i=3}^{k} \frac{iH_i-3}{iH_i}$ in a difference field. We start with the constant field $\mathbb{Q}$ and consider k. Since there is no $g \in \mathbb{Q}$ with $\sigma(g) - g = 1$, we adjoin it in form of the Σ^*-extension $(\mathbb{Q}(k), \sigma)$ of $(\mathbb{Q}, \sigma)$. Similarly, we can construct the Σ^*-extension $(\mathbb{Q}(k)(h), \sigma)$ of $(\mathbb{Q}(k), \sigma)$ since there does not exist a $g \in \mathbb{Q}(k)$ with $\sigma(g) - g = \frac{1}{k+1}$. This fact can be checked algorithmically; see Example 1. Since Karr's algorithm (Thm. 3) tells us that there does not exist a $g \in \mathbb{Q}(k)(h)^*$ and $n \geq 1$ with $\sigma(g) = \left(\frac{h(k+1)-2}{h(k+1)+1}\right)^n g$, we can adjoin this product in form of the Π-extension $(\mathbb{Q}(k)(h)(p), \sigma)$ of $(\mathbb{Q}(k)(h), \sigma)$.

3.3 Some Additional Properties and Notation

The following lemma will be heavily used. The proof is straightforward.

Lemma 1 *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$. Then $\mathbb{F}(t)$ is a field of rational functions over $\mathbb{F}$. Furthermore, σ is an automorphism of the polynomial ring $\mathbb{F}[t]$. Additionally, for all $f, g \in \mathbb{F}[t]$ we have $\deg(\sigma(f)) = \deg(f)$ and $\gcd(\sigma(f), \sigma(g)) = \sigma(\gcd(f, g))$.*

Let $(\mathbb{F}, \sigma)$ be a difference field, $f \in \mathbb{F}^*$, and $k \in \mathbb{Z}$. Then we define the *σ-factorial* $f_{(k)}$ by $\prod_{i=0}^{k-1} \sigma^i(f)$ if $k \geq 0$, and by $\prod_{i=1}^{-k} \sigma^{-i}(1/f)$ if $k < 0$. Clearly, if $(\mathbb{F}, \sigma)$ is a difference field, also $(\mathbb{F}, \sigma^k)$ is a difference field for any $k \in \mathbb{Z}$. Moreover, if $(\mathbb{F}(t), \sigma)$ is a difference field extension of $(\mathbb{F}, \sigma)$ where t is transcendental over $\mathbb{F}$ and $\sigma(t) = \alpha\, t + \beta$ with $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$, also $(\mathbb{F}(t), \sigma^k)$ is a difference field extension of $(\mathbb{F}, \sigma^k)$ where t is transcendental over $\mathbb{F}$. More precisely, we have $\sigma^k(t) = \alpha_{(k)}\, t + \gamma$ for some $\gamma \in \mathbb{F}$. If $\beta = 0$, we have $\sigma^k(t) = \alpha_{(k)}\, t$. Note that this property holds for ΠΣ-extensions.
The next theorem is taken from [12, Thm: page 314]. This result is essential to generalize degree bounds from first order to higher order linear difference equations; see Subsections 5.2 and 6.2.

Theorem 5 *If $(\mathbb{F}, \sigma)$ is a ΠΣ-field, $(\mathbb{F}, \sigma^k)$ is a ΠΣ-field for all $k \in \mathbb{Z}^*$.*

Corollary 1 *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-field and $k \in \mathbb{Z}^*$. Then $(\mathbb{F}(t), \sigma)$ is a Π-extension (resp. Σ-extension) of $(\mathbb{F}, \sigma)$ iff $(\mathbb{F}(t), \sigma^k)$ is a Π-extension (resp. Σ-extension) of $(\mathbb{F}, \sigma)$. Moreover, $(\mathbb{F}(t), \sigma)$ is a simple Σ-extension of $(\mathbb{F}, \sigma)$ iff $(\mathbb{F}(t), \sigma^k)$ is a simple Σ-extension of $(\mathbb{F}, \sigma^k)$.*

Proof Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t$ and $\alpha \in \mathbb{F}^*$. Then by Theorem 5 $(\mathbb{F}(t), \sigma^k)$ is a Π-extension of $(\mathbb{F}, \sigma^k)$ with $\sigma^k(t) = \alpha_{(k)}\, t$. Conversely, if $(\mathbb{F}(t), \sigma^k)$ is a Π-extension of $(\mathbb{F}, \sigma)$ with $\sigma^k(t) = \alpha'\, t$ and $\alpha' \in \mathbb{F}^*$, by Theorem 5 $(\mathbb{F}(t), \sigma)$ is a Π-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha'_{(-k)}\, t$. Hence Π-extensions are in both directions transformed to Π-extension. But then the same must be valid for Σ-extensions by Theorem 5. Now let $(\mathbb{F}(t), \sigma)$ be a simple Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$. Then by the first statement of this corollary $(\mathbb{F}(t), \sigma^k)$ is a Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma^k(t) = \alpha_{(k)}\, t + \gamma$ for some $\gamma \in \mathbb{F}^*$. What remains to show is that there does not exist an $n > 0$ with $\alpha_{(k)}^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$. Construct the difference field extension $(\mathbb{F}(x), \sigma)$ of $(\mathbb{F}, \sigma)$ with the rational function field $\mathbb{F}(x)$ and $\sigma(x) = \alpha\, x$. Since $(\mathbb{F}(t), \sigma)$ is a simple Σ-extension of $(\mathbb{F}, \sigma)$, there is no $n > 0$ with $\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$ and hence $(\mathbb{F}(x), \sigma)$ is a Π-extension of $(\mathbb{F}, \sigma)$ by Theorem 1. Therefore by the first statement of this corollary $(\mathbb{F}(x), \sigma^k)$ is a Π-extension with $\sigma(x) = \alpha_{(k)}\, x$. Thus there is no $n > 0$ with $\alpha_{(k)}^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$ by Theorem 1. The reverse direction is analogous.

Let $\mathbb{F}$ be a field and $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. If $\boldsymbol{c} \in \mathbb{F}^n$, we define the vector product $\boldsymbol{c}\,\boldsymbol{f} = \sum_{i=1}^n c_i\, f_i$. Moreover, for a function $\sigma : \mathbb{F} \to \mathbb{F}$, $\boldsymbol{a} \in \mathbb{F}^m$ and $g \in \mathbb{F}$, we introduce $\sigma_{\boldsymbol{a}} g := a_1\, \sigma^{m-1}(g) + \dots + a_m\, g \in \mathbb{F}$.
Let $\mathbb{F}[t]$ be a polynomial ring. By convention the *zero-polynomial* 0 has degree $-\infty$. Furthermore, if $f = \sum_{i=0}^n f_i\, t^i \in \mathbb{F}[t]$, the i-th *coefficient* f_i of f will be denoted by $[f]_i$, i.e., $[f]_i = f_i$. If $i > n$ or $i < 0$, we have $[f]_i = 0$. Moreover, we define the rank function $\|\ \|$ by $\|f\| := -1$ if $f = 0$, and $\|f\| := \deg(f)$ if $f \neq 0$.
Now we will generalize these definitions from $\mathbb{F}[t]$ to its quotient field $\mathbb{F}(t)$. Decompose $f \in \mathbb{F}(t)$ uniquely by polynomial division into $f = f_p + f_r$ where $f_p \in \mathbb{F}[t]$ and $f_r = \frac{p}{q}$ with $p \in \mathbb{F}[t]$, $q \in \mathbb{F}[t]^*$ and $\deg(p) < \deg(q)$. Then we define the coefficient of f by $[f]_i := [f_p]_i$ and the rank of f by $\|f\| := \|f_p\|$. For $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}(t)^n$ we define $[\boldsymbol{f}]_i := ([f_1]_i, \dots, [f_n]_i) \in \mathbb{F}^n$ and $\|\boldsymbol{f}\| := \max_i \|f_i\|$.

Lemma 2 *Let $f, g \in \mathbb{F}(t)$ with $d := \|f\|$ and $e := \|g\|$, and let $\boldsymbol{f} \in \mathbb{F}(t)^n$. Then the following holds.*

1. $\|f + g\| \leq \max(\|f\|, \|g\|)$.
2. *If* $\|f\|, \|g\| \neq -1$, *we have* $\|f\, g\| = \|f\| + \|g\|$.
3. $\|\boldsymbol{c}\,\boldsymbol{f}\| \leq \|\boldsymbol{f}\|$ *for any* $\boldsymbol{c} \in \mathbb{F}$.
4. $[f + g]_d = [f]_d + [g]_d$ *for any* $d \in \mathbb{N}_0$.
5. *If* $e, d \geq 0$, *then* $[f\, g]_r = \sum_{i+j=r} [f]_i\, [g]_j$ *for any* r *with* $\max(d, e) \leq r$.
6. *If* $\sigma : \mathbb{F}(t) \to \mathbb{F}(t)$ *is a field automorphism with* $\sigma(t) = \alpha\, t + \beta$ $(\alpha, \beta \in \mathbb{F})$, *then* $\left[\sigma^i(f)\right]_r = 0$ *for all* $r > d$ *and* $i \in \mathbb{Z}^*$.

4 The Degree Bound Problem in ΠΣ-Extensions

As motivated in Section 2 we are interested in solving problem *DegB*. More generally, we consider the following problem. Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$, let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$, and define

$$\mathbb{W}_d := \{f \in \mathbb{W} \mid \|f\| \leq d\}$$

which we consider as a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. Furthermore, let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. **Find** $b \in \mathbb{N}_0 \cup \{-1\}$ such that

$$\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W}) = \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W}_b).$$

Such a b is also called a *degree bound* of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$. Note that problem *DegB* is included in the above problem by restricting to the case that $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field and choosing $\mathbb{W} := \mathbb{F}[t]$.

We shall introduce algorithms that compute degree bounds for the first order case, i.e., for $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^2$. More precisely, in Section 5 we give an algorithm for a Π- or a simple Σ-extension if one can solve problem *HG*. Moreover, in Section 6 we obtain an algorithm for Σ-extensions if one can solve problem *PLDE* with $m = 2$. Hence, by applying Theorems 3 and 4, we shall get an algorithm that solves problem *DegB* with $m = 2$ if $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over a σ-computable constant field.

Moreover, generalizations of this result will allow us to find the degree bound for a certain class of linear difference equations of higher order; see Situations 1, 3 and 6.

4.1 A Key Property to Determine Degree Bounds in $\Pi\Sigma$*-Fields*

The following lemma gives the key idea to find our degree bounds. In particular, it is applied in Lemma 7, Proposition 2 and Theorems 8 and 12.

Lemma 3 *Let $(\mathbb{F}(t),\sigma)$ be difference field extension of $(\mathbb{F},\sigma)$ with t transcendental over $\mathbb{F}$ and $\sigma(t) = \alpha\, t + \beta$ ($\alpha \in \mathbb{F}^*, \beta \in \mathbb{F}$). Let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ with $l := \|\boldsymbol{a}\|$, $\boldsymbol{f} \in \mathbb{F}[t]^n$, and define $\boldsymbol{b} := \big([a_1]_l\, \alpha^d_{(m-1)}, \dots, [a_m]_l\, \alpha^d_{(0)}\big) \in \mathbb{F}^m$ where $d \in \mathbb{N}_0$. If there exists a $g = w\, t^d + r \in \mathbb{F}(t)$ with $w \in \mathbb{F}^*$, $\|r\| < d$ and $\|\sigma_{\boldsymbol{a}} g\| < \|\boldsymbol{a}\| + d$ then $\sigma_{\boldsymbol{b}} w = 0$.*

Proof Let $g = w\, t^d + r$ with $w \in \mathbb{F}^*$, $\|r\| < d$, $l := \|\boldsymbol{a}\|$ and $\|\sigma_{\boldsymbol{a}} g\| < l + d$. Then by Lemma 2 it follows that

$$\begin{aligned}
0 = [\sigma_{\boldsymbol{a}} g]_{l+d} &= \big[a_1\, \sigma^{m-1}(w\, t^d + r) + \dots + a_m\, (w\, t^d + r)\big]_{l+d} \\
&= \big[a_1\, \sigma^{m-1}(w\, t_d) + \dots + a_m\, w\, t^d\big]_{l+d} + \underbrace{\big[a_1\, \sigma^{m-1}(r) + \dots + a_m\, r\big]_{l+d}}_{=0} \\
&= \Big[a_1\, \sigma^{m-1}(w)\, \alpha^d_{(m-1)}\, t^d + \dots + a_m\, w\, \alpha^d_{(0)}\, t^d\Big]_{d+l} \\
&= [a_1\,]_l\ \alpha^d_{(m-1)}\, \sigma^{m-1}(w) + \dots + [a_m]_l\ \alpha^d_{(0)}\, w = \sigma_{\boldsymbol{b}} w.
\end{aligned}$$

The following proposition is a direct consequence of Lemma 3.

Proposition 2 *Let* $(\mathbb{F}(t),\sigma)$ *be a* Σ^**-extension of* $(\mathbb{F},\sigma)$ *and* $\mathbb{W}$ *be subspace of* $\mathbb{F}(t)$ *over* $\text{const}_\sigma\mathbb{F}$*. Let* $\boldsymbol{f}\in\mathbb{F}[t]^n$ *and* $\boldsymbol{0}\neq\boldsymbol{a}\in\mathbb{F}[t]^m$ *and define* $\boldsymbol{b}:=[\boldsymbol{a}]_{\|\boldsymbol{a}\|}$*. If there is no* $w\in\mathbb{F}^*$ *with* $\sigma_{\boldsymbol{b}}w=0$ *then* $\max(\|\boldsymbol{f}\|-\|\boldsymbol{a}\|,-1)$ *is a degree bound of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$*.*

Proof Suppose there are a $g\in\mathbb{F}[t]$ and a $\boldsymbol{c}\in\mathbb{K}^n$ with $d:=\|g\|>\max(\|\boldsymbol{f}\|-\|\boldsymbol{a}\|,-1)$ and $\sigma_{\boldsymbol{a}}g=\boldsymbol{c}\,\boldsymbol{f}$. Take such a g with $w:=[g]_d\in\mathbb{F}^*$. Then by case distinction ($d>\|\boldsymbol{f}\|-\|\boldsymbol{a}\|$ or $\|\boldsymbol{f}\|-\|\boldsymbol{a}\|<-1$) it follows that $\|\sigma_{\boldsymbol{a}}g\|=\boldsymbol{c}\,\boldsymbol{f}\leq\|\boldsymbol{f}\|<\|g\|+\|\boldsymbol{a}\|$. By Lemma 3, $\sigma_{\boldsymbol{b}}w=0$.

Let $(\mathbb{F},\sigma)$ be a difference field, $\boldsymbol{0}\neq\boldsymbol{a}\in\mathbb{F}^m$ and $\boldsymbol{f}\in\mathbb{F}^n$. Then this proposition states that there cannot exist a Σ^*-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})\subsetneq\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$ if there does not exist already a $w\in\mathbb{F}^*$ with $\sigma_{\boldsymbol{a}}w=0$. This criterium plays an important role in the theory of sum solutions, a subclass of d'Alembertian and Liouvillian solutions; see [18].

4.2 A Bound Criterion and a Special Case of the Degree Bound Problem

We introduce a bound criterion, namely Corollary 2, which gives a proof strategy to decide if a b is a degree bound of a given solution space. First we state a lemma that follows immediately by the definition of degree bounds.

Lemma 4 *Let* $(\mathbb{F}(t),\sigma)$ *be a* $\Pi\Sigma$*-ext. of* $(\mathbb{F},\sigma)$ *and* $\mathbb{W}$ *be subspace of* $\mathbb{F}(t)$ *over* $\text{const}_\sigma\mathbb{F}$*. Let* $\boldsymbol{0}\neq\boldsymbol{a}\in\mathbb{F}[t]^m$*,* $b\in\mathbb{N}_0\cup\{-1\}$ *and* $f\in\mathbb{F}[t]$*. If* b *is a degree bound of* $\mathrm{V}(\boldsymbol{a},(f),\mathbb{W})$*, then for all* $g\in\mathbb{W}$ *with* $\sigma_{\boldsymbol{a}}g=f$ *we have* $\|g\|\leq b$*.*

Theorem 6 *Let* $(\mathbb{F}(t),\sigma)$ *be a* $\Pi\Sigma$*-extension of* $(\mathbb{F},\sigma)$*,* $\mathbb{W}$ *be subspace of* $\mathbb{F}(t)$ *over* $\text{const}_\sigma\mathbb{F}$*,* $\boldsymbol{0}\neq\boldsymbol{a}\in\mathbb{F}[t]^m$*,* $\boldsymbol{f}\in\mathbb{F}[t]^n$ *and* $b\in\mathbb{N}_0\cup\{-1\}$*. If* b *is a degree bound of* $\mathrm{V}(\boldsymbol{a},(f),\mathbb{W})$ *for all* $f\in\mathbb{F}[t]$ *with* $\|f\|\leq\|\boldsymbol{f}\|$ *then* b *is a degree bound of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$*.*

Proof Assume b is a degree bound of $\mathrm{V}(\boldsymbol{a},(f),\mathbb{W})$ for all $f\in\mathbb{F}[t]$ with $\|f\|\leq\|\boldsymbol{f}\|$. Let $(c_1,\dots,c_n,g)\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$, i.e., $\sigma_{\boldsymbol{a}}g=\boldsymbol{c}\,\boldsymbol{f}$ for $\boldsymbol{c}=(c_1,\dots,c_n)\in\mathbb{K}^n$. Take $f:=\boldsymbol{c}\,\boldsymbol{f}$. By $\|f\|=\|\boldsymbol{c}\,\boldsymbol{f}\|\leq\|\boldsymbol{f}\|$ and $\sigma_{\boldsymbol{a}}g=\boldsymbol{c}\,\boldsymbol{f}$ we may conclude that b is a degree bound of $\mathrm{V}(\boldsymbol{a},(f),\mathbb{W})$ and it follows that $\|g\|\leq b$ by Lemma 4. Consequently for all $(c_1,\dots,c_n,g)\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$ we have $\|g\|\leq b$ and thus $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W}_b)$ which proves the theorem.

The desired corollary follows immediately by Lemma 4 and Theorem 6.

Corollary 2 *Let* $(\mathbb{F}(t),\sigma)$ *be a* $\Pi\Sigma$*-extension of* $(\mathbb{F},\sigma)$*,* $\mathbb{W}$ *be subspace of* $\mathbb{F}(t)$ *over* $\text{const}_\sigma\mathbb{F}$ *and* $\boldsymbol{0}\neq\boldsymbol{a}\in\mathbb{F}[t]^m$*,* $\boldsymbol{f}\in\mathbb{F}[t]^n$*. Let* $b\in\mathbb{N}_0\cup\{-1\}$ *be such that for all* $f\in\mathbb{F}[t]$ *and* $g\in\mathbb{W}$ *with* $\|f\|\leq\|\boldsymbol{f}\|$ *and* $\sigma_{\boldsymbol{a}}g=f$ *we have* $\|g\|\leq b$*. Then* b *is a degree bound of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$*.*

We apply this corollary and obtain the following result which gives a degree bound for linear difference equations that are specified in Situation 1.

Situation 1 Assume $\mathbf{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ with $\|a_r\| = \|\boldsymbol{a}\|$ for some $r \in \{1, \dots, m\}$ and $\|a_i\| < \|\boldsymbol{a}\|$ for all i with $i \neq r$

Lemma 5 *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $f, g \in \mathbb{F}[t]$ with $\sigma_{\boldsymbol{a}} g = f$. Then $\|f\| \leq \|\boldsymbol{a}\| + \|g\|$. If $\boldsymbol{a}$ is as in Situation 1 and $g \neq 0$ then $\|f\| = \|\boldsymbol{a}\| + \|g\|$.*

Proof If $g = 0$, $f = \sigma_{\boldsymbol{a}} g = 0$ and hence $-1 = \|f\| \leq \|\boldsymbol{a}\| + \|g\|$ by $\|g\| = -1$ and $\|\boldsymbol{a}\| \geq 0$. Otherwise assume that $g \neq 0$, i.e., $\|g\| \geq 0$. By Lemma 2,

$$\|f\| = \|\sigma_{\boldsymbol{a}} g\| = \|a_1\, \sigma^{m-1}(g) + \dots + a_m\, g\| \leq \max(\|a_1\, \sigma^{m-1}(g)\|, \dots, \|a_m\, g\|).$$

Note that we have $\|a_i\, \sigma^{m-i}(g)\| \leq \|a_i\| + \|\sigma^{m-i}(g)\|$, if $a_i = 0$; otherwise, if $a_i \neq 0$, we even have equality. Moreover, if $a_i = 0$ and $a_j \neq 0$ then $\|a_i\| + \|\sigma^{m-i}(g)\| < \|a_j\| + \|\sigma^{m-j}(g)\|$. Since $a_j \neq 0$ for some j, we have

$$\max(\|a_1\, \sigma^{m-1}(g)\|, \dots, \|a_m\, g\|) = \max(\|a_1\| + \|\sigma^{m-1}(g)\|, \dots, \|a_m\| + \|g\|).$$

By Lemma 1 we have $\|\sigma^i(g)\| = \|g\|$ for all $i \in \mathbb{Z}$ and thus

$$\max(\|a_1\| + \|\sigma^{m-1}(g)\|, \dots, \|a_m\| + \|g\|) = \|\boldsymbol{a}\| + \|g\|$$

which proves the first statement of the lemma. If there exists additionally an r as in Situation 1, we have

$$\|a_1 \sigma^{m-1}(g) + \dots + a_m g\| = \|a_r \sigma^{m-r}(g)\| = \max(\|a_1 \sigma^{m-1}(g)\|, \dots, \|a_m g\|).$$

Then with similar arguments the second statement follows.

Theorem 7 *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. Let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ as in Situation 1 and $\boldsymbol{f} \in \mathbb{F}[t]^n$. Then $\max(\|\boldsymbol{f}\| - \|\boldsymbol{a}\|, -1)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.*

Proof We will prove the theorem by Corollary 2. Let $f \in \mathbb{F}[t]$ and $g \in \mathbb{W}$ be arbitrary but fixed with $\sigma_{\boldsymbol{a}} g = f$ and $\|f\| \leq \|\boldsymbol{f}\|$. We will show by case distinction that for an appropriate $b \in \mathbb{N}_0 \cup \{-1\}$ it follows that $\|g\| \leq b$ which will prove that b, for the particular case, is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. If $g \neq 0$ then by Lemma 5 it follows that $\|\boldsymbol{f}\| \geq \|f\| = \|\sigma_{\boldsymbol{a}} g\| = \|\boldsymbol{a}\| + \|g\|$ and therefore $\|g\| \leq \|\boldsymbol{f}\| - \|\boldsymbol{a}\|$. Otherwise, if $g = 0$ then $\|g\| = -1$. Altogether we have $\|g\| \leq \max(\|\boldsymbol{f}\| - \|\boldsymbol{a}\|, -1)$ and hence by Cor. 2 $\max(\|\boldsymbol{f}\| - \|\boldsymbol{a}\|, -1)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

5 Degree Bounds for Π- and Simple Σ-Extensions

Based on [11] we develop algorithms that solve the degree bound problem for first order linear difference equations within Π-extensions. Here we noticed that these degree bound techniques not only work for Π-extensions but also for simple Σ-extensions. Finally, we apply these techniques in order to solve the degree bound problem for Situation 3.

In this section let $(\mathbb{F}(t), \sigma)$ be a Π- or a simple Σ-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{K} := \mathrm{const}_\sigma \mathbb{F}$ and $\sigma(t) = \alpha\, t + \beta$ ($\alpha \in \mathbb{F}^*, \beta \in \mathbb{F}$). Hence for all $n > 0$ we have $\alpha^n \notin \mathrm{H}_{(\mathbb{F},\sigma)}$. Furthermore, let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$.

5.1 Degree Bounds of First Order Linear Difference Equations

In the sequel we consider the degree bound problem for $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ where $\mathbf{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}[t]^2$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. If $\|a_1\| \neq \|a_2\|$, Theorem 7 provides a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. Hence what remains is $\|a_1\| = \|a_2\| \geq 0$. More precisely, we deal with the following case.

Situation 2 Assume $\boldsymbol{a} = (a_1, a_2) \in \mathbb{F}[t]^2$ with $a_1 = t^p + r_1$ and $a_2 = -c\,t^p + r_2$ for $c \in \mathbb{F}^*$, $p \geq 0$ and $r_1, r_2 \in \mathbb{F}[t]$ with $\|r_1\|, \|r_2\| < p$.

The main idea of the following section is taken from Theorem 15 of [11]. Whereas in Karr's version theoretical and computational aspects are mixed, we separate his theorem in several parts to achieve more transparency.

Theorem 8 *Let $(\mathbb{F}(t), \sigma)$ be difference field extension of $(\mathbb{F}, \sigma)$ with t transcendental over $\mathbb{F}$ and $\sigma(t) = \alpha\, t + \beta$ ($\alpha \in \mathbb{F}^*$, $\beta \in \mathbb{F}$). Let $a_1, a_2 \in \mathbb{F}[t]$ as in Situation 2. If there exists a $g \in \mathbb{F}(t)$ with $\|g\| \geq 0$ such that*

$$\|a_1\, \sigma(g) + a_2\, g\| < \|g\| + p \tag{18}$$

then $\frac{c}{\alpha^{\|g\|}} \in \mathrm{H}_{(\mathbb{F},\sigma)}$.

Proof Let $g \in \mathbb{F}(t)$ with $d := \|g\| \geq 0$, i.e., $g = w\, t^d + r \in \mathbb{F}(t)$ with $w \in \mathbb{F}^*$, $\|r\| < d$. By Lemma 3, $\sigma(w)\, \alpha^d - c\, w = 0$, hence $\frac{c}{\alpha^d} = \frac{\sigma(w)}{w}$, thus $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$.

Next we exploit the properties of Π- and simple Σ-extensions.

Lemma 6 *Let $(\mathbb{F}(t), \sigma)$ be a Π- or a simple Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ ($\alpha \in \mathbb{F}^*, \beta \in \mathbb{F}$). Let $c \in \mathbb{F}^*$ and assume that there is a $d \in \mathbb{Z}$ such that $c\, \alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$. Then such a d is uniquely determined.*

Proof Assume there are $d_1, d_2 \in \mathbb{Z}$ with $d_1 < d_2$ and $c\, \alpha^{d_1} \in \mathrm{H}_{(\mathbb{F},\sigma)}$, $c\, \alpha^{d_2} \in \mathrm{H}_{(\mathbb{F},\sigma)}$ i.e., there are $g_1, g_2 \in \mathbb{F}^*$ such that $\frac{\sigma(g_1)}{g_1} = c\, \alpha^{d_1}$ and $\frac{\sigma(g_2)}{g_2} = c\, \alpha^{d_2}$. Since $d_2 - d_1 > 0$, it follows that $\alpha^{d_2 - d_1} = \frac{\sigma(g_2)/g_2}{\sigma(g_1)/g_1} = \frac{\sigma(g_2/g_1)}{g_2/g_1}$ and thus $\alpha^{d_2 - d_1} \in \mathrm{H}_{(\mathbb{F},\sigma)}$; a contradiction by Def. 4 or Thm. 1.

Combining the previous results gives a recipe to compute a degree bound.

Theorem 9 *Let $(\mathbb{F}(t), \sigma)$ be a Π- or a simple Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ ($\alpha \in \mathbb{F}^*, \beta \in \mathbb{F}$), $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over $\mathrm{const}_\sigma \mathbb{F}$, $\boldsymbol{f} \in \mathbb{F}[t]^n$ and assume $a_1, a_2 \in \mathbb{F}[t]$ as in Situation 2. If there is a $d \in \mathbb{N}_0$ with $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$, d is uniquely determined and $\max(\|\boldsymbol{f}\| - p, d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. If there is not exist such a d then $\max(\|\boldsymbol{f}\| - p, -1)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.*

Proof We will prove the theorem by Corollary 2. Let $f \in \mathbb{F}[t]$ and $g \in \mathbb{W}$ with $a_1\, \sigma(g) + a_2\, g = f$ and $\|f\| \leq \|\boldsymbol{f}\|$. We will show by case distinction that for an appropriate $b \in \mathbb{N}_0 \cup \{-1\}$ it follows that $\|g\| \leq b$ which will prove that b for the particular case is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

1. Assume there is a $d \geq 0$ such that $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$. Then d is uniquely determined by Lemma 6. If $\|g\| + p > \|f\|$ and $\|g\| \geq 0$, it follows by Theorem 8 that $\|g\| = d$ and consequently $\|g\| = d = \max(\|f\| - p, d) \leq \max(\|\boldsymbol{f}\| - p, d)$. Otherwise, if $\|g\| + p \leq \|f\|$ or $\|g\| = -1$, we have $\|g\| \leq \max(\|f\| - p, d) \leq \max(\|\boldsymbol{f}\| - p, d)$. Thus for both cases we may apply Corollary 2 and hence $\max(\|\boldsymbol{f}\| - p, d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.
2. Assume there does not exist such a d. Then by Theorem 8 it follows that $\|g\|+p = \|f\| \leq \|\boldsymbol{f}\|$ or $\|g\| = -1$ and thus by Corollary 2 $\max(\|\boldsymbol{f}\|-p, -1)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

Example 14 (Cont. Exp. 10) Define $\boldsymbol{a} = (a_1, a_2) = (1, -\frac{(k\,h-3)((k+1)h-2)^3}{k\,h((k+1)h+1)^3})$ and $\boldsymbol{f} = (0)$; see (7). Hence $c = -a_2$. By solving problem *HG* with $\alpha = \frac{h(k+1)-2}{h(k+1)+1}$ we get $c\,\alpha^{-d} \in \mathrm{H}_{(\mathbb{Q}(k)(h),\sigma)}$ with $d = 4$. (More precisely, we have $\sigma(g)/g = c\,\alpha^{-4}$ with $g = \frac{kh}{kh-3}$.) Thus by Theorem 9 $\max(\|\boldsymbol{f}\| - 0, d) = 4$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{Q}(k)(h)[p])$.

Looking closer at the previous theorem, one obtains a degree bound for the case $\boldsymbol{a} = (1, -1)$ and $\boldsymbol{f} \in \mathbb{F}(t)^n$ which amounts to indefinite summation.

Corollary 3 *Let* $(\mathbb{F}(t), \sigma)$ *be a* Π- *or simple* Σ*-extension of* $(\mathbb{F}, \sigma)$*, let* $\mathbb{W}$ *be a subspace of* $\mathbb{F}(t)$ *over* $\mathrm{const}_\sigma\mathbb{F}$*,* $a \in \mathbb{F}[t]^*$ *and* $\boldsymbol{f} \in \mathbb{F}[t]^n$*. Then* $\max(\|\boldsymbol{f}\| - \|a\|, 0)$ *is a degree bound of* $\mathrm{V}((a, -a), \boldsymbol{f}, \mathbb{W})$.

Proof Suppose that $\sigma(t) = \alpha\,t+\beta$ with $\alpha \in \mathbb{F}^*$, $\beta \in \mathbb{F}$. Since $\frac{1}{\alpha^0} = 1 \in \mathrm{H}_{(\mathbb{F},\sigma)}$, by Theorem 9 $\max(0, \|\boldsymbol{f}\| - \|a\|)$ is a degree bound of $\mathrm{V}((a, -a), \boldsymbol{f}, \mathbb{W})$.

Example 15 (Cont. Exp. 9) Let $\boldsymbol{f} = (\frac{h(3k(k+1)h+3k-1)}{k^2(k\,h-3)}p)$. Then by Corollary 3 $\max(\|f\| - 0, 0) = 1$ is a degree bound of $\mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{Q}(k)(h)[p])$.

Summarizing, we obtain the following algorithm. The correctness follows by the previous corollaries and theorems.

Algorithm 1 Compute a degree bound for Π-extensions or simple Σ-extensions.
b=`Π-DegreeBound(`$(\mathbb{F}(t), \sigma), \boldsymbol{a}, \boldsymbol{f}$`)`

Input: A Π- or a simple Σ-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\,t + \beta$ in which one can solve problem *HG*; $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}[t]^2$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$.
Output: A degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ for any subspace $\mathbb{W}$ of $\mathbb{F}(t)$ over $\mathrm{const}_\sigma\mathbb{F}$.

(1) IF $\|a_1\| \neq \|a_2\|$ THEN RETURN $\max(\|\boldsymbol{f}\| - \|\boldsymbol{a}\|, -1)$.
(2) IF $a_1 + a_2 = 0$ THEN RETURN $\max(\|\boldsymbol{f}\| - \|a_1\|, 0)$.
(3) Set $p := \|\boldsymbol{a}\|$ and $c := -\frac{[a_2]_p}{[a_1]_p}$.
(4) IF there is a $d \in \mathbb{N}_0$ with $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$, take it and RETURN $\max(\|\boldsymbol{f}\| - p, d)$.
(5) OTHERWISE RETURN $\max(\|\boldsymbol{f}\| - p, -1)$.

Note that by Theorem 3 problem *DegB* is solved for a Π- or simple Σ-extension $(\mathbb{F}(t), \sigma)$ if $(\mathbb{F}, \sigma)$ is a ΠΣ-field over a σ-computable constant field.

5.2 A Generalization for Higher Order Linear Difference Equations

Finally we look at the degree bound problem for $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ with $\boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$ for the more general Situation 3 that contains Situation 2.

Situation 3 Assume $\mathbf{0} \neq \boldsymbol{a} = (a_1, \dots, a_\lambda, \dots, a_\mu \dots, a_m) \in \mathbb{F}[t]^m$ with $\lambda < \mu$, $\|a_\lambda\| = \|a_\mu\| = p$ and

$$\|a_i\| < p \quad \forall i \neq \lambda, \mu.$$

In particular, suppose that $a_\lambda = t^p + r_1$ and $a_\mu = -c\,t^p + r_2$ for $c \in \mathbb{F}^*$, $p \geq 0$ and $r_1, r_2 \in \mathbb{F}[t]$ with $\|r_1\|, \|r_2\| < p$.

First we generalize Theorem 8.

Theorem 10 *Let $(\mathbb{F}(t), \sigma)$ be difference field ext. of $(\mathbb{F}, \sigma)$ with t transcendental over $\mathbb{F}$ and $\sigma(t) = \alpha\,t + \beta$ ($\alpha \in \mathbb{F}^*, \beta \in \mathbb{F}$). Let $\boldsymbol{a} \in \mathbb{F}[t]^m$ as in Sit. 3. If there is a $g \in \mathbb{F}(t)$ with $\|g\| \geq 0$ and $\|\sigma_{\boldsymbol{a}} g\| < \|g\| + p$, $\frac{\sigma^{\mu-m}(c)}{\alpha_{(\mu-\lambda)}^{\|g\|}} \in \mathrm{H}_{(\mathbb{F}, \sigma^{\mu-\lambda})}$.*

Proof Let $d := \|g\| \geq 0$. By Lemma 2 and Situation 3, $\|a_\lambda\, \sigma^{m-\lambda}(g)\| = \|a_\mu\, \sigma^{m-\mu}(g)\| = p + d$ with $\lambda \neq \mu$ and $\|a_i\, \sigma^{m-i}(g)\| < p + d$ for all $i \neq \mu, \lambda$. Hence,

$$0 = [\sigma_{\boldsymbol{a}} g]_{p+d} = \left[\sum_{i=1}^{m} a_i\, \sigma^{m-i}(g)\right]_{p+d} = \left[a_\lambda\, \sigma^{m-\lambda}(g) + a_\mu\, \sigma^{m-\mu}(g)\right]_{p+d}$$

and thus $\left[\sigma^{\mu-m}(a_\lambda)\, \sigma^{\mu-\lambda}(g) + \sigma^{\mu-m}(a_\mu)\, g\right]_{p+d} = 0$ by Lemma 2. By

$$\sigma^{\mu-m}(a_\lambda) = \alpha_{(\mu-m)}^{p} t^p + \sigma^{\mu-m}(r_1),$$
$$\sigma^{\mu-m}(a_\mu) = -\sigma^{\mu-m}(c)\, \alpha_{(\mu-m)}^{p} t^p + \sigma^{\mu-m}(r_2)$$

it follows that $\left[b_1\, \sigma^{\mu-\lambda}(g) + b_2\, g\right]_{p+d} = 0$ where

$$b_1 := t^p + \sigma^{\mu-m}(r_1)/\alpha_{(\mu-m)}^{p}, \quad b_2 := -\sigma^{\mu-m}(c)\, t^p + \sigma^{\mu-m}(r_2)/\alpha_{(\mu-m)}^{p}.$$

Since $\|b_1\, \sigma^{\mu-\lambda}(g) + b_2\, g\| \leq p+d$, we have $\|b_1\, \sigma^{\mu-\lambda}(g) + b_2\, g\| < p+d$. Hence we may apply Theorem 8 and thus we obtain $\frac{\sigma^{\mu-m}(c)}{\alpha_{(\mu-\lambda)}^{d}} \in \mathrm{H}_{(\mathbb{F}, \sigma^{\mu-\lambda})}$.

Finally we obtain a degree bound method for Situation 3.

Theorem 11 *Let $(\mathbb{F}(t), \sigma)$ be a Π- or a simple Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\,t + \beta$ ($\alpha \in \mathbb{F}^*, \beta \in \mathbb{F}$), $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over $\mathrm{const}_\sigma \mathbb{F}$, $\boldsymbol{f} \in \mathbb{F}[t]^n$ and $\boldsymbol{a} \in \mathbb{F}[t]^m$ as in Situation 3. Suppose that $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a Π- or simple Σ-extension of $(\mathbb{F}, \sigma^{\mu-\lambda})$. If there is a $d \in \mathbb{N}_0$ with $\frac{\sigma^{\mu-m}(c)}{\alpha_{(\mu-\lambda)}^{d}} \in \mathrm{H}_{(\mathbb{F}, \sigma^{\mu-\lambda})}$, d is uniquely determined and $\max(\|\boldsymbol{f}\| - p, d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. If there is not such a d, $\max(\|\boldsymbol{f}\| - p, -1)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.*

Proof We will prove the theorem by Corollary 2. Let $f \in \mathbb{F}[t]$ and $g \in \mathbb{F}[t]$ with $\sigma_{\boldsymbol{a}} g = f$ and $\|f\| \leq \|\boldsymbol{f}\|$. We will show by case distinction that for an appropriate $b \in \mathbb{N}_0 \cup \{-1\}$ it follows that $\|g\| \leq b$ which will prove that b for the particular case is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$.

1. Assume there exists a $d \geq 0$ such that $\frac{\sigma^{\mu-m}(c)}{\alpha^d_{(\mu-\lambda)}} \in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$. Then by Lemma 6 d is uniquely determined. If $\|g\|+p > \|f\|$ and $\|g\| \geq 0$, by Theorem 10 it follows that $\|g\| = d$ and therefore $\|g\| = d = \max(\|f\|, d) \leq \max(\|\boldsymbol{f}\|, d)$. Otherwise, if $\|g\| + p \leq \|f\|$ or $\|g\| = -1$, we have $\|g\| \leq \max(\|f\|, d) \leq \max(\|\boldsymbol{f}\|, d)$. Consequently in both cases we may apply Corollary 2 and $\max(\|\boldsymbol{f}\|, d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.
2. Assume there does not exist such a d. Then by Theorem 10 it follows that $\|g\|+p = \|f\| \leq \|\boldsymbol{f}\|$ or $\|g\| = -1$ and thus by Corollary 2 $\max(\|\boldsymbol{f}\|-p, -1)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

Suppose that $(\mathbb{F}(t), \sigma)$ is a ΠΣ-field over a σ-computable constant field where the extension t is a Π- or a simple Σ-extension. Then by Theorem 5 and Corollary 1 $(\mathbb{F}(t), \sigma^k)$ is a ΠΣ-field for any $k \in \mathbb{Z}^*$, in particular t is a Π- or a simple Σ-extension of $(\mathbb{F}, \sigma^k)$. Hence one can decide if $\frac{\sigma^{\mu-m}(c)}{\alpha^d_{(\mu-\lambda)}} \in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$ for some d and can compute such a d by Theorem 3 if it exists. Therefore we can apply Theorem 11 to compute a degree bound for Situation 3.

6 Degree Bounds for Σ-Extensions

We deliver degree bounds of parameterized linear difference equations for Σ-extensions. Moreover, we extend these degree bound techniques introduced in [11]. This allows us to solve the degree bound problem for Situation 6. As a side remark note that for simple Σ-extensions one can choose either the degree bounds of this section or those of the previous section — it turns out that the degree bounds of the previous section are slightly better.

In this section let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{K} := \mathrm{const}_\sigma \mathbb{F}$ and $\sigma(t) = \alpha\, t + \beta$ $(\alpha, \beta \in \mathbb{F}^*)$. Let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$.

6.1 Degree Bounds of First Order Linear Difference Equations

We will find degree bounds of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ with $\mathbf{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}[t]^2$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. If $\|a_1\| \neq \|a_2\|$, Theorem 7 provides a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. What remains is the case $\|a_1\| = \|a_2\| \geq 0$.
Similarly to the Π-extension case the main idea is taken from Theorem 14 of [11]. In the sequel we separate this result into theoretical and algorithmic aspects and give detailed proofs. First we will consider the following case.

Situation 4 Assume $\boldsymbol{a} = (a_1, a_2) \in \mathbb{F}^2$ with $a_1 \neq 0 \neq a_2$.

Lemma 7 (Corrected version) *Let $(\mathbb{F}(t),\sigma)$ be a Σ-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$ ($\alpha,\beta\in\mathbb{F}^*$) and $\boldsymbol{a}\in(\mathbb{F}[t]^*)^2$. If there is a $g\in\mathbb{F}(t)$ with $\|g\|>0$ and $\|a_1\,\sigma(g)+a_2\,g\|<\|g\|+\|\boldsymbol{a}\|-1$ then $\|a_1\|=\|a_2\|>0$.*

Proof Let $g\in\mathbb{F}(t)$ with $d:=\|g\|>0$ as stated above. By Theorem 7, $\|a_1\|=\|a_2\|$. Assume $\|a_1\|=\|a_2\|=0$, i.e., $a_1,a_2\in\mathbb{F}^*$. Then for $u=-a_1/a_2$,

$$[\sigma(g)-u\,g]_d=0 \quad \text{and} \quad [\sigma(g)-u\,g]_{d-1}=0. \tag{19}$$

Write $g=g_dt^d+g_{d-1}t^{d-1}+r$ with $\|r\|<d-1$. By (19) we get $u=\frac{\sigma(g_d)}{g_d}\alpha^d$ and $\sigma(g_d)d\alpha^{d-1}\beta+\sigma(g_{d-1})\alpha^{d-1}-ug_{d-1}=0$. Hence $\sigma(w)-\alpha w=\beta$ for $w=\frac{-g_{d-1}}{dg_d}\in\mathbb{F}$, a contradiction by Thm. 2.

Corollary 4 *Let $(\mathbb{F}(t),\sigma)$ be a Σ-extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$. Let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$, $\boldsymbol{f}\in\mathbb{F}[t]^n$ and $\boldsymbol{a}$ as in Situation 4. Then $\|\boldsymbol{f}\|+1$ is a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$.*

Example 16 (Cont. Exp. 8) Take $f=h(h\,k-1)/k^2$. Then by Corollary 4 $b=3$ is a degree bound of $\mathrm{V}((1,-1),(f),\mathbb{Q}(k)[h])$.

For the case $\|a_1\|=\|a_2\|=0$ Corollary 4 delivers a degree bound. What remains is the case $\|a_1\|=\|a_2\|>0$. More precisely we deal with Situation 5.

Situation 5 Assume $\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}[t]^2$ with

$$a_1=(t^p+u_1\,t^{p-1}+r_1) \qquad \text{and} \qquad a_2=c\,(t^p+u_2\,t^{p-1}+r_2)$$

for some $c\in\mathbb{F}^*$, $u_1,u_2\in\mathbb{F}$, $p\geq 1$ and $r_1,r_2\in\mathbb{F}[t]$ with $\|r_1\|,\|r_2\|<p-1$.

The following considerations lead us to an algorithm that computes a degree bound of the solution space $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$ if one can compute a basis of the solution space $\mathrm{V}(\boldsymbol{b},\boldsymbol{v},\mathbb{F})$ for any $\boldsymbol{0}\neq\boldsymbol{b}\in\mathbb{F}^2$ and $\boldsymbol{v}\in\mathbb{F}^2$.

Theorem 12 *Let $(\mathbb{F}(t),\sigma)$ be difference field extension of $(\mathbb{F},\sigma)$ with t transcendental over $\mathbb{F}$ and $\sigma(t)=\alpha\,t+\beta$ ($\alpha\in\mathbb{F}^*,\beta\in\mathbb{F}$). Assume $a_1,a_2\in\mathbb{F}[t]$ as in Situation 5. If there is a $g\in\mathbb{F}(t)$ with $d:=\|g\|>0$ and*

$$\|a_1\sigma(g)+a_2\,g\|<\|g\|+p-1 \tag{20}$$

then there exists a $w\in\mathbb{F}$ such that

$$\sigma(w)-\alpha\,w=(u_2-u_1)\,\alpha-d\,\beta. \tag{21}$$

Proof Let $g\in\mathbb{F}(t)$ with $d=\|g\|>0$ as stated above. By (20) it follows that

$$[a_1\,\sigma(g)+a_2\,g]_{p+d}=0 \quad \text{and} \quad [a_1\,\sigma(g)+a_2\,g]_{p+d-1}=0.$$

Write $g=\sum_{i=0}^d g_i\,t^i+r$ where $g_i\in\mathbb{F}$, $g_d\neq 0$ and $r\in\mathbb{F}(t)$ with $\|r\|=-1$. By Lemma 3 it follows that $\sigma(g_d)\,\alpha^d+c\,g_d=0$ and therefore

$$c=-\frac{\sigma(g_d)}{g_d}\,\alpha^d. \tag{22}$$

By $\sigma(g) = \sum_{i=0}^{d} \sigma(g_i)\,(\alpha\, t + \beta)^i + \sigma(r)$ and Lemma 2 we obtain

$$\begin{aligned}
[a_2 g]_{p+d-1} &= [a_2]_p\,[g]_{d-1} + [a_2]_{p-1}\,[g]_d = c g_{d-1} + c u_2 g_d = c(g_{d-1} + u_2 g_d),\\
[a_1\,\sigma(g)]_{p+d-1} &= [a_1]_{p-1}\,[\sigma(g)]_d + [a_1]_p\,[\sigma(g)]_{d-1}\\
&= u_1\,\alpha^d\,\sigma(g_d) + \left[\sigma(g_d)\,(\alpha\, t+\beta)^d + \sigma(g_{d-1})\,(\alpha\, t+\beta)^{d-1}\right]_{d-1}\\
&= u_1\,\alpha^d\,\sigma(g_d) + d\,\alpha^{d-1}\,\beta\,\sigma(g_d) + \alpha^{d-1}\,\sigma(g_{d-1}).
\end{aligned}$$

Hence by Lemma 2 we have $0 = [a_1\,\sigma(g) + a_2\,g]_{p+d-1} = [a_1\,\sigma(g)]_{p+d-1} + [a_2\,g]_{p+d-1}$ and therefore

$$u_1\,\alpha^d\,\sigma(g_d) + d\,\alpha^{d-1}\,\beta\,\sigma(g_d) + \alpha^{d-1}\,\sigma(g_{d-1}) + c\,(g_{d-1} + u_2\,g_d) = 0.$$

Using (22) we get

$$\begin{aligned}
&u_1\,\alpha^d\,\sigma(g_d) + d\,\alpha^{d-1}\,\beta\,\sigma(g_d) + \alpha^{d-1}\,\sigma(g_{d-1}) - \frac{\sigma(g_d)}{g_d}\,\alpha^d\,(g_{d-1} + u_2\,g_d) = 0\\
&\quad\Leftrightarrow\quad \sigma(g_d)\,(u_1\,\alpha + d\,\beta - \alpha\,\frac{g_{d-1}}{g_d} - \alpha\,u_2) = -\sigma(g_{d-1})\\
&\quad\Leftrightarrow\quad \sigma(\frac{g_{d-1}}{g_d}) - \alpha\,\frac{g_{d-1}}{g_d} = (u_2 - u_1)\,\alpha - d\,\beta
\end{aligned}$$

and thus for $w := \frac{g_{d-1}}{g_d}$ the theorem is proven.

Corollary 5 *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$. Assume $a_1, a_2 \in \mathbb{F}[t]$ as in Situation 5. If there is a $g \in \mathbb{F}(t)$ with $\|g\| > 0$ and (20) then $u_1 \neq u_2$.*

Proof Suppose that $\sigma(t) = \alpha\, t + \beta$ and $d := \|g\| > 0$. By Thm. 12 it follows that (21) for some $w \in \mathbb{F}$. Now assume that $u_1 = u_2$. Then $\sigma(w) - \alpha\, w = -d\,\beta$, thus $\sigma(\frac{w}{-d}) - \alpha\,\frac{w}{-d} = \beta$ and therefore by Theorem 2 $(\mathbb{F}(t), \sigma)$ is not a Σ-extension of $(\mathbb{F}, \sigma)$, a contradiction.

Looking closer at the previous corollary, one obtains a degree bound for the case $\boldsymbol{a} = (1, -1)$ and $\boldsymbol{f} \in \mathbb{F}(t)^n$ which amounts to indefinite summation.

Corollary 6 *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$, $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathrm{const}_\sigma\mathbb{F}$, $a \in \mathbb{F}[t]^*$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. Then $\max(\|\boldsymbol{f}\| - \|a\| + 1, 0)$ is a degree bound of $\mathrm{V}((a, -a), \boldsymbol{f}, \mathbb{W})$.*

Example 17 (Cont. Exp. 7) Take $\boldsymbol{f} = (1)$ and $a = k + 1$. By Cor. 6 a degree bound of $\mathrm{V}((a, -a), \boldsymbol{f}, \mathbb{Q}[k])$ is $\max(\|\boldsymbol{f}\| - \|a\| + 1, 0) = \max(0 - 1 + 1, 0) = 0$.

Example 18 (Cont. Exp. 11) Take $\mathbf{f} = (h\,h', h(h' + \frac{1}{n-k+1}))$ and $a = 1$. By Corollary 6 $b = 2$ is a degree bound of $\mathrm{V}((a, -a), \boldsymbol{f}, \mathbb{Q}(k)(h)[h'])$.

The next lemma allows us to find a degree bound for Situation 5.

Lemma 8 *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ $(\alpha, \beta \in \mathbb{F}^*)$, and let $u \in \mathbb{F}$. Assume there is a $d \in \mathbb{Z}$ such that $\sigma(w) - \alpha\, w = u\,\alpha + d\,\beta$ holds for some $w \in \mathbb{F}$. Then such a d is uniquely determined.*

Proof Assume there are $w_1, w_2 \in \mathbb{F}$ and $d_1, d_2 \in \mathbb{Z}$ with $d_1 < d_2$, $\sigma(w_1) - \alpha\, w_1 = u\,\alpha + d_1\,\beta$ and $\sigma(w_2) - \alpha\, w_2 = u\,\alpha + d_2\,\beta$. Then it follows that $\sigma(w_2 - w_1) - \alpha\,(w_2 - w_1) = (d_2 - d_1)\,\beta$, consequently $\sigma(\frac{w_2-w_1}{d_2-d_1}) - \alpha\,\frac{w_2-w_1}{d_2-d_1} = \beta$, and hence by Theorem 2 $(\mathbb{F}(t), \sigma)$ is not a Σ-extension of $(\mathbb{F}, \sigma)$, a contradiction.

As for the case of Π-extensions in Theorem 9 one obtains a method that solves the degree bound problem for Σ-extensions.

Theorem 13 *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\,t + \beta$ $(\alpha, \beta \in \mathbb{F}^*)$, $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathrm{const}_\sigma\mathbb{F}$, $\boldsymbol{f} \in \mathbb{F}[t]^n$ and $a_1, a_2 \in \mathbb{F}[t]$ as in Situation 5. If $u_1 = u_2$, $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. Otherwise, if there is a $d \in \mathbb{N}_0$ with (21) for some $w \in \mathbb{F}$, d is uniquely determined and $\max(\|\boldsymbol{f}\| - p + 1, d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. If there is not such a d, $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.*

Proof We will prove the theorem by Corollary 2. Let $f \in \mathbb{F}[t]$ and $g \in \mathbb{W}$ with $a_1\,\sigma(g) + a_2\,g = f$ and $\|f\| \leq \|\boldsymbol{f}\|$. We will show by case distinction that for an appropriate $b \in \mathbb{N}_0 \cup \{-1\}$ it follows that $\|g\| \leq b$ which will prove that b for the particular case is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.
If $u_1 = u_2$ then by Cor. 5 it follows either that $\|g\| \leq 0$ or that $\|g\| \leq \|f\| - p + 1 \leq \|\boldsymbol{f}\| - p + 1$ holds. Hence by Cor. 2 $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. Otherwise, assume $u_1 \neq u_2$.

1. Assume there is a $d \geq 0$ such that (21) holds for some $w \in \mathbb{F}$. Then by Lemma 8 d is uniquely determined. If $\|g\| + p - 1 > \|f\|$ and $\|g\| > 0$, by Theorem 12 the inequality $\|g\| = d = \max(\|f\| - p + 1, d) \leq \max(\|\boldsymbol{f}\| - p + 1, d)$ holds. Otherwise, if $\|g\| + p - 1 \leq \|f\|$ or $\|g\| \leq 0$ then we obtain $\|g\| \leq \max(\|f\| - p + 1, d) \leq \max(\|\boldsymbol{f}\| - p + 1, d)$. Thus in both cases, by Corollary 2, $\max(\|\boldsymbol{f}\| - p + 1, d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.
2. Assume there does not exist such a d. Then by Theorem 12 it follows that $\|g\| \leq \|f\| - p + 1 \leq \|\boldsymbol{f}\| - p + 1$ or $\|g\| \leq 0$ and thus $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ by Corollary 2.

Example 19 (Cont. Exp. 9) Looking at (17) we need a degree bound of $\mathrm{V}(\boldsymbol{a}, (f), \mathbb{Q}(k)[h])$ with $f = h((k+1)h + 1)(3k(k+1)h + 3k - 1)/k^3$ and

$$\boldsymbol{a} = (h - \frac{3}{k}, -\frac{k+1}{k}(h + \frac{1}{k+1}));$$

we have $u_1 = -\frac{3}{k}$ and $u_2 = \frac{1}{k+1}$. In order to accomplish this task, we apply Theorem 13. Namely, since $u_1 \neq u_2$, we solve the following *PLDE*-problem: find all $c_1, c_2 \in \mathbb{Q}$ and all $g \in \mathbb{Q}(k)$ with

$$\sigma(g) - g = c_1(\frac{1}{k+1} + \frac{3}{k}) + c_2\frac{-1}{k+1}.$$

We get the basis $\{(1, 4, -\frac{3}{k}), (0, 0, 1)\}$ of $\mathrm{V}((1, -1), (\frac{1}{k+1} + \frac{3}{k}, \frac{-1}{k+1}), \mathbb{Q}(k))$. Hence for (21) we get the solution $w = -\frac{3}{k}$ and $d = 4$. This gives the degree bound $\max(\|f\| - \|\boldsymbol{a}\| + 1, d) = \max(3 - 1 + 1, 4) = 4$ of $\mathrm{V}(\boldsymbol{a}, (f), \mathbb{Q}(k)[h])$.

Summarizing, we obtain the following algorithm. The correctness follows by the previous corollaries and theorems.

Algorithm 2 Compute a degree bound for Σ-extensions.
b=`Σ-DegreeBound`$((\mathbb{F}(t),\sigma),\boldsymbol{a},\boldsymbol{f})$
Input: A Σ-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$ in which one can solve problem *PLDE* for $m=2$; $\mathbf{0}\neq\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}[t]^2$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$.
Output: A degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$ for any subspace $\mathbb{W}$ of $\mathbb{F}(t)$ over $\mathrm{const}_\sigma\mathbb{F}$.

(1) IF $\|a_1\|\neq\|a_2\|$ THEN RETURN $\max(\|\boldsymbol{f}\|-\|\boldsymbol{a}\|,-1)$.
(2) IF $a_1+a_2=0$ THEN RETURN $\max(\|\boldsymbol{f}\|-\|a_1\|+1,0)$.
(3) Set $p:=\|\boldsymbol{a}\|$;
(4) IF $p=0$ THEN RETURN $\|\boldsymbol{f}\|+1$.
(5) Set $u_1:=\frac{[a_1]_{p-1}}{[a_1]_p}$, $u_2:=\frac{[a_2]_{p-1}}{[a_2]_p}$.
(6) IF $u_1=u_2$ THEN RETURN $\max(\|\boldsymbol{f}\|-p+1,0)$.
(7) IF there are $d\in\mathbb{N}_0$, $w\in\mathbb{F}$ with (21), take d and RETURN $\max(\|\boldsymbol{f}\|-p+1,d)$.
(8) OTHERWISE RETURN $\max(\|\boldsymbol{f}\|-p+1,0)$.

By Theorem 4 problem *DegB* is completely solved for a Σ-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ if $(\mathbb{F},\sigma)$ is a ΠΣ-field over a σ-computable constant field.

6.2 A Generalization for Higher Order Linear Difference Equations

In the end we solve the degree bound problem of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$ with $\boldsymbol{a}\in\mathbb{F}[t]^m$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$ for Situation 6 that contains Situation 5.

Situation 6 Assume $\mathbf{0}\neq\boldsymbol{a}=(a_1,\dots,a_\lambda,\dots,a_\mu\;\dots,a_m)\in\mathbb{F}[t]^m$ with $\lambda<\mu$, $\|a_\lambda\|=\|a_\mu\|=p$ and

$$\|a_i\|<p-1\quad\forall i\neq\lambda,\mu.$$

In particular, suppose that

$$a_\lambda=(t^p+u_1\,t^{p-1}+r_1)\qquad\text{and}\qquad a_\mu=c\,(t^p+u_2\,t^{p-1}+r_2)$$

for some $c\in\mathbb{F}^*$, $u_1,u_2\in\mathbb{F}$, $p>0$ and $r_1,r_2\in\mathbb{F}[t]$ with $\|r_1\|,\|r_2\|<p-1$.

In order to accomplish this task, we generalize Theorem 12 to Theorem 14.

Lemma 9 *Let $(\mathbb{F}(t),\sigma)$ be difference field extension of $(\mathbb{F},\sigma)$ with t transcendental over $\mathbb{F}$ and $\sigma(t)=\alpha\,t+\beta$ ($\alpha\in\mathbb{F}^*,\beta\in\mathbb{F}$). Set $\beta_k:=\sigma^k(t)-\alpha_{(k)}\,t\in\mathbb{F}$. Assume $\boldsymbol{a}\in\mathbb{F}[t]^m$ as in Situation 6. If there is a $g\in\mathbb{F}(t)$ with $\|g\|\geq 0$ and $\|\sigma_{\boldsymbol{a}}g\|<\|g\|+p-1$ then $\|b_1\,\sigma^{\mu-\lambda}(g)+b_2\,g\|<\|g\|+p-1$ where*

$$b_1:=t^p+t^{p-1}\,(p\,\beta_{\mu-m}+\sigma^{\mu-m}(u_1))/\alpha_{(\mu-m)}\in\mathbb{F}[t]^*,$$
$$b_2:=\sigma^{\mu-m}(c)\,t^p+t^{p-1}\,(p\,\beta_{\mu-m}\,\sigma^{\mu-m}(c)+\sigma^{\mu-m}(u_2))/\alpha_{(\mu-m)}\in\mathbb{F}[t]^*.$$

Proof Let $d := \|g\| \geq 0$. By Lemma 2 and Sit. 6 we have $\|a_\lambda\, \sigma^{m-\lambda}(g)\| = \|a_\mu\, \sigma^{m-\mu}(g)\| = p + d$ with $\lambda \neq \mu$ and $\|a_i\, \sigma^{m-i}(g)\| < p + d - 1$ for all $i \neq \mu, \lambda$. Hence for $i \in \{0, 1\}$ we have

$$0 = [\sigma_{\boldsymbol{a}} g]_{p+d-i} = \big[a_\lambda\, \sigma^{m-\lambda}(g) + a_\mu\, \sigma^{m-\mu}(g)\big]_{p+d-i}$$

and thus $0 = \big[\sigma^{\mu-m}(a_\lambda)\, \sigma^{\mu-\lambda}(g) + \sigma^{\mu-m}(a_\mu)\, g\big]_{p+d-i}$ by Lemma 2. By

$$\begin{aligned}
\sigma^{\mu-m}(a_\lambda) &= (\alpha_{(\mu-m)} t + \beta_{\mu-m})^p \\
&\quad + \sigma^{\mu-m}(u_1)(\alpha_{(\mu-m)} t + \beta_{\mu-m})^{p-1} + \sigma^{\mu-m}(r_1) \\
&= \alpha^p_{(\mu-m)} t^p + t^{p-1} \alpha^{p-1}_{(\mu-m)} (p\beta_{\mu-m} + \sigma^{\mu-m}(u_1)) + \tilde{r}_1, \\
\sigma^{\mu-m}(a_\mu) &= \sigma^{\mu-m}(c)(\alpha_{(\mu-m)} t + \beta_{\mu-m})^p \\
&\quad + \sigma^{\mu-m}(u_2)(\alpha_{(\mu-m)} t + \beta_{\mu-m})^{p-1} + \sigma^{\mu-m}(r_2) \\
&= \sigma^{\mu-m}(c) \alpha^p_{(\mu-m)} t^p + t^{p-1} \alpha^{p-1}_{(\mu-m)} (p\beta_{\mu-m} \sigma^{\mu-m}(c) + \sigma^{\mu-m}(u_2)) + \tilde{r}_2
\end{aligned}$$

for some $\tilde{r}_1, \tilde{r}_2 \in \mathbb{F}[t]$ with $\|\tilde{r}_1\|, \|\tilde{r}_2\| < p - 2$ it follows, as above, that $\big[b_1\, \sigma^{\mu-\lambda}(g) + b_2\, g\big]_{p+d-i} = 0$ for $i \in \{0, 1\}$ with $b_1, b_2 \in \mathbb{F}[t]^*$. Hence by $\|b_1\, \sigma^{\mu-\lambda}(g) + b_2\, g\| \leq \|g\| + p$ we have $\|b_1\, \sigma^{\mu-\lambda}(g) + b_2\, g\| < \|g\| + p - 1$.

Theorem 14 *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ and set $\beta_k := \sigma^k(t) - \alpha_{(k)}\, t \in \mathbb{F}$. Assume $\boldsymbol{a} \in \mathbb{F}[t]^m$ as in Situation 6 and suppose that $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a Σ-extension of $(\mathbb{F}, \sigma^{\mu-\lambda})$. Define*

$$\begin{aligned}
v_1 &:= (p\, \beta_{\mu-m} + \sigma^{\mu-m}(u_1))/\alpha_{(m-\mu)} \in \mathbb{F}, \\
v_2 &:= (p\, \beta_{\mu-m}\, \sigma^{\mu-m}(c) + \sigma^{\mu-m}(u_2))/\alpha_{(\mu-m)} \in \mathbb{F}.
\end{aligned} \tag{23}$$

If there is a $g \in \mathbb{F}(t)$ with $d := \|g\| > 0$ and $\|\sigma_{\boldsymbol{a}} g\| < \|g\| + p - 1$ then there is a $w \in \mathbb{F}$ with

$$\sigma^{\mu-\lambda}(w) - \alpha_{(\mu-\lambda)}\, w = (v_2 - v_1)\, \alpha_{(\mu-\lambda)} - d\, \beta_{\mu-\lambda} \tag{24}$$

Moreover, $v_1 \neq v_2$.

Proof Assume there is a $g \in \mathbb{F}(t)$ with $d := \|g\| > 0$ and $\|\sigma_{\boldsymbol{a}} g\| < \|g\| + p - 1$. By Lemma 9 there are $b_1 := t^p + v_1\, t^{p-1}$, $b_2 := \sigma^{\mu-m}(c)\, t^p + v_2\, t^{p-1}$ s.t.

$$\|b_1\, \sigma^{\mu-\lambda}(g) + b_2\, g\| < \|g\| + p - 1.$$

As $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a Σ-extension $(\alpha_{(\mu-\lambda)}, \beta_{\mu-\lambda} \in \mathbb{F}^*)$, we may apply Theorem 12 and obtain (24) for some $w \in \mathbb{F}$. Now assume that $v_1 = v_2$. Then $\sigma^{\mu-\lambda}(\frac{w}{-d}) - \alpha_{(\mu-\lambda)}\, \frac{w}{-d} = \beta_{\mu-\lambda}$. By Theorem 2 $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is not a Σ-extension of $(\mathbb{F}, \sigma^{\mu-\lambda})$, a contradiction.

Finally one obtains a degree bound method for Situation 6.

Theorem 15 *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ and set $\beta_k := \sigma^k(t) - \alpha_{(k)}\, t \in \mathbb{F}$. Let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathrm{const}_\sigma\mathbb{F}$, $\boldsymbol{f} \in \mathbb{F}[t]^n$, and assume $\boldsymbol{a} \in \mathbb{F}[t]^m$ as in Situation 6. Suppose that $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a Σ-extension of $(\mathbb{F}, \sigma^{\mu-\lambda})$. Define $v_1, v_2 \in \mathbb{F}$ as in (23).*
If $v_1 = v_2$, $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. Otherwise, if there is a $d \in \mathbb{N}_0$ with (24) for some $w \in \mathbb{F}$, d is uniquely determined and $\max(\|\boldsymbol{f}\| - p + 1, d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. If there is not such a d, $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

Proof We will prove the theorem by Corollary 2. Let $f \in \mathbb{F}[t]$ and $g \in \mathbb{W}$ with $a_1\, \sigma(g) + a_2\, g = f$ and $\|f\| \leq \|\boldsymbol{f}\|$. We will show by case distinction that for an appropriate $b \in \mathbb{N}_0 \cup \{-1\}$ it follows that $\|g\| \leq b$ which will prove that b for the particular case is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. If $v_1 = v_2$ then by Theorem 14 it follows that $\|g\| \leq \|f\| - p + 1 \leq \|\boldsymbol{f}\| - p + 1$ or $\|g\| \leq 0$ and thus by Corollary 2 $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. Otherwise, assume $v_1 \neq v_2$.

1. Assume there is a $d \geq 0$ such that (24) holds for some $w \in \mathbb{F}$. Then by Lemma 8 d is uniquely determined. If $\|g\| + p - 1 > \|f\|$ and $\|g\| > 0$, it follows that $\|g\| = d = \max(\|f\| - p + 1, d) \leq \max(\|\boldsymbol{f}\| - p + 1, d)$ by Theorem 14. Otherwise, if $\|g\| + p - 1 \leq \|f\|$ or $\|g\| \leq 0$ then clearly we have $\|g\| \leq \max(\|f\| - p + 1, d) \leq \max(\|\boldsymbol{f}\| - p + 1, d)$. Thus by Corollary 2 $\max(\|\boldsymbol{f}\| - p + 1, d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.
2. Assume there does not exist such a d. Then by Theorem 14 it follows that $\|g\| \leq 0$ or $\|g\| \leq \|f\| - p + 1 \leq \|\boldsymbol{f}\| - p + 1$ and therefore by Corollary 2 $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

Suppose that $(\mathbb{F}(t), \sigma)$ is a ΠΣ-field over a σ-computable constant field where t is a Σ-extension. Then by Theorem 5 and Corollary 1 $(\mathbb{F}(t), \sigma^k)$ is a ΠΣ-field for any $k \in \mathbb{Z}^*$, in particular t is a Σ-extension of $(\mathbb{F}, \sigma^k)$. Hence by Theorem 4 we can decide if there are $d \in \mathbb{N}_0$ and $w \in \mathbb{F}$ with (24); in case of existence we can compute them. Consequently we can apply Theorem 11 to compute a degree bound for the special case described in Situation 6.

7 Extension Stable Degree Bounds for the First Order Case

Combining the previous results we arrive at the following algorithm.

Algorithm 3 Compute a degree bound for ΠΣ-extensions.
b=`DegreeBound(`$(\mathbb{F}(t), \sigma), \boldsymbol{a}, \boldsymbol{f}$`)`
Input: A ΠΣ-ext. $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ in which problems *HG* and *PLDE* $(m = 2)$ are solvable; $\mathbf{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}[t]^2$, $\boldsymbol{f} \in \mathbb{F}[t]^n$.
Output: A degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ for any subspace $\mathbb{W}$ of $\mathbb{F}(t)$ over $\mathrm{const}_\sigma\mathbb{F}$.

(1) IF $\beta = 0$ THEN RETURN Π-`DegreeBound(`$(\mathbb{F}(t), \sigma), \boldsymbol{a}, \boldsymbol{f}$`)`
(2) ELSE RETURN Σ-`DegreeBound(`$(\mathbb{F}(t), \sigma), \boldsymbol{a}, \boldsymbol{f}$`)`.

Theorem 16 *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ with $\mathbb{K}:=\mathrm{const}_\sigma\mathbb{F}$ in which one can solve problems HG and PLDE ($m=2$); let $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{F}[t]^2$, $\boldsymbol{f}\in\mathbb{F}[t]^n$, and $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. Then there exists an algorithm that computes a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$.*

Remark 1 Theorem 4 is proven in [19] by induction on the number of extensions e in the $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ over $\mathbb{K}$ with $\mathbb{F}:=\mathbb{K}(t_1)\dots(t_e)$. Inside of the induction step, say in the $\Pi\Sigma$-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$, the reduction technique given in Section 2 is applied. Namely, by results from [20] we bound the denominator of the solutions in $\mathbb{F}(t)$ and reduce the problem to find solutions in $\mathbb{F}[t]$. Next, we use the induction assumption that one can solve problem *PLDE* ($m=2$) in $(\mathbb{F},\sigma)$. Together with Theorem 3 we can apply Theorem 16 and can compute a degree bound. Finally, we apply the reduction techniques given in [19] by solving various problems *PLDE* ($m=2$) in $(\mathbb{F},\sigma)$. This finally gives an algorithm to solve problem *PLDE* ($m=2$) for the $\Pi\Sigma$-field $(\mathbb{F}(t),\sigma)$.

As illustrated in Example 2–5, one can reduce sum and product quantifiers by transforming nested multi-sums into $\Pi\Sigma$-fields. We want to emphasize that the construction of the appropriate $\Pi\Sigma$-field might be quite subtle like for instance in Example 2. Namely, in a naive way we would just construct the $\Pi\Sigma$-field $(\mathbb{Q}(k)(h),\sigma)$ with $\sigma(k)=k+1$ and $\sigma(h)=h+\frac{1}{k+1}$ in order to find a solution $g\in\mathbb{Q}(k)(h)$ for (3) with $f=h(h\,k-1)/k^2$. But this does not suffice. More precisely, we have to extend the difference field for instance with the Σ^*-extension $(\mathbb{Q}(k)(h)(h_3),\sigma)$ with $\sigma(h_3)=h_3+\frac{1}{(k+1)^3}$. Then we are able to find the solution $g=\frac{1}{3k^2}\big[3h-3k\,h^2+k^2h^3-k^2h_3\big]$.

Our refined summation algorithm given in [22] can handle such problematic cases. Namely it decides algorithmically if such a telescoper g in terms of a "nice" Σ^*-extension exists; if yes, it can compute such a solution. One of the important ingredients for this refined algorithm is the following result stated in Theorem 17. Namely, we need the property that Algorithm 3 is *extension-stable*. This means that extending the underlying difference field by certain additional $\Pi\Sigma$-extensions does not change the output result.

Let $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=\alpha_i\,t_i+\beta_i$ and suppose that there is a permutation $\tau:\{1,\dots,e\}\to\{1,\dots,e\}$ with $\alpha_{\tau(i)},\beta_{\tau(i)}\in\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(i-1)})$ for all $1\leq i\leq e$. Then for such a τ the generators of the $\Pi\Sigma$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ can be reordered without changing the $\Pi\Sigma$-nature of the extension. In short, we say that $(\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(e)}),\sigma)$ *is equal to* $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ *up to reordering* if there exists such a permutation τ. On the rational function field level we identify two such fields, i.e., $\mathbb{F}(t_1)\dots(t_e)=\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(e)})$.

Lemma 10 *Let $(\mathbb{F}(t),\sigma)$ be a Π-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t$ and $\boldsymbol{a}\in\mathbb{F}[t]$ as in Situation 2. If there is an $h\in\mathbb{F}[t]^*$ with $\sigma_{\boldsymbol{a}}h=0$ then there is a $d\in\mathbb{N}_0$ with $\frac{c}{\alpha^d}\in\mathrm{H}_{(\mathbb{F},\sigma)}$.*

Proof Let $\boldsymbol{a}\in\mathbb{F}[t]$ be as in Situation 2, in particular $p:=\|\boldsymbol{a}\|\geq 0$, and assume that there does not exist a $d\in\mathbb{N}_0$ such that $\frac{c}{\alpha^d}\in\mathrm{H}_{(\mathbb{F},\sigma)}$. Then by Theorem 9 it follows that $\max(\|0\|-p,-1)=-1$ is a degree bound of $\mathrm{V}(\boldsymbol{a},(0),\mathbb{F}[t])$. Hence there does not exist an $h\in\mathbb{F}[t]^*$ such that $\sigma_{\boldsymbol{a}}h=0$.

Lemma 11 *Let $(\mathbb{F}(t),\sigma)$ be a Σ-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$, let $\boldsymbol{a}\in(\mathbb{F}[t]^*)^2$ be as in Situation 5 and suppose that there is an $h\in\mathbb{F}[t]^*$ with $\sigma_{\boldsymbol{a}}h=0$. If $u_1\neq u_2$ then there exist a $w\in\mathbb{F}^*$ and a $d\in\mathbb{N}_0$ with (21).*

Proof Let $\boldsymbol{a}$ be as in Situation 5, in particular $p:=\|\boldsymbol{a}\|>0$, and let $h\in\mathbb{F}[t]^*$ with $\sigma_{\boldsymbol{a}}h=0$. Furthermore, assume that $u_1\neq u_2$ and suppose that there do not exist a $w\in\mathbb{F}^*$ and a $d\in\mathbb{N}_0$ with (21). Hence by Theorem 13 $\max(\|0\|-p+1,0)=0$ is a degree bound of $\mathrm{V}(\boldsymbol{a},(0),\mathbb{F}[t])$ and thus $h\in\mathbb{F}^*$. Consequently $c\,(t^p+u_2\,t^{p-1}+r_2)=a_2=-\frac{\sigma(h)}{h}\,a_1=-\frac{\sigma(h)}{h}\,(t^p+u_1\,t^{p-1}+r_1)$ and therefore $c=-\frac{\sigma(h)}{h}$ and $u_1=u_2$, a contradiction.

Theorem 17 *Let $(\mathbb{F}(x_1)\dots(x_e)(t)(s),\sigma)$, $(\mathbb{F}(s)(x_1)\dots(x_e)(t),\sigma)$ be $\Pi\Sigma$-extensions of $(\mathbb{F},\sigma)$ which are equal up to reordering; in short, write $\mathbb{G}=\mathbb{F}(x_1)\dots(x_e)$ and $\mathbb{E}=\mathbb{F}(s)(x_1)\dots(x_e)$. Suppose that problem HG is solvable in $(\mathbb{G},\sigma)$ and $(\mathbb{E},\sigma)$, and problem PLDE ($m=2$) is solvable in $(\mathbb{G},\sigma)$ and $(\mathbb{E},\sigma)$. Let $\boldsymbol{a}\in(\mathbb{G}[t]^*)^2$ such that there is an $h\in\mathbb{G}(t)^*$ with $\sigma_{\boldsymbol{a}}h=0$, and let $\boldsymbol{f}\in\mathbb{G}[t]^n$. Then we have*

$$\texttt{DegBound}((\mathbb{G}(\texttt{t}),\sigma),\boldsymbol{a},\boldsymbol{f})=\texttt{DegBound}((\mathbb{E}(\texttt{t}),\sigma),\boldsymbol{a},\boldsymbol{f}).$$

Proof In the following we will consider the computation steps for both difference fields $(\mathbb{G}(t),\sigma)$ and $(\mathbb{E}(t),\sigma)$ and will prove that the output will be always the same. Assume that $\sigma(t)=\alpha\,t+\beta$. If we have $\|a_1\|\neq\|a_2\|$ or $a_1+a_2=0$, in both cases the output is the same.
Now assume that $\beta=0$. Then by Lemma 10 we find a $d\in\mathbb{N}_0$ such that $\frac{c}{\alpha^d}\in\mathrm{H}_{(\mathbb{G},\sigma)}$. For this d we also have $\frac{c}{\alpha^d}\in\mathrm{H}_{(\mathbb{E},\sigma)}$. Since d is unique by Lemma 6, in both cases we find the same d in Alg. 1 and consequently the output is in both cases the same.
Now assume that $\beta\neq 0$. If $p=0$ or $u_1=u_2$, in both cases we compute the same output in Alg. 2. Now assume that $u_1\neq u_2$. Then by Lemma 11 one can find a $w\in\mathbb{G}^*$ and a $d\in\mathbb{N}_0$ with (21). Then we find also a $w\in\mathbb{E}^*$ for the same d with (21). Since d is unique by Lemma 8, in both cases we find the same d in Alg. 2 and thus the output is in both cases the same.

If $(\mathbb{F}(t),\sigma)$ and $(\mathbb{G}(t),\sigma)$ are $\Pi\Sigma$-fields which are equal up to reordering, the difference fields are essential the same. In particular, it is a direct consequence that Algorithm 3 computes the same result in $(\mathbb{F}(t),\sigma)$ or $(\mathbb{G}(t),\sigma)$.

Proposition 3 *Let $(\mathbb{F}(t),\sigma)$ and $(\mathbb{G}(t),\sigma)$ be $\Pi\Sigma$-fields over a σ-computable constant field which are the same up to reordering. Then for any $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{F}[t]^2$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$ we have*

$$\texttt{DegreeBound}((\mathbb{F}(\texttt{t}),\sigma),\boldsymbol{a},\boldsymbol{f})=\texttt{DegreeBound}((\mathbb{G}(\texttt{t}),\sigma),\boldsymbol{a},\boldsymbol{f}).$$

In work under development we need this last result in order to refine the summation algorithms in [22] further.

8 Some Further Results

In [18] there are various investigations to find degree bounds. In particular in [18, Cor. 3.4.12] the following result pops up.

Theorem 18 *Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma$-field over $\mathbb{K}$ and let $(\mathbb{F}(t),\sigma)$ be a Σ^*-extension of $(\mathbb{F},\sigma)$; let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^m$ and $f \in \mathbb{F}[t]$ with $l := \|f\|$. If there is a $g \in \mathbb{F}[t]^*$ with $\sigma_{\boldsymbol{a}} g = f$ and $n := \deg(g)$ then there are $g_i \in \mathbb{F}[t]$ with $0 \leq i \leq n-l-1$ such that $\sigma_{\boldsymbol{a}} g_i = 0$ and $\deg(g_i) = i$.*

Example 20 Looking at Example 6 there is the solution $g = -\frac{1}{6}(h^3 + 3hh_2 + 2h_3)$ for $\sigma_{\boldsymbol{a}} g = -1$. Theorem 18 predicts $\sigma_{\boldsymbol{a}} g_i = 0$ with $\deg(g_i) = i$ for $0 \leq i \leq 2$. Indeed, we have the solutions $g_0 = 1$, $g_1 = h$ and $g_2 = h^2 + h_2$.

For a $\Pi\Sigma$-field $(\mathbb{F}(t),\sigma)$ where $(\mathbb{F}(t),\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$ this result delivers a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$, if $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$.

Corollary 7 *Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma$-field over $\mathbb{K}$ and let $(\mathbb{F}(t),\sigma)$ be a Σ^*-extension of $(\mathbb{F},\sigma)$. Let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^m$, $\boldsymbol{f} \in \mathbb{F}[t]^n$. If there is no $g \in \mathbb{F}^*$ with $\sigma_{\boldsymbol{a}} g = 0$ then $\|\boldsymbol{f}\|$ is a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$. Otherwise assume there are at least $k \geq 1$ linearly independent $g \in \mathbb{F}^*$ over $\mathbb{K}$ with $\sigma_{\boldsymbol{a}} g = 0$. Then $m + \|\boldsymbol{f}\| - k$ is a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$.*

Proof If there is no $g \in \mathbb{F}^*$ with $\sigma_{\boldsymbol{a}} g = 0$, then by Prop. 2 or Thm. 18 $\|\boldsymbol{f}\|$ is a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$. Otherwise assume that there are at least $k \geq 1$ linearly independent $g \in \mathbb{F}^*$ over $\mathbb{K}$ with $\sigma_{\boldsymbol{a}} g = 0$. Moreover suppose that $b := m + \|\boldsymbol{f}\| - k$ is not a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$, i.e., there is a $(c_1,\dots,c_n,g) \in \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]) \setminus \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_b)$. This means that $d := \deg(g) > b$ and $\sigma_{\boldsymbol{a}} g = \boldsymbol{c}\boldsymbol{f} =: f$ for $\boldsymbol{c} = (c_1,\dots,c_n)$. Clearly we have $l := \|f\| \leq \|\boldsymbol{f}\|$. By Theorem 18 there are at least $d - l + k - 1$ linearly independent solutions $g \in \mathbb{F}[t]^*$ over $\mathbb{K}$ with $\sigma_{\boldsymbol{a}} g = 0$. Thus by $d - l + k - 1 > (m + \|\boldsymbol{f}\| - k) - l + k - 1 \geq m - 1$ it follows that there is a subspace of $\mathrm{V}(\boldsymbol{a},(0),\mathbb{F}[t])$ which is generated by a basis of the form $B := \{(0,g_1),\dots,(0,g_m)\}$. Since $(1,0) \in \mathrm{V}(\boldsymbol{a},(0),\mathbb{F}[t])$, $B \cup \{(1,0)\}$ forms a basis of a subspace of $\mathrm{V}(\boldsymbol{a},(0),\mathbb{F}[t])$ over $\mathbb{K}$. Hence $\mathrm{V}(\boldsymbol{a},(0),\mathbb{F}(t))$ has at least dimension $m+1$; a contradiction to Prop. 1.

Example 21 (Cont. Exp. 12) By Corollary 7 we obtain the degree bound 3 for $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{Q}(n)(h_2)(h_3)(h))$.

Note that Corollary 7 contains Corollary 4 if one restricts to Σ^*-extensions and sets $k = 1$, i.e., $m - 1 + \|\boldsymbol{f}\|$ is a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$.

Remark 2 In many cases the dimension of $\mathbb{V} := \{g \in \mathbb{F} \,|\, \sigma_{\boldsymbol{a}} g = 0\}$ is respectably high. As pointed out in [18, Section 3.4.9], one first has to compute this vector space $\mathbb{V}$ if one wants to find a basis of the whole solution space $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$ according to the reduction techniques given in [19]. Hence one obtains the dimension for free. In particular, one can apply Corollary 7 with k as big as possible.

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