SYMBOLIC SUMMATION WITH SINGLE-NESTED SUM EXTENSIONS (EXTENDED VERSION)

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Abstract. We present a streamlined and refined version of Karr's summation algorithm. Karr's original approach constructively decides the telescoping problem in $\Pi\Sigma$-fields, a very general class of difference fields that can describe rational terms of arbitrarily nested indefinite sums and products. More generally, our new algorithm can decide constructively if there exists a so called single-nested $\Pi\Sigma$-extension over a given $\Pi\Sigma$-field in which the telescoping problem for f can be solved in terms that are not more nested than f itself. This allows to eliminate an indefinite sum over f by expressing it in terms of additional sums that are not more nested than f. Moreover, our refined algorithm contributes to definite summation: it can decide constructively if the creative telescoping problem for a fixed order can be solved in single-nested Σ^*-extensions that are less nested than the definite sum itself.

1. Introduction

Let $(\mathbb{F}, \sigma)$ be a difference field, i.e., a field[1] $\mathbb{F}$ together with a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$, and let $\mathbb{K}$ be its constant field, i.e., $\mathbb{K} = \text{const}_\sigma\mathbb{F} := \{k \in \mathbb{F} \mid \sigma(k) = k\}$. Then Problem *PFLDE* plays an important role in symbolic summation.

Problem *PFLDE*: Solving Parameterized First Order Linear Difference Equations

- *Given* $a_1, a_2 \in \mathbb{F}^*$ and $(f_1, \dots, f_n) \in \mathbb{F}^n$.
- *Find all* $g \in \mathbb{F}$ and $(c_1, \dots, c_n) \in \mathbb{K}^n$ with $a_1\,\sigma(g) + a_2\,g = \sum_{i=1}^{n} c_i\,f_i$.

For instance, if one takes the field of rational functions $\mathbb{F} = \mathbb{K}(k)$ with the shift $\sigma(k) = k+1$ and specializes to $n = 1$, $a_1 = 1$ and $a_2 = -1$, one considers the telescoping problem for a rational function $f_1 = f'(k) \in \mathbb{K}(k)$. Moreover, if $\mathbb{K} = \mathbb{K}'(m)$ and $f_i = f'(m+i-1, k) \in \mathbb{K}'(m)(k)$ for $1 \le i \le n$, one formulates the creative telescoping problem [14] of order $n-1$ for definite rational sums.

More generally, $\Pi\Sigma$-fields, introduced in [6, 7], are difference fields $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ where $\mathbb{F} := \mathbb{K}(t_1)\dots,(t_e)$ is a rational function field and the application of σ on the t_i's is recursively defined over $1 \le i \le e$ with $\sigma(t_i) = \alpha_i\,t_i + \beta_i$ for $\alpha_i, \beta_i \in \mathbb{K}(t_1)\dots(t_{i-1})$; we omitted some technical conditions given in Section 2. Note that $\Pi\Sigma$-fields enable to describe a huge class of sequences, like hypergeometric terms, as shown in [13], or most d'Alembertian solutions [1, 9], a subclass of Liouvillian solutions [5] of linear recurrences. More generally, $\Pi\Sigma$-fields allow to describe rational terms consisting of arbitrarily nested indefinite sums and products. We want to emphasize that the nested depth of these sums and products gives a

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[1] Throughout this paper all fields will have characteristic 0.

measure of the complexity of expressions. This can be carried over to $\Pi\Sigma$-fields by introducing the *depth* of t_i as the number of recursive definition steps that are needed to describe the application of σ on t_i; for more details see Section 2. Moreover, the depth of $f \in \mathbb{F}$ is the maximum depth of the t_i's that occur in f, and the depth of $(\mathbb{F}, \sigma)$ is the maximum depth of all the t_i.
The main result in [6] is an algorithm that solves Problem *PFLDE* and therefore the telescoping and creative telescoping problem for a given $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ where the constant field $\mathbb{K}$ is *σ-computable*. This means that **(1)** for any $k \in \mathbb{K}$ one can decide if $k \in \mathbb{Z}$, **(2)** polynomials in $\mathbb{K}[t_1, \dots, t_n]$ can be factored over $\mathbb{K}$, and **(3)** one knows how to compute a basis of $\{(n_1, \dots, n_k) \in \mathbb{Z}^k \mid c_1^{n_1} \dots c_k^{n_k} = 1\}$ for $(c_1, \dots, c_k) \in \mathbb{K}^k$ which is a submodule of $\mathbb{Z}^k$ over $\mathbb{Z}$. For instance, any rational function field $\mathbb{K} = \mathbb{A}(x_1, \dots, x_r)$ over an algebraic number field $\mathbb{A}$ is σ-computable; see [13].
In this paper we will present a streamlined and simplified version of Karr's original algorithm [6] for Problem *PFLDE* using Bronstein's denominator bound [2] and results from [6, 12, 10, 11]. Afterwards we will extend this approach to an algorithm that can solve

Problem *RS*: **R**efined **S**ummation

- Given a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with depth d, constant field $\mathbb{K}$ and $(f_1, \dots, f_n) \in \mathbb{F}^n$.
- *Decide* constructively if there are $(0, \dots, 0) \neq (c_1, \dots, c_n) \in \mathbb{K}^n$ and $g \in \mathbb{F}(x_1) \dots (x_e)$ for $\sigma(g) - g = \sum_{i=1}^{n} c_i f_i$ in an extended $\Pi\Sigma$-field $(\mathbb{F}(x_1) \dots (x_e), \sigma)$ with depth d and $\sigma(x_i) = \alpha_i x_i + \beta_i$ where $\alpha_i, \beta_i \in \mathbb{F}$.

Suppose we fail to find a solution g with $\sigma(g) - g = f$ in a given $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with depth d and $f \in \mathbb{F}^*$ with depth d, but there exists such an extended $\Pi\Sigma$-field $(\mathbb{F}(x_1) \dots (x_e), \sigma)$ and a solution g with depth d for $\sigma(g) - g = f$. Then our new algorithm can compute such an extension with such a solution g. As a side result we will show that it suffices to restrict to the sum case, i.e., $\sigma(x_i) - x_i \in \mathbb{F}$. In some sense our results shed new constructive light on Karr's Fundamental Theorem [6].

For instance, in Karr's approach [6] one can find the right hand side in (1) only by setting up manually the corresponding $\Pi\Sigma$-field in terms of the harmonic numbers $H_n := \sum_{i=1}^{n} \frac{1}{i}$ and the generalized versions $H_n^{(r)} := \sum_{i=1}^{n} \frac{1}{i^r}$, $r > 1$, whereas with our new algorithm the underlying $\Pi\Sigma$-field is constructed completely automatically. Additional examples are

$$\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j} \sum_{i=1}^{j} \frac{1}{i} = \frac{1}{6}\big[H_n^3 + 3H_n\, H_n^{(2)} + 2\, H_n^{(3)}\big], \tag{1}$$

$$\sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{H_i} = -n + H_n \sum_{i=1}^{n} \frac{1}{H_i} + \sum_{i=1}^{n} \frac{1}{iH_i},$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i} \right)^2 = (n-a)\binom{n}{a} \sum_{i=0}^{a} \binom{n}{i} - \frac{n-2a-2}{2} \left(\sum_{i=0}^{a} \binom{n}{i} \right)^2 - \frac{n}{2} \sum_{i=1}^{n} \binom{n}{i}^2.$$

Our new approach also refines creative telescoping: we might find a recurrence of smaller order by introducing additional sums with depths smaller than the definite sum.

All these algorithms have been implemented in form of the summation package *Sigma* in the computer algebra system Mathematica. The wide applicability of this new approach is illustrated for instance in [9, 8, 4].

2. Refined Summation in ΠΣ-Fields

First we introduce some notations and definitions. Let $(\mathbb{F},\sigma)$ be a difference field with $\mathbb{K}=\text{const}_\sigma\mathbb{F}$, $\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}^2$ and $\boldsymbol{f}=(f_1,\dots,f_n)\in\mathbb{F}^n$. For any $\boldsymbol{h}=(h_1,\dots,h_n)\in\mathbb{F}^n$ and $p\in\mathbb{F}$ we write $\boldsymbol{f}\,\boldsymbol{h}:=\sum_{i=1}^n f_i\,h_i$, $\sigma(\boldsymbol{h}):=(\sigma(h_1),\dots,\sigma(h_n))$, and $\boldsymbol{h}\,p:=(h_1\,p,\dots,h_n\,p)$. We define $\boldsymbol{0_n}:=(0,\dots,0)\in\mathbb{K}^n$, and write $\boldsymbol{0}=\boldsymbol{0_n}$ if it is clear form the context. We call $\boldsymbol{a}$ *homogeneous over* $\mathbb{F}$ if $a_1\,a_2\neq 0$ and $a_1\,\sigma(g)+a_2\,g=0$ for some $g\in\mathbb{F}^*$.
Now let $\mathbb{V}$ be a subspace of $\mathbb{F}$ over $\mathbb{K}$ and suppose that $\boldsymbol{a}\neq\boldsymbol{0}$. Then we define the *solution space* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{V})$ as the subspace $\{(c_1,\dots,c_n,g)\in\mathbb{K}^n\times\mathbb{V}\,|\,a_1\sigma(g)+a_2g=\sum_{i=1}^n c_if_i\}$ of the vector space $\mathbb{K}^n\times\mathbb{F}$ over $\mathbb{K}$. By difference field theory [3], the dimension is at most $n+1$; see also [9, 10]. Therefore Problem *PFLDE* is equivalent to find a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$.
A difference field $(\mathbb{E},\sigma')$ is a *difference field extension* of $(\mathbb{F},\sigma)$ if $\mathbb{F}$ is a subfield of $\mathbb{E}$ and $\sigma'(g)=\sigma(g)$ for $g\in\mathbb{F}$; note that from now σ and σ' are not distinguished anymore.
A difference field extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ is a Π- *(resp. Σ^*-) extension* if $\mathbb{F}(t)$ is a rational function field, $\sigma(t)=a\,t$ ($\sigma(t)=t+a$ resp.) for some $a\in\mathbb{F}^*$ and $\text{const}_\sigma\mathbb{F}(t)=\text{const}_\sigma\mathbb{F}$. A difference field extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ is a *Σ-extension* if $\mathbb{F}(t)$ is a rational function field, $\sigma(t)=\alpha\,t+\beta$ for some $\alpha,\beta\in\mathbb{F}^*$, $\text{const}_\sigma\mathbb{F}(t)=\text{const}_\sigma\mathbb{F}$, and the following two properties hold: (1) there does not exist a $g\in\mathbb{F}(t)\setminus\mathbb{F}$ with $\sigma(g)/g\in\mathbb{F}$, and (2) if there is a $g\in\mathbb{F}^*$ and $n\neq 0$ with $\sigma(g)/g=\alpha^n$ then there is also a $g\in\mathbb{F}^*$ with $\sigma(g)/g=\alpha$. Note that any Σ^*-extension is also a Σ-extension; for more details see [6, 7, 2, 9, 13]. A *ΠΣ-extension* is either a Π- or a Σ-extension. A difference field extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ is a *(nested) Σ^*/ΠΣ-extension* if $(\mathbb{F}(t_1)\dots(t_i),\sigma)$ is a Σ^*/ΠΣ-extension of $(\mathbb{F}(t_1)\dots(t_{i-1}),\sigma)$ for all $1\leq i\leq e$; for $i=0$ we define $\mathbb{F}(t_1)\dots(t_{i-1})=\mathbb{F}$. Note that $e=0$ gives the trivial extension.
For $\mathbb{H}\subseteq\mathbb{F}$, a ΠΣ-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ is *single-nested over* $\mathbb{H}$, or in short *over* $\mathbb{H}$, if $\sigma(t_i)=\alpha_i\,t_i+\beta_i$ with $\alpha_i,\beta_i\in\mathbb{H}$ for all $1\leq i\leq e$. A ΠΣ-extension of $(\mathbb{F},\sigma)$ is called single-nested, if it is single-nested over $\mathbb{F}$.
Finally, a *ΠΣ-field* $(\mathbb{F},\sigma)$ *over* $\mathbb{K}$ is a ΠΣ-extension of $(\mathbb{K},\sigma)$ with $\text{const}_\sigma\mathbb{K}=\mathbb{K}$, i.e., $\text{const}_\sigma\mathbb{F}=\mathbb{K}$.

In [6] alternative definitions of ΠΣ-extensions are introduced that allow to decide constructively if an extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ is a ΠΣ-extension under the assumption that $(\mathbb{F},\sigma)$ is a ΠΣ-field over a σ-computable $\mathbb{K}$. For instance, for Σ^*-extensions there is the following result given in [7, Theorem 2.3] or [9, Corollary 2.2.4].

Theorem 1. *Let $(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$. Then this is a Σ^*-extension iff $\sigma(t)=t+\beta$, $t\notin\mathbb{F}$, $\beta\in\mathbb{F}$, and there is no $g\in\mathbb{F}$ with $\sigma(g)-g=\beta$.*

In particular, this result states that indefinite summation and building up Σ^*-extensions are closely related. Namely, if one fails to find a $g\in\mathbb{F}$ with $\sigma(g)-g=\beta\in\mathbb{F}$, i.e., one cannot solve the telescoping problem in $\mathbb{F}$, one can adjoin the solution t with $\sigma(t)+t=\beta$ to $\mathbb{F}$ in form of the Σ^*-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$.

Our refined simplification strategy for a given sum is as follows: If we fail to solve the telescoping problem, we do not adjoin immediately the sum in form of a Σ^*-extension, but we first try to find an appropriate ΠΣ-extension in which the sum can be formulated less nested.

These ideas can be clarified further with the depth-function. Let $\mathbb{F} = \mathbb{K}(t_1, \dots, t_e)$ be a function field over $\mathbb{K}$. Then for $g = \frac{g_1}{g_2} \in \mathbb{F}^*$ with $g_i \in \mathbb{K}[t_1, \dots, t_e]$ and $\gcd_{\mathbb{K}[t_1,\dots,t_e]}(g_1, g_2) = 1$ we define the support of g, in short $\text{supp}_{\mathbb{F}}(g)$, as those t_i that occur in g_1 or g_2. Then for a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ over $\mathbb{K}$ with $\mathbb{F} := \mathbb{K}(t_1)\dots(t_e)$ and $\sigma(t_i) = \alpha_i\, t_i + \beta_i$ for $\alpha_i, \beta_i \in \mathbb{K}(t_1)\dots(t_{i-1})$, the depth-function $\text{depth} : \mathbb{F} \to \mathbb{N}_0$ is defined recursively as follows. For any $g \in \mathbb{K}$ set $\text{depth}(g) = 0$. If the depth-function is defined for $(\mathbb{K}(t_1)\dots(t_{i-1}), \sigma)$ with $i > 1$, we define $\text{depth}(t_i) = \max(\text{depth}(\alpha_i), \text{depth}(\beta_i)) + 1$ and for $g \in \mathbb{K}(t_1)\dots(t_i)$ we define $\text{depth}(g) = \max(\{\text{depth}(x) \mid x \in \text{supp}_{\mathbb{K}(t_1,\dots,t_i)}(g)\} \cup \{0\})$. The depth of $(\mathbb{F}, \sigma)$, in short $\text{depth}(\mathbb{F})$, is the maximal depth of all elements in $\mathbb{F}$, i.e., $\text{depth}(\mathbb{F})$ is equal to $\max(0, \text{depth}(t_1), \dots, \text{depth}(t_e))$. We say that a $\Pi\Sigma/\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ of a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ has maximal depth d if $\text{depth}(t_i) \le d$ for all $1 \le i \le e$.

Now we can reformulate Problem *RS* as follows. *Given* a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with depth d and $\boldsymbol{f} \in \mathbb{F}^n$. *Decide* constructively if there is a single-nested $\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with maximal depth d, $g \in \mathbb{E}$ and $\mathbf{0} \neq \boldsymbol{c} \in \mathbb{K}^n$ such that $\sigma(g) - g = \boldsymbol{c}\,\boldsymbol{f}$.

Example 1. *Denote the left side in* (1) *with* $S_n^{(3)}$ *and define* $S_n^{(1)} := \sum_{i=1}^n \frac{1}{i}$ *and* $S_n^{(2)} := \sum_{j=1}^n S_j^{(1)}/j$. *In the straightforward summation approach one applies usual telescoping which results in the* $\Pi\Sigma$*-field* $(\mathbb{Q}(t_1)(t_2)(t_3)(t_4), \sigma)$ *over* $\mathbb{Q}$ *with* $\sigma(t_1) = t_1 + 1$, $\sigma(t_2) = t_2 + \frac{1}{t_1+1}$, $\sigma(t_3) = t_3 + \sigma(\frac{t_2}{t_1})$ *and* $\sigma(t_4) = t_4 + \sigma(\frac{t_3}{t_1})$, *i.e., there is no* $g \in \mathbb{Q}(t_1)$ *with* $\sigma(g) - g = \frac{1}{t_1+1}$ *and no* $g \in \mathbb{Q}(t_1)\dots(t_r)$ *with* $\sigma(g) - g = \sigma(\frac{t_r}{t_1})$ *for* $r = 2, 3$. *Then* t_r *represents* $S_n^{(r-1)}$ *with* $\text{depth}(t_r) = r$ *for* $r = 2, 3, 4$, *and* $\text{depth}(\mathbb{Q}(t_1)\dots(t_4)) = 4$. *But with our refined summation approach we obtain the following improvement starting from the* $\Pi\Sigma$*-field* $(\mathbb{F}, \sigma)$ *with* $\mathbb{F} := \mathbb{Q}(t_1)(t_2)$. *We find the* Σ^**-extension* $(\mathbb{F}(s), \sigma)$ *of* $(\mathbb{F}, \sigma)$ *with* $\sigma(s) = s + \frac{1}{(t_1+1)^2}$ *with the solution* $g := \frac{t_2^2 + s}{2}$ *for* $\sigma(g) - g = \sigma(\frac{t_2}{t_1})$ *that represents the sum* $S_n^{(2)}$. *Moreover, we find the* Σ^**-extension* $(\mathbb{F}(s)(s')), \sigma)$ *of* $(\mathbb{F}(s), \sigma)$ *with* $\sigma(s') = s' + \frac{1}{(t_1+1)^3}$ *and the solution* $g' = \frac{1}{6}(t_2^3 + 3t_2\, s + 2s')$ *for* $\sigma(g') - g' = \sigma(g/t_1)$. *Then* $S_n^{(3)}$ *is represented by* g' *with* $\text{depth}(g') = 2$ *which gives the right hand side of identity* (1).

Besides refined indefinite summation, we obtain a generalized version of creative telescoping in $\Pi\Sigma$-fields. Suppose that the sequences $f'(m + i - 1, k)$ can be represented with $f_i \in \mathbb{F}$ for $i \ge 1$ in a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ over $\mathbb{K}(m)$ with $\text{depth}(f_i) = d$. Moreover assume that we do not find a $g \in \mathbb{F}$ and $\mathbf{0} \neq \boldsymbol{c} \in \mathbb{K}(m)^n$ with $\sigma(g) - g = \boldsymbol{c}\,\boldsymbol{f}$ for $\boldsymbol{f} = (f_1, \dots, f_n)$. Then the usual strategy is to increase n, i.e., the order of the possibly resulting creative telescoping recurrence. But if we find a solution for Problem *RS*, we derive a recurrence of order $n - 1$ in terms of sum extensions with maximal depth d.

Summarizing, for telescoping and creative telescoping we are interested in finding a single-nested $\Pi\Sigma$-extension in which a nontrivial linear combination of $(f_1, \dots, f_n)$ in the solution space exists. More generally, we will ask for those extensions that will give us additional linear combinations. To make this more precise, we define for any $\mathbb{A} \subseteq \mathbb{F}^{n+1}$ the set $\Pi_n(\mathbb{A}) := \{(a_1, \dots, a_n) \mid (a_1, \dots, a_n, a_{n+1}) \in \mathbb{A}\}$.

Definition 1. *Let* $(\mathbb{E}, \sigma)$ *be a* $\Pi\Sigma$*-field over* $\mathbb{K}$ *with depth* d, $1 \le \delta \le d + 1$, *and* $\boldsymbol{f} \in \mathbb{E}^n$. *We call a* $\Pi\Sigma$*-extension* $(\mathbb{G}, \sigma)$ *of* $(\mathbb{E}, \sigma)$ single-nested δ-complete *for* $\boldsymbol{f}$ *if for all single-nested* $\Pi\Sigma$*-extensions* $(\mathbb{H}, \sigma)$ *of* $(\mathbb{E}, \sigma)$ *with maximal depth* δ *we have*

$$\Pi_n(\mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{H})) \subseteq \Pi_n(\mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{G})). \tag{2}$$

In this paper we solve the following problem. *Given* a $\Pi\Sigma$-field $(\mathbb{E},\sigma)$ over a σ-computable $\mathbb{K}$ with depth d, $\boldsymbol{f}\in\mathbb{E}^n$ and $\delta\in\mathbb{N}$ with $1\leq\delta\leq d+1$; *compute* a single-nested Σ^*-extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ with maximal depth δ which is single-nested δ-complete for $\boldsymbol{f}$, and compute a basis of $\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{G})$. Note that Problem *RS* for single-nested $\Pi\Sigma$-extension is contained in this problem by setting $\delta:=d$.

3. A more general problem

In order to treat the problem stated in the previous paragraph, we solve the more general problem to find an $\mathbb{F}$-complete extension of $(\mathbb{E},\sigma)$ for $\boldsymbol{f}$ defined in

Definition 2. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ and $\boldsymbol{f}\in\mathbb{E}^n$. We call a $\Pi\Sigma$-extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$* single-nested $\mathbb{F}$-complete for $\boldsymbol{f}$*, or in short* $\mathbb{F}$-complete for $\boldsymbol{f}$*, if* (2) *holds for all $\Pi\Sigma$-extensions $(\mathbb{H},\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$.*

The following lemma is crucial in order to show in Theorem 2 that there exists a Σ^*-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ which is $\mathbb{F}$-complete for $\boldsymbol{f}$. This means that it suffices to restrict to Σ^*-extensions. Moreover this lemma is needed to prove Theorem 6 which gives us the essential idea how one can compute such $\mathbb{F}$-complete extensions.

Lemma 1. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ and $f\in\mathbb{E}^*$. If there exists a single-nested $\Pi\Sigma$-extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ with a $g\in\mathbb{G}\setminus\mathbb{E}$ such that $\sigma(g)-g=f$ then there exists a Σ^*-extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ with a $w\in\mathbb{E}$ such that $\sigma(s+w)-(s+w)=f$.*

Proof. Let $(\mathbb{G},\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$, i.e., $\mathbb{G}=\mathbb{E}(t_1)\dots(t_e)$ with $\sigma(t_i)=\alpha_i\,t_i+\beta_i$ and $\alpha_i,\beta_i\in\mathbb{F}$, and suppose that there is a $g\in\mathbb{G}\setminus\mathbb{E}$ with $\sigma(g)-g=f$. Then by Karr's Fundamental Theorem [6, Theorem 24], see also [7, Section 4], it follows that $g=\sum_{i=0}^{e}c_i\,t_i+w$ for some $w\in\mathbb{E}$ and $c_i\in\mathbb{K}$, where $c_i=0$ if $\sigma(t_i)-t_i\notin\mathbb{F}$. In particular, $\mathbf{0}\neq(c_1,\dots,c_e)$, since $g\notin\mathbb{E}$. Now let $\mathbb{E}(s)$ be a rational function field and suppose that the difference field extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(s)-s=\sum_{i=1}^{e}c_i\,(\sigma(t_i)-t_i)=:\beta\in\mathbb{F}$ is not a Σ^*-extension. Then by Theorem 1 we can take a $g'\in\mathbb{E}$ with $\sigma(g')-g'=\beta$. Let j be maximal such that $c_j\neq 0$. Then we have $\sigma(v)-v=\sigma(t_j)-t_j\in\mathbb{F}$ for $v:=\frac{1}{c_j}\big(g'-\sum_{i=1}^{j-1}c_i\,t_i\big)\in\mathbb{E}(t_1)\dots(t_{j-1})$, and thus $(\mathbb{E}(t_1)\dots(t_{j-1})(t_j),\sigma)$ is not a Σ^*-extension of $(\mathbb{E}(t_1)\dots(t_{j-1}),\sigma)$ by Theorem 1, a contradiction. Hence $(\mathbb{E}(s),\sigma)$ is a Σ^*-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$, and $\sigma(s+w)-(s+w)=\sum_{i=1}^{e}c_i(\sigma(t_i)-t_i)+\sigma(w)-w=\sigma(g)-g=f$. □

Observe that Lemma 1 follows immediately by Theorem 1 if one restricts to the special case $\mathbb{E}=\mathbb{F}$. For the case $\mathbb{F}\subsetneq\mathbb{E}$, in which we are actually interested, we needed Karr's Fundamental Theorem [6].

Theorem 2. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ and $\boldsymbol{f}\in\mathbb{E}^n$. Then there is a Σ^*-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ which is $\mathbb{F}$-complete for $\boldsymbol{f}$.*

Proof. Let $(\mathbb{G},\sigma)$ be a Σ^*-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ which is not $\mathbb{F}$-complete for $\boldsymbol{f}$. Then we can take a $\boldsymbol{c}\in\mathbb{K}^n$ such that $\sigma(g)-g=\boldsymbol{c}\,\boldsymbol{f}\in\mathbb{E}$ has a solution in some $\Pi\Sigma$-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$, but no solution in $\mathbb{E}$. Then by Lemma 1 it follows that there is a Σ^*-extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ with $\sigma(s+w)-(s+w)=f$ for some $w\in\mathbb{E}$. Observe that there also does not exist an $h\in\mathbb{G}$ with $\sigma(h)-h=\beta\in\mathbb{F}$. Otherwise we would have $\sigma(h+w)-(h+w)=\boldsymbol{c}\,\boldsymbol{f}$ with $h+w\in\mathbb{G}$, a contradiction. Consequently, by Theorem 1 also $(\mathbb{G}(s),\sigma)$ is a Σ^*-extension of $(\mathbb{G},\sigma)$ with $\sigma(s)=s+\beta$ and therefore a Σ^*-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$. Since $\Pi_n(\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{G}))$ is a proper subspace of $\Pi_n(\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{G}(s)))$ and those spaces have dimension at most n, this argument can be repeated at most n times before an $\mathbb{F}$-complete Σ^*-extension is reached. □

In the following we will represent the $\Pi\Sigma$-field $(\mathbb{E}, \sigma)$ in such a way that one can find a single-nested δ-complete extension of $(\mathbb{E}, \sigma)$ for $\boldsymbol{f}$ by finding an $\mathbb{F}$-complete extension over a certain subfield $\mathbb{F} \subseteq \mathbb{E}$.

Let $\mathbb{G} := \mathbb{F}(s_1) \dots (s_u)(x)(t_1) \dots (t_v)$ be a field of rational functions. Then the field $\mathbb{H} := \mathbb{F}(x)(s_1) \dots (s_u)(t_1) \dots (t_v)$ is isomorphic with $\mathbb{G}$ by the field isomorphism $\tau : \mathbb{G} \to \mathbb{H}$ with $\tau(f) = f$ for all $f \in \mathbb{F}$, $\tau(s_i) = s_i$, $\tau(x) = x$ and $\tau(t_i) = t_i$. More sloppily, we write $\tau(f) = f$ for $f \in \mathbb{G}$, or $\mathbb{G} = \mathbb{H}$. Now suppose that in addition we consider a $\Pi\Sigma$-extension $(\mathbb{G}, \sigma)$ of $(\mathbb{F}, \sigma)$. Then we can define the automorphism $\sigma' : \mathbb{H} \to \mathbb{H}$ with $\sigma'(f) = \tau(\sigma(\tau^{-1}(f)))$ for all $f \in \mathbb{H}$. In a more sloppy way, we write $\sigma = \sigma'$. Then obviously, $(\mathbb{H}, \sigma)$ is a difference field extension of $(\mathbb{F}, \sigma)$ with $\mathrm{const}_\sigma\mathbb{G} = \mathrm{const}_\sigma\mathbb{H} = \mathrm{const}_\sigma\mathbb{F}$. But in general, $(\mathbb{H}, \sigma)$ is not anymore a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$. But if we have $\sigma(x) = \alpha\, x + \beta$ with $\alpha, \beta \in \mathbb{F}$ then this reordering of the variables gives us again a $\Pi\Sigma$-extension which is isomorphic to the original one with the trivial difference field isomorphism $\tau : \mathbb{G} \to \mathbb{H}$ with $\tau(f) = f$ and $\sigma(\tau(f)) = \tau(\sigma(f))$. The proof of this statement can be carried out rigorously with techniques used in [9, Section 2.4]. Observe that one can reorder a $\Pi\Sigma$-field $(\mathbb{E}, \sigma)$ over $\mathbb{K}$ with depth d and $1 \leq \delta \leq d + 1$ to a $\Pi\Sigma$-field $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ with $\mathrm{depth}(\mathbb{F}) = \delta - 1$ and $\mathrm{depth}(t_i) \geq \delta$ for all $1 \leq i \leq e$. This construction is possible, since any $\Pi\Sigma$-extension in $\mathbb{F}$ has smaller depth than the t_i and is therefore free of the t_i in the definition of σ. In addition, we obtain the *difference field isomorphism* $\tau : \mathbb{E} \to \mathbb{F}(t_1) \dots (t_e)$ where $\tau(f) = f$ for all $f \in \mathbb{E}$. With this reordered $\Pi\Sigma$-field one obtains

Lemma 2. *Let $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ be a $\Pi\Sigma$-field with $\delta := \mathrm{depth}(\mathbb{F}) + 1$ and $\mathrm{depth}(t_i) \geq \delta$ for $1 \leq i \leq e$, and let $(\mathbb{H}, \sigma)$ be a single-nested $\Pi\Sigma$-extension of $(\mathbb{F}(t_1) \dots (t_e), \sigma)$. Then this extension has maximal depth δ iff it is over $\mathbb{F}$.*

Proof. Write $\mathbb{H} := \mathbb{F}(t_1) \dots (t_e)(s_1) \dots (s_u)$. First assume that the extension is over $\mathbb{F}$, i.e., $\sigma(s_i) = \alpha_i s_i + \beta_i$ with $\alpha_i, \beta_i \in \mathbb{F}$. Then, because of $\mathrm{depth}(\mathbb{F}) = \delta - 1$ it follows that $\mathrm{depth}(\beta_i) \leq \delta - 1$ and $\mathrm{depth}(\alpha_i) \leq \delta - 1$, thus $\mathrm{depth}(s_i) = \max(\mathrm{depth}(\alpha_i), \mathrm{depth}(\beta_i)) + 1 \leq \delta$, and therefore the extension has maximal depth δ. Conversely, suppose that this extension has maximal depth δ, i.e. $\mathrm{depth}(s_i) \leq \delta$. Then $\mathrm{depth}(\alpha_i) \leq \delta - 1$ and $\mathrm{depth}(\beta_i) \leq \delta - 1$, and consequently $\alpha_i, \beta_i \in \mathbb{F}$. □

Theorem 3. *Let $(\mathbb{E}, \sigma)$ with $\mathbb{E} := \mathbb{F}(t_1) \dots (t_e)$ be a $\Pi\Sigma$-field where $\delta := \mathrm{depth}(\mathbb{F}) + 1$ and $\mathrm{depth}(t_i) \geq \delta$ for $1 \leq i \leq e$, and $\boldsymbol{f} \in \mathbb{E}^n$. Then a $\Pi\Sigma$-extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$ which is $\mathbb{F}$-complete for $\boldsymbol{f}$ has maximal depth δ and is single-nested δ-complete for $\boldsymbol{f}$.*

Proof. Assume such an extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ is not single-nested δ-complete for $\boldsymbol{f}$. Then take a single-nested $\Pi\Sigma$-extension $(\mathbb{H}, \sigma)$ of $(\mathbb{E}, \sigma)$ with maximal depth δ and $\boldsymbol{c} \in \Pi_n(\mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{H})) \setminus \Pi_n(\mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{G}))$. Since $\delta = \mathrm{depth}(\mathbb{F}) + 1$ and $\mathrm{depth}(t_i) \geq \delta$, $(\mathbb{H}, \sigma)$ is an extension of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$ by Lemma 2, and thus the extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ is not $\mathbb{F}$-complete for $\boldsymbol{f}$, a contradiction. Moreover, the extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ is single-nested with maximal depth δ by Lemma 2. □

In Section 5 we will develop an algorithm that computes an $\mathbb{F}$-complete Σ^*-extension of $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ over $\mathbb{F}$ for $\boldsymbol{f}$. Then by Theorem 3 this extension will be also single-nested δ-complete for $\boldsymbol{f}$ with maximal depth δ.

4. A Reduction Strategy

We develop a streamlined version of Karr's summation algorithm [6] based on results of [2] and [9, 12, 10, 11] that solves Problem *PFLDE*. In particular, this approach will assist in finding $\mathbb{F}$-complete extensions over $\mathbb{F}$.

More precisely, let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$, $\mathbb{K}=\text{const}_\sigma\mathbb{F}$, $\boldsymbol{0}\neq\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}(t)^2$ and $\boldsymbol{f}\in\mathbb{F}(t)^n$. We will introduce a simplified version of Karr's reduction strategy [6] that helps in finding a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ over $\mathbb{K}$. If $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field, this reduction turns into a complete algorithm. Moreover, this reduction technique will deliver all the information to compute an $\mathbb{F}$-complete extension.

A special case. If $a_1\,a_2=0$, we have $\boldsymbol{g}=\boldsymbol{c}\,\sigma^{-1}(\frac{\boldsymbol{f}}{a_1})$ with $a_1\neq 0$ or $\boldsymbol{g}=\boldsymbol{c}\frac{\boldsymbol{f}}{a_2}$ with $a_2\neq 0$. Then it follows with $\boldsymbol{g}=(g_1,\dots,g_n)$ and the i-th unit vector $(0\dots,1,\dots,0)\in\mathbb{K}^n$ that $\{(0\dots,1,\dots,0,g_i)\}_{1\leq i\leq n}\subseteq\mathbb{K}^n\times\mathbb{F}(t)$ is a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$. Hence from now on we suppose $\boldsymbol{a}\in(\mathbb{F}(t)^*)^2$.

Clearing denominators and cancelling common factors. Compute $\boldsymbol{a'}=(a'_1,a'_2)\in(\mathbb{F}[t]^*)^2$, $\boldsymbol{f'}=(f'_1,\dots,f'_n)\in\mathbb{F}[t]^n$ such that $\gcd_{\mathbb{F}[t]}(f'_1,\dots,f'_n,a'_1,a'_2)=1$ and $\boldsymbol{a'}=\boldsymbol{a}\,q$, $\boldsymbol{f'}=\boldsymbol{f}\,q$ for some $q\in\mathbb{F}(t)^*$. Then we have $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(t))$. Therefore we may suppose that $\boldsymbol{a}\in(\mathbb{F}[t]^*)^2$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$ where the entries have no common factors.

In Karr's original approach [6] the solutions $g=p+q\in\mathbb{F}(t)$ in $(c_1,\dots,c_n,g)\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ are computed by deriving first the polynomial part $p\in\mathbb{F}[t]$ and afterwards the fractional part $q\in\mathbb{F}(t)$, i.e., the degree of the numerator is smaller than the degree of the denominator. We simplify this approach substantially by first computing a common denominator of all the possible solutions in $\mathbb{F}(t)$ and afterwards computing the “numerator” of the solutions over this common denominator.

Denominator bounding. In the first important reduction step one looks for a *denominator bound* d of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$, i.e. a polynomial $d\in\mathbb{F}[t]^*$ that fulfills

$$\forall(c_1,\dots,c_n,g)\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t)):\ d\,g\in\mathbb{F}[t].$$

Since $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ is a finite dimensional vector space over $\mathbb{K}$, a denominator bound must exist. Now suppose that we have given such a d and define $\boldsymbol{a'}:=(\frac{a_1}{\sigma(d)},\frac{a_2}{d})$. Then it follows that $\{(c_{i1},\dots,c_{in},g_i)\}_{1\leq i\leq r}$ is a basis of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{F}[t])$ if and only if $\{(c_{i1},\dots,c_{in},\frac{g_i}{d})\}_{1\leq i\leq r}$ is a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$. For a proof we refer to [9, 12]. Hence, given a denominator bound d of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$, we can reduce the problem to search for a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ to look for a basis of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{F}[t])$. By clearing denominators and cancelling common factors in $\boldsymbol{a}$ and $\boldsymbol{f}$, as above, we may also suppose that $\boldsymbol{a}\in(\mathbb{F}[t]^*)^2$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$.

Polynomial degree bounding. The next step consists of bounding the polynomial degrees in $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$. For convenience we introduce $\mathbb{F}[t]_b:=\{f\in\mathbb{F}[t]\mid\deg(f)\leq b\}$ for $b\in\mathbb{N}_0$ and $\mathbb{F}[t]_{-1}:=\{0\}$. Moreover, we define $\|b\|:=\deg b$ for $b\in\mathbb{F}[t]^*$, $\|0\|:=-1$, and $\|\boldsymbol{b}\|:=\max_{1\leq i\leq l}\|b_i\|$ for $\boldsymbol{b}=(b_1,\dots,b_l)\in\mathbb{F}[t]^l$. Then we look for a *polynomial degree bound* b of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$, i.e., a $b\in\mathbb{N}_0\cup\{-1\}$ such that

$$\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_b)=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]),\ b\geq\max(-1,\|\boldsymbol{f}\|-\|\boldsymbol{a}\|). \tag{3}$$

Again, since $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$ is finite dimensional over $\mathbb{K}$, a degree bound must exist. Note that by the second condition in (3) it follows that $\boldsymbol{f}\in\mathbb{F}[t]_{\|\boldsymbol{a}\|+b}$ which is needed to proceed with the degree elimination technique below.

Due to [6, 7, 2] the problem to determine a denominator bound or degree bound is completely constructive if $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$. The proofs and sub-algorithms of these results can be found in [2, 10, 11].

Theorem 4. *If* $(\mathbb{F}(t), \sigma)$ *is a* $\Pi\Sigma$*-field over a* σ*-computable* $\mathbb{K}$*,* $\boldsymbol{a} \in (\mathbb{F}[t]^*)^2$ *and* $\boldsymbol{f} \in \mathbb{F}[t]^n$ *then there are algorithms that compute a denominator bound of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ *or a degree bound of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$*.*

Polynomial degree reduction. Finally we have to deal with the problem to compute a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_\delta)$ for some $\delta \in \mathbb{N}_0 \cup \{-1\}$ where $\boldsymbol{f} \in \mathbb{F}[t]^n_{\delta+l}$ with $l := \|\boldsymbol{a}\|$; this is guaranteed if δ is a polynomial degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$. Here we follow exactly the idea in [6]. Namely, we first find the candidates of the leading coefficients $g_\delta \in \mathbb{F}$ for the solutions $(c_1, \dots, c_n, g) \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_\delta)$ with $g = \sum_{i=0}^{\delta} g_i\, t^i$, plugging back its solution space and go on recursively to derive the candidates of the missing coefficients $g_i \in \mathbb{F}$.

This reduction idea is graphically illustrated in Figure 1 which has to be read as follows. The problem of finding a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_\delta)$ is reduced to **(i)** searching for a basis of $\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{\delta}}, \tilde{\boldsymbol{f}}_{\boldsymbol{\delta}}, \mathbb{F})$ for some specifically determined $\boldsymbol{0} \neq \tilde{\boldsymbol{a}}_{\boldsymbol{\delta}} \in \mathbb{F}^2$ and $\tilde{\boldsymbol{f}}_{\boldsymbol{\delta}} \in \mathbb{F}^n$ and **(ii)** finding a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}_{\boldsymbol{\delta-1}}, \mathbb{F}[t]_{\delta-1})$ for some particular chosen $\boldsymbol{f}_{\boldsymbol{\delta-1}} \in \mathbb{F}[t]^\lambda_{\delta-1}$. Then **(iii)**, the original problem $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_\delta)$ can be reconstructed by the two bases of the corresponding subproblems. Intuitively, the solution in $\mathbb{F}[t]_\delta$ is reconstructed by sub-solutions in $\mathbb{F}$ (the leading coefficients) and $\mathbb{F}[t]_{\delta-1}$ (the polynomial with the remaining coefficients) which is reflected by the vector space isomorphism $\mathbb{F}[t]_\delta \simeq \mathbb{F}[t]_{\delta-1} \oplus t^\delta\, \mathbb{F}$. In the sequel we explain this reduction in more details. Define

$$\begin{aligned} \tilde{\boldsymbol{a}}_{\boldsymbol{\delta}} &= (\tilde{a}_1, \tilde{a}_2) := \big(\mathrm{coeff}(a_1, l)\, \alpha^\delta, \mathrm{coeff}(a_2, l)\big) \\ \tilde{\boldsymbol{f}}_{\boldsymbol{\delta}} &:= (\mathrm{coeff}(f_1, \delta + l), \dots, \mathrm{coeff}(f_n, \delta + l)). \end{aligned} \tag{4}$$

where $\boldsymbol{0} \neq \tilde{\boldsymbol{a}}_{\boldsymbol{\delta}} \in \mathbb{F}^2$ and $\tilde{\boldsymbol{f}}_{\boldsymbol{\delta}} \in \mathbb{F}^n$; $\mathrm{coeff}(p, l)$ gives the l-th coefficient of $p \in \mathbb{F}[t]$. Then there is the following crucial observation for a solution $\boldsymbol{c} \in \mathbb{K}^n$ and $g = \sum_{i=0}^{\delta} g_i\, t^i \in \mathbb{F}[t]_\delta$ of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_\delta)$; see [9, 12]: Since t is transcendental over $\mathbb{F}$, it follows by coefficient comparison that $\tilde{a}_1\, \sigma(g_\delta) + \tilde{a}_2\, g_\delta = \boldsymbol{c}\, \tilde{\boldsymbol{f}}_{\boldsymbol{\delta}}$ which means that $(c_1, \dots, c_n, g_\delta) \in \mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{\delta}}, \tilde{\boldsymbol{f}}_{\boldsymbol{\delta}}, \mathbb{F})$. Therefore, the right linear combinations of a basis of $\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{\delta}}, \tilde{\boldsymbol{f}}_{\boldsymbol{\delta}}, \mathbb{F})$ enable one to construct partially the solutions $(c_1, \dots, c_n, g) \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_\delta)$, namely $(c_1, \dots, c_n) \in \mathbb{K}^n$ with the δ-th coefficient g_δ in $g \in \mathbb{F}[t]_\delta$. So, the basic idea is to find first a basis B_1 of $\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{\delta}}, \tilde{\boldsymbol{f}}_{\boldsymbol{\delta}}, \mathbb{F})$.

• CASE A: $B_1 = \{\}$. Then there are no $g \in \mathbb{F}[t]_\delta$ and $\boldsymbol{0} \neq \boldsymbol{c} \in \mathbb{K}^n$ with $a_1\, \sigma(g) + a_2\, g = \boldsymbol{c}\, \boldsymbol{f}$, and thus $\boldsymbol{c} = \boldsymbol{0}$ and $g \in \mathbb{F}[t]_{\delta-1}$ with $a_1\, \sigma(g) + a_2\, g = 0$ give the only solutions; see [12]. Hence, take a basis B_2 of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}_{\boldsymbol{\delta-1}}, \mathbb{F}[t]_{\delta-1})$ with

$$\boldsymbol{f}_{\boldsymbol{\delta-1}} := (0)$$

and try to extract such a $g \in \mathbb{F}[t]^*_{\delta-1}$ from B_2. If possible, a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_\delta)$ is $(0, \dots, 0, g)$. Otherwise, $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_\delta) = \{\boldsymbol{0}_{\boldsymbol{n+1}}\}$.

• CASE B: $B_1 \neq \{\}$, say $B_1 = \{(c_{i1}, \dots, c_{in}, w_i)\}_{1 \leq i \leq \lambda}$. Then define $\boldsymbol{C} := (c_{ij}) \in \mathbb{K}^{\lambda \times n}$, $\boldsymbol{g} := (w_1\, t^\delta, \dots, w_\lambda\, t^\delta) \in t^\delta\, \mathbb{F}^\lambda$ and consider

$$\boldsymbol{f}_{\boldsymbol{\delta-1}} := \boldsymbol{C}\, \boldsymbol{f} - (a_1\, \sigma(\boldsymbol{g}) + a_2\, \boldsymbol{g}). \tag{5}$$

By construction it follows that $\boldsymbol{f}_{\boldsymbol{\delta-1}} \in \mathbb{F}[t]^\lambda_{\delta+l-1}$. Now we proceed as follows. We try to determine exactly those $h \in \mathbb{F}[t]_{\delta-1}$ and $\boldsymbol{d} \in \mathbb{K}^\lambda$ that fulfill $a_1\, \sigma(h + \boldsymbol{d}\, \boldsymbol{g}) + a_2\, (h + \boldsymbol{d}\, \boldsymbol{g}) = \boldsymbol{d}\, \boldsymbol{C}\, \boldsymbol{f}$ which is equivalent to $a_1\, \sigma(h) + a_2\, h = \boldsymbol{d}\, \boldsymbol{f}_{\boldsymbol{\delta-1}}$. For this task, we take a basis B_2 of

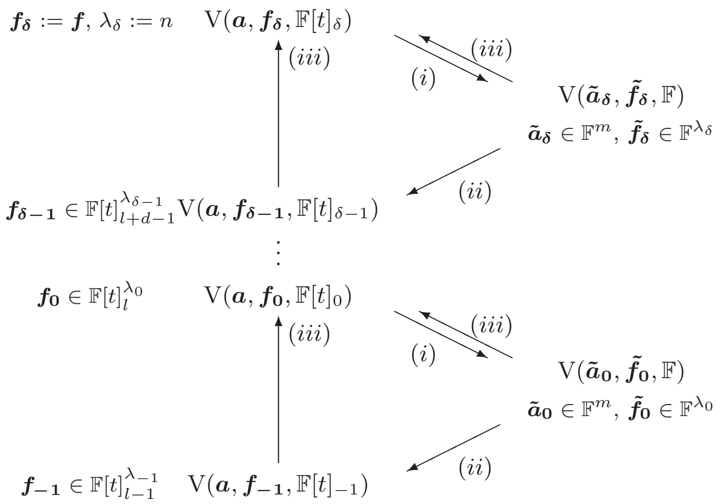

FIGURE 1. Incremental reduction

$\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_{\delta-1}}, \mathbb{F}[t]_{\delta-1})$.

$\star$ CASE $B.a$: $B_2 = \{\}$. Then $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_\delta}, \mathbb{F}[t]_\delta) = \{\boldsymbol{0_{n+1}}\}$.

$\star$ CASE $B.b$: $B_2 \neq \{\}$, say $B_2 = \{(d_{i1}, \dots, d_{i\lambda}, h_i)\}_{1 \le i \le \mu}$. Then define $\boldsymbol{D} := (d_{ij}) \in \mathbb{K}^{\mu \times \lambda}$, $\boldsymbol{h} := (h_1, \dots, h_\mu) \in \mathbb{F}[t]_{\delta-1}^{\mu}$ which gives $a_1\, \sigma(\boldsymbol{h} + \boldsymbol{D}\, \boldsymbol{g}) + a_2\, (\boldsymbol{h} + \boldsymbol{D}\, \boldsymbol{g}) = \boldsymbol{D}\, \boldsymbol{C}\, \boldsymbol{f}$. Now define $\kappa_{ij} \in \mathbb{K}$ and $p_i \in \mathbb{F}[t_e]_\delta^\mu$ with

$$\begin{pmatrix} \kappa_{11} & \dots & \kappa_{1n} \\ \vdots & & \vdots \\ \kappa_{\mu 1} & \dots & \kappa_{\mu n} \end{pmatrix} := \boldsymbol{D}\, \boldsymbol{C}, \qquad (p_1, \dots, p_\mu) := \boldsymbol{h} + \boldsymbol{D}\, \boldsymbol{g}. \tag{6}$$

Then by the above considerations it follows that $B_3 := \{(\kappa_{i1}, \dots, \kappa_{in}, p_i)\}_{1 \le i \le \mu}$ spans a subspace of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_\delta)$ over $\mathbb{K}$. By linear algebra arguments one can even show that B_3 is a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_\delta)$ over $\mathbb{K}$. This polynomial degree reduction is the inner core of Karr's summation algorithm given in [6]. A complete proof can be found in [12].

Summarizing, let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$, $\boldsymbol{a} \in (\mathbb{F}[t]^*)^2$ with $l := \|\boldsymbol{a}\|$ and $\boldsymbol{f} \in \mathbb{F}[t]_{\delta+l}^n$ for some $\delta \in \mathbb{N}_0 \cup \{-1\}$. Then we can apply this reduction technique step by step and obtain an *incremental reduction* of $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_\delta)$ given in Figure 1. We call $\{(\boldsymbol{a}, \boldsymbol{f_\delta}, \mathbb{F}[t]_\delta), \dots, (\boldsymbol{a}, \boldsymbol{f_{-1}}, \mathbb{F}[t]_{-1})\}$ the *incremental tuples* and $\{(\tilde{\boldsymbol{a}}_{\boldsymbol{\delta}}, \tilde{\boldsymbol{f}}_{\boldsymbol{\delta}}, \mathbb{F}), \dots, (\tilde{\boldsymbol{a}}_{\boldsymbol{0}}, \tilde{\boldsymbol{f}}_{\boldsymbol{0}}, \mathbb{F})\}$ the *coefficient tuples* of such an incremental reduction.

Base case I. In the incremental reduction we finally reach the problem to find a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_{-1})$ with $\mathbb{F}[t]_{-1} = \{0\}$. Then we have $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \{0\}) = \mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f}) \times \{0\}$ where $\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f}) = \{\boldsymbol{k} \in \mathbb{K}^n \,|\, \boldsymbol{f}\, \boldsymbol{k} = 0\}$. Note that a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \{0\})$ can be computed by linear algebra if $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$; for more details see [12].

Example 2. *Take the* $\Pi\Sigma$*-field* $(\mathbb{Q}(t_1)(t_2), \sigma)$ *over* $\mathbb{Q}$ *from Example 1 and write* $\mathbb{F} := \mathbb{Q}(t_1)$. *With our reduction strategy we will find a basis of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t_2))$ *for* $\boldsymbol{a} = (1, -1) \in \mathbb{F}(t_2)^2$ *and* $\boldsymbol{f} = (\sigma(t_2/t_1)) = (\frac{1+(t_1+1)\, t_2}{(t_1+1)^2}) \in \mathbb{F}(t_2)^1$. *Clearing denominators gives the vectors* $\boldsymbol{a} = ((t_1+1)^2, -(t_1+1)^2) \in \mathbb{F}[t_2]^2$, $\boldsymbol{f} = (1 + (t_1+1)\, t_2) \in \mathbb{F}[t_2]^1$. *A denominator bound of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t_2))$ *is* 1, *and a degree bound of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t_2])$ *is* 2. *Now we start the incremental reduction of* $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t_2]_2)$. *For the incremental tuple* $(\boldsymbol{a}, \boldsymbol{f_2}, \mathbb{F}[t_2]_2)$ *with* $\boldsymbol{f_2} := \boldsymbol{f} \in \mathbb{F}[t_2]_2^1$

we obtain the coefficient tuple $(\boldsymbol{a}, (0), \mathbb{F})$. *The basis* $\{(1,0),(0,1)\}$ *of* $\mathrm{V}(\boldsymbol{a},(0),\mathbb{F})$ *gives* $\boldsymbol{C} = \binom{1}{0} \in \mathbb{K}^{2\times 1}$, $\boldsymbol{g} = (0, t_2^2) \in \mathbb{F}[t_2]_2^2$. *This defines the incremental tuple* $(\boldsymbol{a}, \boldsymbol{f_1}, \mathbb{F}[t_2]_1)$ *with* $\boldsymbol{f_1} = (1 + (t_1+1)\,t_2, -1-2(t_1+1)\,t_2) \in \mathbb{F}[t_2]_1^2$ *and the coefficient tuple* $(\boldsymbol{a}, (t_1+1, -2(t_1+1)), \mathbb{F})$. *Then taking the basis* $\{(2,1,0),(0,0,1)\}$ *of* $\mathrm{V}(\boldsymbol{a},(1,-2),\mathbb{F})$, *one obtains* $\boldsymbol{f_0} = (1, -t_1-1) \in \mathbb{F}[t_2]_0^2$, *the incremental tuple* $(\boldsymbol{a}, \boldsymbol{f_0}, \mathbb{F}[t_2]_0)$ *and the coefficient tuple* $(\boldsymbol{a}, \boldsymbol{f_0}, \mathbb{F})$. *A basis of the solution space* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_0}, \mathbb{F})$ *is* $\{(0,0,1)\}$ *which defines* $\boldsymbol{f_{-1}} = (0)$. *Finally, a basis of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_{-1}}, \{0\})$ *is* $\{(1,0)\}$. *This gives the basis* $\{(0,0,1)\}$ *of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_i}, \mathbb{F}[t_2]_i)$ *for* $i \in \{0,1\}$ *and therefore the basis* $\{(0,1)\}$ *of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_2}, \mathbb{F}[t_2]_2)$ *and* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t_2))$.

A reduction to $\mathbb{F}$**.** Suppose that we have given not only a single but a nested $\Pi\Sigma$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ where we write $\mathbb{F}_i := \mathbb{F}(t_1)\dots(t_i)$ for $0 \le i \le e$, i.e., $\mathbb{F}_0 = \mathbb{F}$. Let $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}_e$ and $\boldsymbol{f} \in \mathbb{F}_e^n$. Then we understand by a *reduction of* $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}_e)$ *to* $\mathbb{F}$ a recursive application of the above reductions. More precisely, if $e = 0$, we do nothing. Otherwise, suppose that $e > 0$. If $a_1\,a_2 = 0$, we just apply the special case from above. Otherwise, within our reduction there is a denominator bound $d \in \mathbb{F}_{e-1}[t_e]^*$ which reduces the problem to find a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}_e)$ to find one for $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}_{e-1}[t_e])$ for some $\boldsymbol{a'} \in (\mathbb{F}_{e-1}[t_e]^*)^2$ and $\boldsymbol{f'} \in \mathbb{F}_{e-1}[t_e]^n$; those are given by setting $\boldsymbol{a}' := (a_1/\sigma(d), a_2/d)$, $\boldsymbol{f'} := \boldsymbol{f}$ and clearing denominators and cancelling common factors. Next, with a degree bound b of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}_{e-1}[t_e])$ the incremental reduction of $(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}_{e-1}[t_e]_b)$ is applied. Within this reduction the coefficient tuples $(\boldsymbol{a_i}, \boldsymbol{f_i}, \mathbb{F}_{e-1})$ for $0 \le i \le b$ give the subreductions of $(\boldsymbol{a_i}, \boldsymbol{f_i}, \mathbb{F}_{e-1})$ to $\mathbb{F}$ for $0 \le i \le b$ that define recursively the whole reduction of $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}_e)$ to $\mathbb{F}$.

We call T the *tuple set* of a reduction of $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}_e)$ to $\mathbb{F}$ if besides $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}_e) \in T$ the set T contains exactly all those coefficient tuples that occur in the recursively applied incremental reductions. Moreover, for $\boldsymbol{a_e} := \boldsymbol{a}$ and $\boldsymbol{f_e} := \boldsymbol{f}$ we call $\{(\boldsymbol{a_i}, \boldsymbol{f_i}, \mathbb{F}_i)\}_{r \le i \le e} \subseteq T$ *path-tuples* of $(\boldsymbol{a_r}, \boldsymbol{f_r}, \mathbb{F}_r) \in T$ if in the subreduction of $(\boldsymbol{a_{i+1}}, \boldsymbol{f_{i+1}}, \mathbb{F}_{i+1})$ to $\mathbb{F}$ the coefficient tuple $(\boldsymbol{a_i}, \boldsymbol{f_i}, \mathbb{F}_i)$ occurs for each $r \le i < e$ in the incremental reduction. Finally, we introduce the $\mathbb{F}_r$*-critical tuple set* S in a reduction of $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}_e)$ to $\mathbb{F}$ as that subset of the tuple set T of the reduction to $\mathbb{F}$ that contains all $(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}_r) \in T$ with the following property: for its path-tuples $\{(\boldsymbol{a_i}, \boldsymbol{f_i}, \mathbb{F}_i)\}_{r \le i \le e}$ we have that $\boldsymbol{a_i}$ is homogeneous for all $r \le i \le e$. Summarizing, we obtain the following method that generates a reduction to $\mathbb{F}$.

Algorithm 1. `SolveSolutionSpace`*(*$(\mathbb{F}(t_1)\dots(t_e),\sigma), \boldsymbol{a}, \boldsymbol{f}$*)*

Input: *A* $\Pi\Sigma$*-extension* $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ *of* $(\mathbb{F},\sigma)$ *with* $\mathbb{K} := \mathrm{const}_\sigma\mathbb{F}$*;* $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}(t_1)\dots(t_e)^2$ *and* $\boldsymbol{f} \in \mathbb{F}(t_1)\dots(t_e)^n$.

Output: *A basis of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t_1)\dots(t_e))$ *over* $\mathbb{K}$.

(1) IF $e = 0$ *compute a basis* B *of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$ *and RETURN* B*. FI*
Let $\mathbb{H} := \mathbb{F}(t_1)\dots(t_{e-1})$*, i.e.* $(\mathbb{H}(t_e),\sigma)$ *is a* $\Pi\Sigma$*-ext. of* $(\mathbb{H},\sigma)$.

(2) IF $a_1\,a_2 = 0$ *THEN set* $\boldsymbol{g} := \frac{\boldsymbol{f}}{a_2}$ *if* $a_2 \neq 0$*, otherwise set* $\boldsymbol{g} := \frac{\sigma(\boldsymbol{f})}{a_2}$*; with* $\boldsymbol{g} = (g_1, \dots, g_n)$ *RETURN* $\{(0\dots,1,\dots,0,g_i)\}_{1 \le i \le n}$*. FI*

(3) Clear denominators and common factors s.t. $\boldsymbol{a} \in (\mathbb{H}[t_e]^*)^2$*,* $\boldsymbol{f} \in \mathbb{H}[t_e]^n$.

(4) Compute a denominator bound $d \in \mathbb{H}[t_e]^*$ *of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{H}(t_e))$.

(5) Set $\boldsymbol{a}' := (a_1/\sigma(d), a_2/d) \in \mathbb{H}(t_e)^2$*,* $\boldsymbol{f'} := \boldsymbol{f}$ *and clear denominators and common factors s.t.* $\boldsymbol{a}' \in (\mathbb{H}[t_e]^*)^2$ *and* $\boldsymbol{f'} \in \mathbb{H}[t_e]^n$.

(6) Compute a degree bound b *of* $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{H}[t_e])$.

(7) Compute a basis $B :=$ `IncrementalReduction`$((\mathbb{H}(t_e),\sigma), b, \boldsymbol{a'}, \boldsymbol{f'})$ *by using Algorithm 2; say* $B = \{(\kappa_{i1}, \dots, \kappa_{in}, p_i)\}_{1 \le i \le \mu}$.

(8) IF $B=\{\}$ *THEN RETURN* $\{\}$ *ELSE RETURN* $\{(\kappa_{i1},\dots,\kappa_{in},\frac{p_i}{d})\}_{1\leq i\leq\mu}$. *FI*

Algorithm 2. `InrementalReduction`$((\mathbb{F}(t_1)\dots(t_e),\sigma),\delta,\boldsymbol{a},\boldsymbol{f})$

Input: *A* $\Pi\Sigma$*-extension* $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ *of* $(\mathbb{F},\sigma)$ *with* $\mathbb{K}:=\text{const}_\sigma\mathbb{F}$; $\delta\in\mathbb{N}_0\cup\{-1\}$; $\boldsymbol{a}=(a_1,a_2)\in(\mathbb{F}(t_1)\dots(t_{e-1})[t_e]^*)^2$ *with* $l:=\|\boldsymbol{a}\|$ *and* $\boldsymbol{f}\in\mathbb{F}(t_1)\dots(t_{e-1})[t_e]^n_{l+\delta}$.

Output: *A basis of* $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_\delta)$ *over* $\mathbb{K}$.

(1) IF $d=-1$, *RETURN a basis of* $\text{Nullspace}_{\mathbb{K}}(\boldsymbol{f})\times\{0\}$ *over* $\mathbb{K}$. *FI*
Let $\mathbb{H}:=\mathbb{F}(t_1)\dots(t_{e-1})$, *i.e.* $(\mathbb{H}(t_e),\sigma)$ *is a* $\Pi\Sigma$*-ext. of* $(\mathbb{H},\sigma)$.
(2) Define $\mathbf{0}\neq\tilde{\boldsymbol{a}}_{\boldsymbol{\delta}}\in\mathbb{H}^2$ *and* $\tilde{\boldsymbol{f}}_{\boldsymbol{\delta}}\in\mathbb{H}^n$ *as in* (4).
(3) Compute $B_1:=$ `SolveSolutionSpace`$((\mathbb{H},\sigma),\tilde{\boldsymbol{a}}_{\boldsymbol{\delta}},\tilde{\boldsymbol{f}}_{\boldsymbol{\delta}})$ *with Alg. 1.*
(4) IF $B_1=\{\}$ *THEN*
(5) Compute $B_2:=$ `IncrementalReduction`$((\mathbb{H}(t_e),\sigma),\delta-1,\boldsymbol{a},(0))$.
(6) IF an $h\in\mathbb{H}[t_e]_{\delta-1}$ *with* $a_1\,\sigma(h)+a_2\,h=0$ *is found THEN*
RETURN $\{(0,\dots,0,h)\}\subset\mathbb{K}^n\times\mathbb{H}[t_e]_{\delta-1}$ *ELSE RETURN* $\{\}$ *FI*
FI
(7) With $B_1=\{(c_{i1},\dots,c_{in},w_i)\}_{1\leq i\leq\lambda}$ *define* $\boldsymbol{C}:=(c_{ij})\in\mathbb{K}^{\lambda\times n}$, $\boldsymbol{g}:=(w_1\,t_e^\delta,\dots,w_\lambda\,t_e^\delta)\in t_e^\delta\,\mathbb{H}^\lambda$, *and* $\boldsymbol{f_{\delta-1}}\in\mathbb{H}[t_e]^\lambda_{\delta-1}$ *as in* (5).
(8) Compute $B_2:=$ `IncrementalReduction`$((\mathbb{H}(t_e),\sigma),\delta-1,\boldsymbol{a},\boldsymbol{f_{\delta-1}})$.
(9) IF $B_2=\{\}$ *THEN RETURN* $\{\}$ *FI*
(10) Let $B_2=\{(d_{i1},\dots,d_{i\lambda},h_i)\}_{1\leq i\leq\mu}$ *and define* $\boldsymbol{D}:=(d_{ij})\in\mathbb{K}^{\mu\times\lambda},\boldsymbol{h}:=(h_1,\dots,h_\mu)\in\mathbb{H}[t_e]^\mu_{\delta-1}$. *Define* $\kappa_{ij}\in\mathbb{K}$ *for* $1\leq i\leq\mu$, $1\leq j\leq n$ *and* $p_i\in\mathbb{H}[t_e]_\delta$ *for* $1\leq i\leq\mu$ *as in* (6).
(11) RETURN $\{(\kappa_{i1},\dots,\kappa_{in},p_i)\}_{1\leq i\leq\mu}$

If the denominator bound problem and polynomial degree bound problem can be solved in the $\Pi\Sigma$-extensions $(\mathbb{F}_i,\sigma)$ of $(\mathbb{F}_{i-1},\sigma)$ for $1\leq i\leq e$ and one can compute a basis of any solution space in $(\mathbb{F},\sigma)$, Algorithms 1 and 2 give an algorithm to compute a basis of a solution space $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_e)$. In particular these algorithms give a reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_e)$ to $\mathbb{F}$. Moreover, by taking all $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_i)$ when calling Algorithm 1, one gets the reduction tuple set of this reduction. Furthermore, if one stops collecting tuples in the subreductions of $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_i)$ to $\mathbb{F}$ when $\boldsymbol{a}$ is inhomogeneous, one can extract the $\mathbb{F}_r$-critical tuples in this reduction.
Now assume that $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$, i.e., $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma$-field over $\mathbb{K}$. Then by Theorem 4 there are algorithms to solve the denominator and polynomial degree bound problem. Moreover, for the special case $\mathbb{F}=\mathbb{K}$ there is the following

Base case II. If $\text{const}_\sigma\mathbb{K}=\mathbb{K}$, $\mathbf{0}\neq\boldsymbol{a}=(a_1,a_2)\in\mathbb{K}^2$ and $\boldsymbol{f}=(f_1,\dots,f_n)\in\mathbb{K}^n$ then $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{K})=\text{Nullspace}_{\mathbb{K}}(\boldsymbol{f'})$ for $\boldsymbol{f'}=(f_1,\dots,f_n,-(a_1+a_2))$. A basis can be computed by linear algebra; see [10].

Hence, with Algorithms 1 and 2 one can compute a basis of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_1)\dots(t_e))$ in a $\Pi\Sigma$-field $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ over a σ-computable $\mathbb{K}$ and can extract the $\mathbb{F}$-critical tuples of the corresponding reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_1)\dots(t_e))$ to $\mathbb{F}$.
Finally, we introduce reductions to $\mathbb{F}$ that are extension-stable. Let $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$, $\boldsymbol{a}\in(\mathbb{H}[t_e]^*)^2$ and $\boldsymbol{f}\in\mathbb{H}[t_e]^n$ for $\mathbb{H}:=\mathbb{F}(t_1)\dots(t_{e-1})$. We call a denominator bound $d\in\mathbb{H}[t_e]^*$ of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t_e))$ or a degree bound b of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}[t_e])$ *extension-stable over* $\mathbb{F}$ if $\boldsymbol{a}$ is inhomogeneous over $\mathbb{H}(t_e)$ or the following holds. Take any Σ^*-extension $(\mathbb{F}(t_1)\dots(t_e)(s),\sigma)$ of $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ over $\mathbb{F}$, and embed $\boldsymbol{a}$, $\boldsymbol{f}$ in the reordered

$\Pi\Sigma$-ext. $(\mathbb{F}(s)(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$. Then also d embedded in $\mathbb{F}(s)(t_1)\dots(t_e)$ must be a denominator bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(s)(t_1)\dots(t_e))$. Similarly, b must be a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(s)(t_1)\dots(t_{e-1})[t_e])$.
We call a reduction of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_1)\dots(t_e))$ to $\mathbb{F}$ *extension-stable* if all denominator and degree bounds within the reduction to $\mathbb{F}$ are extension-stable over $\mathbb{F}$.
It has been shown in [10, Theorem 8.2] and [11, Theorem 7.3] that the algorithms proposed in [6] already compute extension-stable denominator and degree bounds in a $\Pi\Sigma$-field. Summarizing, we obtain

Theorem 5. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$, $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{E}^2$ and $\boldsymbol{f}\in\mathbb{E}^n$. Then with Algorithms 1 and 2 one can compute a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$ with an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$ to $\mathbb{F}$. Moreover, during this computation, one can extract the $\mathbb{F}$-critical tuples.*

Example 3. *In Example 2 the denominator and degree bounds are extension-stable. Consequently, this reduction of $((1,-1),(\sigma(t_2/t_1)),\mathbb{F}(t_2))$ to $\mathbb{F}$ is extension-stable. The $\mathbb{F}$-critical tuples are $(((t_1+1)^2,-(t_1+1)^2),\boldsymbol{f},\mathbb{F})$ for $\boldsymbol{f}\in\{(0),(t_1+1,-2(t_1+1)),(1,-(t_1+1))\}$.*

5. Refined Summation Algorithms

In the sequel let $(\mathbb{E},\sigma)$ with $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$ and $\boldsymbol{f}\in\mathbb{E}^n$. Then in Theorem 6 we will develop a constructive criterium which tells us if a Σ^*-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ is $\mathbb{F}$-complete for $\boldsymbol{f}$ and how such an extension can be constructed. For this task we first compute a basis of $\mathbb{V}:=\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E})$ with Algorithms 1 and 2 together with an extension-stable reduction of $((1,-1),\boldsymbol{f},\mathbb{E})$ to $\mathbb{F}$; see Theorem 5. If the dimension of $\mathbb{V}$ is $n+1$, the trivial extension $(\mathbb{E},\sigma)$ of $(\mathbb{E},\sigma)$ is clearly $\mathbb{F}$-complete for $\boldsymbol{f}$. Otherwise, we extract the $\mathbb{F}$-critical tuple set in our extension-stable reduction; see Theorem 5. Then the crucial observation is stated in Proposition 1 that depends on Lemma 3. This lemma is a special case of Karr's Fundamental Theorem [6, 7]; for a proof see [9, Proposition 4.1.2].

Lemma 3. *If $(\mathbb{E},\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$, $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{F}^2$ inhomogeneous over $\mathbb{F}$ and $\boldsymbol{f}\in\mathbb{F}^n$ then $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$.*

Proposition 1. *Let $(\mathbb{E}(s),\sigma)$ with $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ with $\sigma(s)-s\in\mathbb{F}$ and consider the reordered $\Pi\Sigma$-extension $(\mathbb{F}(s)(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$. Let $\boldsymbol{a}\in\mathbb{E}^2$ be homogeneous over $\mathbb{E}$, $\boldsymbol{f}\in\mathbb{E}^n$, and let S be an $\mathbb{F}$-critical tuple set of an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$ to $\mathbb{F}$. If for all $(\boldsymbol{a}',\boldsymbol{f}',\mathbb{F})\in S$ we have $\mathrm{V}(\boldsymbol{a}',\boldsymbol{f}',\mathbb{F})=\mathrm{V}(\boldsymbol{a}',\boldsymbol{f}',\mathbb{F}(s))$ then $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(s))=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(s)(t_1)\dots(t_e))$.*

Proof. The proof will be done by induction on the number e of extensions $\mathbb{F}(t_1)\dots(t_e)$. First consider the case $e=0$. Since $\boldsymbol{a}$ is homogeneous, $(\boldsymbol{a},\boldsymbol{f},\mathbb{F})\in S$ and therefore $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(s))=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$. Now assume that the proposition holds for $e\geq 0$. Let $(\mathbb{F}(t_1)\dots(t_e)(t_{e+1})(s),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ with $\sigma(s)-s\in\mathbb{F}$ and consider the reordered $\Pi\Sigma$-extension $(\mathbb{F}(s)(t_1)\dots(t_e)(t_{e+1}),\sigma)$ of $(\mathbb{F},\sigma)$. We write $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ and $\mathbb{H}:=\mathbb{F}(s)(t_1)\dots(t_e)$ as shortcut. Let $\boldsymbol{a}\in\mathbb{E}(t_{e+1})^2$ be homogeneous over $\mathbb{E}(t_{e+1})$, $\boldsymbol{f}\in\mathbb{E}(t_{e+1})^n$, and take any extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t_{e+1}))$ to $\mathbb{F}$ with the $\mathbb{F}$-critical tuple set S. Now suppose that $\mathrm{V}(\boldsymbol{a}',\boldsymbol{f}',\mathbb{F})=\mathrm{V}(\boldsymbol{a}',\boldsymbol{f}',\mathbb{F}(s))$ for all $(\boldsymbol{a}',\boldsymbol{f}',\mathbb{F})\in S$. Then we will show that

$$\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t_{e+1}))=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t_{e+1})). \tag{7}$$

In the extension-stable reduction let $d\in\mathbb{E}[t_{e+1}]^*$ be the denominator bound of the solution space $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t_{e+1}))$. Since $\boldsymbol{a}$ is homogeneous over $\mathbb{E}(t_{e+1})$, $d\in\mathbb{H}[t_{e+1}]$ is also a denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{H}(t_{e+1}))$. After clearing denominators and cancelling common factors, we get $\boldsymbol{a'} := (a_1/\sigma(d), a_2/d)\, q \in \mathbb{E}[t_{e+1}]^2$ and $\boldsymbol{f'} := \boldsymbol{f}\, q \in \mathbb{E}[t_{e+1}]^n$ for some $q \in \mathbb{E}(t_{e+1})^*$ in our reduction. Note that $\boldsymbol{a'}$ is still homogeneous over $\mathbb{E}(t_{e+1})$. This follows from the fact that if for $h \in \mathbb{E}(t_{e+1})$ we have $a_1\,\sigma(h) + a_2\, h = 0$ then $a'_1\,\sigma(h\,d) + a'_2\, h\, d = 0$. Now it suffices to show that $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{H}[t_{e+1}]) = \mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{E}[t_{e+1}])$, in order to show (7). In the given reduction let b be the degree bound of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{E}[t_{e+1}])$. Since $\boldsymbol{a'}$ is homogeneous over $\mathbb{E}(t_{e+1})$, b is a degree bound of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{H}[t_{e+1}])$ too. Hence, if $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{E}[t_{e+1}]_b) = \mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{H}[t_{e+1}]_b)$, also (7) is proven. Let $((\boldsymbol{a}, \boldsymbol{f_b}, \mathbb{E}[t_{e+1}]_b), \dots, (\boldsymbol{a}, \boldsymbol{f_{-1}}, \mathbb{E}[t_{e+1}]_{-1}))$ be the incremental tuples and $((\boldsymbol{\tilde{a}_b}, \boldsymbol{\tilde{f}_b}, \mathbb{E}), \dots, (\boldsymbol{\tilde{a}_0}, \boldsymbol{\tilde{f}_0}, \mathbb{E}))$ be the coefficient-tuples in the incr. reduction of $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{E}[t_{e+1}]_b)$. We show that $\mathrm{V}(\boldsymbol{\tilde{a}_i}, \boldsymbol{\tilde{f}_i}, \mathbb{E}) = \mathrm{V}(\boldsymbol{\tilde{a}_i}, \boldsymbol{\tilde{f}_i}, \mathbb{H})$ for all $0 \le i \le b$. By reordering of the difference field $(\mathbb{F}(t_1)\dots(t_{e+1})(s), \sigma)$ we get the $\Pi\Sigma$-extension $(\mathbb{F}(t_1)\dots(t_e)(s)(t_{e+1}), \sigma)$ of $(\mathbb{F}, \sigma)$. First suppose that $\boldsymbol{\tilde{a}_i}$ is inhomogeneous over $\mathbb{E}$. Hence, $\mathrm{V}(\boldsymbol{\tilde{a}_i}, \boldsymbol{\tilde{f}_i}, \mathbb{E}) = \mathrm{V}(\boldsymbol{\tilde{a}_i}, \boldsymbol{\tilde{f}_i}, \mathbb{E}(s))$ by Lemma 3, and therefore $\mathrm{V}(\boldsymbol{\tilde{a}_i}, \boldsymbol{\tilde{f}_i}, \mathbb{E}) = \mathrm{V}(\boldsymbol{\tilde{a}_i}, \boldsymbol{\tilde{f}_i}, \mathbb{H})$ by $(\mathbb{F}(t_1)\dots(t_e)(s), \sigma) \simeq (\mathbb{F}(s)(t_1)\dots(t_e), \sigma)$. Otherwise, assume that $\boldsymbol{\tilde{a}_i}$ is homogeneous over $\mathbb{E}$. Then the extension-stable reduction of $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{E}(t_{e+1}))$ to $\mathbb{F}$ contains an extension-stable reduction of $(\boldsymbol{\tilde{a}_i}, \boldsymbol{\tilde{f}_i}, \mathbb{E})$ to $\mathbb{F}$ and the $\mathbb{F}$-critical tuple set of the reduction of $(\boldsymbol{\tilde{a}_i}, \boldsymbol{\tilde{f}_i}, \mathbb{E})$ is a subset of S. Hence with the induction assumption it follows that $\mathrm{V}(\boldsymbol{\tilde{a}_i}, \boldsymbol{\tilde{f}_i}, \mathbb{E}) = \mathrm{V}(\boldsymbol{\tilde{a}_i}, \boldsymbol{\tilde{f}_i}, \mathbb{H})$. Since $\mathbb{E}[t_{e+1}]_{-1} = \mathbb{H}[t_{e+1}]_{-1} = \{0\}$, we have $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_{-1}}, \mathbb{E}[t_{e+1}]_{-1}) = \mathrm{V}(\boldsymbol{a}, \boldsymbol{f_{-1}}, \mathbb{H}[t_{e+1}]_{-1})$. Then by the construction of the incremental reduction we can conclude that $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_i}, \mathbb{E}[t_{e+1}]_i) = \mathrm{V}(\boldsymbol{a}, \boldsymbol{f_i}, \mathbb{H}[t_{e+1}]_i)$ for all $-1 \le i \le b$ and therefore we have proven (7). With reordering $(\mathbb{F}(s)(t_1)\dots(t_{e+1}), \sigma) \simeq (\mathbb{F}(t_1)\dots(t_{e+1})(s), \sigma)$, it follows $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{E}(t_{e+1})) = \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{E}(t_{e+1})(s))$. □

Consequently we have $\mathrm{V}((1,-1), \boldsymbol{f}, \mathbb{E}) \subsetneq \mathrm{V}((1,-1), \boldsymbol{f}, \mathbb{E}(s))$ for a Σ^*-extension $(\mathbb{E}(s), \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$ if $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}) \subsetneq \mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}(s))$ in one of its $\mathbb{F}$-critical tuples $(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F})$ in an extension-stable reduction to $\mathbb{F}$.

Example 4. *Consider the $\Pi\Sigma$-fields from Example 1, 2 and 3. By Example 1 it follows that $\mathrm{V}((1,-1), (\frac{\sigma(t_2)}{t_1}), \mathbb{F}(t_2))$ is a proper subset of $\mathrm{V}((1,-1), (\frac{\sigma(t_2)}{t_1}), \mathbb{F}(t_2)(s))$. Hence looking at the $\mathbb{F}$-critical tuples of our extension stable reduction in Example 3, we know by Proposition 1 that there is an $\boldsymbol{f} \in \{(0), (t_1+1, -2(t_1+1)), (1, -(t_1+1))\}$ such that $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$ with $\boldsymbol{a} = ((t_1+1)^2, -(t_1+1)^2)$ is a proper subset of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(s))$. Indeed, we can choose $\boldsymbol{f} = (1, -(t_1+1))$ since there does not exist a $g \in \mathbb{F}$ with $\sigma(g) - g = \frac{1}{(t_1+1)^2}$, but there is the solution $g = s \in \mathbb{F}(s)$.*

Next we provide a sufficient condition in Proposition 2 which tells us if a Σ^*-extension cannot contribute further to a given solution space.

Proposition 2. *Let $(\mathbb{F}, \sigma)$ be a difference field with $\boldsymbol{a} = (a_1, a_2) \in \mathbb{F}^2$ homogeneous over $\mathbb{F}$ and $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. If for all $1 \le i \le n$ there is a $g \in \mathbb{F}^*$ with $a_1\,\sigma(g) + a_2\, g = f_i$ then for any difference field (ring) extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathrm{const}_\sigma \mathbb{E} = \mathrm{const}_\sigma \mathbb{F}$ we have $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}) = \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{E})$.*

Proof. Let $g_i \in \mathbb{F}$ with $a_1\,\sigma(g_0) + a_2\, g_0 = 0$ and $a_1\,\sigma(g_i) + a_2\, g_i = f_i$ for $1 \le i \le n$. Then observe that $(0, \dots, 0, g_0)$, $(1, 0, \dots, 0, g_1), \dots, (0, \dots, 0, 1, g_n)\}$ forms a basis of $\mathbb{V} := \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$ over $\mathbb{K} := \mathrm{const}_\sigma \mathbb{F}$. Since $\mathbb{V}$ is a subspace of $\mathbb{W} := \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{E})$ over $\mathbb{K}$ and the dimension of $\mathbb{W}$ is at most $n+1$, it follows that $\mathbb{V} = \mathbb{W}$. □

This result allows us to specify a criterium in Theorem 6 if a Σ^*-extension of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$ is $\mathbb{F}$-complete for $\boldsymbol{f}$.

Theorem 6. *Let $(\mathbb{E},\sigma)$ with $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ and $\boldsymbol{f}\in\mathbb{E}^n$. Let $\{(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F})\}_{1\leq i\leq k}$ with $\boldsymbol{a_i}=(a_{i1},a_{i2})$ and $\boldsymbol{f_i}=(f_{i1},\dots,f_{ir_i})\in\mathbb{F}^{r_i}$ be the $\mathbb{F}$-critical tuple set of an extension-stable reduction of $\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E})$ to $\mathbb{F}$. If $(\mathbb{G},\sigma)$ is a Σ^*-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ where for any $1\leq i\leq k$ and $1\leq j\leq r_i$ there is a $g\in\mathbb{G}^*$ with $a_{i1}\,\sigma(g)-a_{i2}\,g=f_{ij}$ then the extension is $\mathbb{F}$-complete for $\boldsymbol{f}$.*

Proof. Suppose such an extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ is not $\mathbb{F}$-complete for $\boldsymbol{f}$. Then we can take a $\boldsymbol{c}\in\mathbb{K}^n$ such that $\sigma(g)-g=\boldsymbol{c}\,\boldsymbol{f}$ has a solution in some $\Pi\Sigma$-extension of $(\mathbb{E},\sigma)$, but no solution in $(\mathbb{G},\sigma)$ and therefore no solution in $(\mathbb{E},\sigma)$. Hence, by Lemma 1 there is a Σ^*-extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ and a $g\in\mathbb{E}(s)$ with $\sigma(g)-g=\boldsymbol{c}\,\boldsymbol{f}$. Consequently, by Proposition 1 there exists an i with $1\leq i\leq k$ such that $\mathrm{V}(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F})\subsetneq\mathrm{V}(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F}(s))$ holds for the Σ^*-extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$. But by Proposition 2 we have $\mathrm{V}(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F})=\mathrm{V}(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F}(s))$, a contradiction. □

Example 5. *Consider Examples 2 and 3. Since for any $f\in\{0,t_1+1,-2(t_1+1),1,-(t_1+1)\}$ there is a $g\in\mathbb{F}(t_2)(s)$ with $\sigma(g)-g=f$, it follows that the Σ^*-extension $(\mathbb{F}(t_2)(s),\sigma)$ of $(\mathbb{F}(t_2),\sigma)$ is $\mathbb{F}$-complete for $(\sigma(t_2)/t_1)$.*

Finally, in Proposition 3 we show that such an extension can be constructed that fulfills our sufficient criterium.

Proposition 3. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$, $(a_{i1},a_{i2})\in\mathbb{F}^2$ be homogeneous over $\mathbb{F}$ and $f_i\in\mathbb{F}$ for $1\leq i\leq n$. Then there is a Σ^*-extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ such that there is a $g\in\mathbb{G}^*$ with $a_{i1}\,\sigma(g)+a_{i2}\,g=f_i$ for all $1\leq i\leq n$. If $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$, such a $\Pi\Sigma$-field $(\mathbb{G},\sigma)$ can be computed.*

Proof. Suppose that we have shown the existence for such a Σ^*-extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ for $1\leq i\leq n$. Now let $(a_1,a_2)\in\mathbb{F}^2$ be homogeneous over $\mathbb{F}$ and $f\in\mathbb{F}$. If there is a $g\in\mathbb{G}$ with $a_1\,\sigma(g)+a_2\,g=f$, we have shown the induction step. Otherwise, construct the extension $(\mathbb{G}(s),\sigma)$ of $(\mathbb{G},\sigma)$ with s transcendental over $\mathbb{F}$ and $\sigma(s)=s-\frac{f}{h\,a_2}\in\mathbb{F}$ where $h\in\mathbb{F}^*$ with $a_1\,\sigma(h)+a_2\,h=0$. Now suppose there is a $g'\in\mathbb{G}^*$ with $\sigma(g')-g'=-\frac{f}{h\,a_2}$. Then for $w:=h\,g'\in\mathbb{G}^*$ we have $f=-a_2\,h(\sigma(g')-g')=a_1\,\sigma(h)\,\sigma(g')+a_2\,h\,g'=a_1\,\sigma(w)+a_2\,w$, a contradiction. Hence by Theorem 1 $(\mathbb{G}(s),\sigma)$ is a Σ^*-extension of $(\mathbb{G},\sigma)$ over $\mathbb{F}$. Furthermore, for $v:=h\,s\in\mathbb{G}(s)$ we have that $a_1\,\sigma(v)+a_2\,v=f$, which follows by similar arguments as above for w. This closes the induction step.
Now suppose that $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$. Then by Theorem 5 one can decide if there exists a $g\in\mathbb{G}^*$ with $a_1\,\sigma(g)+a_2\,g=f$ and can compute an $h\in\mathbb{F}^*$ with $a_1\,\sigma(h)+a_2\,h=0$. This shows, that the proof above becomes completely constructive. □

Summarizing, we first compute a basis of $\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E})$ with an extension-stable reduction and extract the $\mathbb{F}$-critical tuples; this is possible by Theorem 5. Next we construct with Proposition 3 a Σ^*-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ that fulfills the criterium in Theorem 6.

Example 6. *Looking at Example 3 we obtain immediately the Σ^*-extension $(\mathbb{F}(t_2)(s),\sigma)$ of $(\mathbb{F}(t_2),\sigma)$ with $\sigma(s)=s+\frac{1}{(t_1+1)^2}$ which is $\mathbb{F}$-complete for $(\sigma(t_2/t_1))\in\mathbb{F}(t_2)^1$ by following this strategy. Finally we restart our computation in this extension and obtain for $\mathrm{V}((1,-1),(\sigma(t_2/t_1)),\mathbb{F}(t_2)(s))$ the basis $\{(0,1),(2,t_2+s)\}$ which gives the result $g=\frac{t_2+s}{2}$ in Example 1.*
Now we proceed, and try to find a $g'\in\mathbb{F}(t_2)(s)$ such that $\sigma(g')-g'=\sigma(g/t_1)$, but we fail.

Therefore, we extract the $\mathbb{F}$*-critical tuples* $(((t_1+1)^3, -(t_1+1)^3), \boldsymbol{f}, \mathbb{F})$ *with*

$$\boldsymbol{f} \in \{(-(t_1+1)^2, \frac{t_1+1}{2}, -2(t_1+1)), (2(t_1+1)^2, 0, 0), \\ (0,0), (-3(t_1+1)^2, (t_1+1)^2, 0), ((t_1+1), 2, (t_1+1)^2)\} \quad (8)$$

from our extension stable reduction to $\mathbb{F}$*. Following Theorem 6 we construct a* Σ^**-extension* $(\mathbb{G}, \sigma)$ *of* $(\mathbb{F}(t_2)(s), \sigma)$ *over* $\mathbb{F}$ *such that there are* $h \in \mathbb{G}$ *with* $\sigma(h) - h = \frac{f}{(t_1+1)^2}$ *for all* $f \in \mathbb{F}$ *from (8). Following the algorithm given in the proof of Proposition 3 we obtain the* Σ^**-extension* $(\mathbb{F}(t_2)(s)(s'), \sigma)$ *of* $(\mathbb{F}(t_2)(s), \sigma)$ *with* $\sigma(s') = s' + \frac{2}{(t_1+1)^3}$*; afterwards we cancel the constant factor 2. By Theorem 6 this extension is* $\mathbb{F}$*-complete for* $(\sigma(g/t_1)) \in \mathbb{F}(t_2)(s)^1$*. To this end we compute for the solution space* $\mathrm{V}((1,-1), (\sigma(g/t_1)), \mathbb{F}(t_2)(s)(s'))$ *the basis* $\{(0,1), (6, (t_2^3 + 3t_2\, s + 2s'))\}$ *which gives the final result in Example 1.*

Let $I \subseteq \{0, \dots, e\}$. Restricting Algorithm 3 to $I = \{0\}$ gives just the above strategy. In addition, $\mathbb{F}_i := \mathbb{F}(t_1)\dots(t_i)$-complete extensions can be searched for all $i \in I$. This can be motivated as follows. $\mathbb{F}_i$-complete extensions $(\mathbb{E}_i, \sigma)$ of $(\mathbb{E}, \sigma)$ with bigger i can give more solutions $\mathbb{W}_i := \Pi_n(\mathrm{V}((1,-1), \boldsymbol{f}, \mathbb{E}_i)$; but they might be also more complicated, since they depend on more t_j (which are usually more nested). Hence, one should look for extensions with smallest possible i that give still interesting solutions in $\mathbb{W}_i$. Algorithm 3 enables one to search in one stroke for all those $\mathbb{F}_i$-complete extensions with $i \in I$.

Algorithm 3. `SingleNestedCompleteExtensions`*(*$(\mathbb{E}_0, \sigma), \boldsymbol{f}$*)*
Input: *A* $\Pi\Sigma$*-field* $(\mathbb{E}_0, \sigma)$ *with* $\mathbb{E}_0 := \mathbb{F}(t_1)\dots(t_e)$ *over a* σ*-computable* $\mathbb{K}$*,* $I = \{j_1 < \dots < j_\lambda\} \subseteq \{0, \dots, e\}$ *and* $\boldsymbol{f} \in \mathbb{E}_0^n$*.*
Output: Σ^**-extensions* $(\mathbb{E}_i, \sigma)$ *of* $(\mathbb{E}_{i-1}, \sigma)$ *over* $\mathbb{F}(t_1)\dots(t_{j_i})$ *which are single-nested* $\mathbb{F}(t_1)\dots(t_{j_i})$*-complete for* $\boldsymbol{f}$ *for* $1 \le i \le \lambda$*; a basis of* $\mathrm{V}((1,-1), \boldsymbol{f}, \mathbb{E}_\lambda)$*.*

(1) Compute a basis B *of* $\mathrm{V}((1,-1), \boldsymbol{f}, \mathbb{E}_0)$ *with an extension-stable reduction to* $\mathbb{F}$*. Let* $d := \dim \mathrm{V}((1,-1), \boldsymbol{f}, \mathbb{E}_0)$*.*
(2) IF $d = n+1$ *RETURN* $((\mathbb{E}_0, \sigma), B)$ *FI*
(3) FOR $i = 1$ *TO* λ *DO*
(4) Extract the $\mathbb{F}(t_1)\dots(t_{j_\lambda})$*-critical tuple set, say* $\{\boldsymbol{a_i}, \boldsymbol{f_i}, \mathbb{F}\}_{1\le i\le k}$ *where* $\boldsymbol{a_i} = (a_{i1}, a_{i2})$ *and* $\boldsymbol{f_i} = (f_{i1}, \dots, f_{ir_i}) \in \mathbb{F}^{r_i}$ *with* $r_i > 0$*. Construct a single-nested* Σ^**-extension* $(\mathbb{E}_i, \sigma)$ *of* $(\mathbb{E}_{i-1}, \sigma)$ *over* $\mathbb{F}(t_1)\dots(t_{j_\lambda})$ *such that for any* $1 \le i \le k$ *and* $1 \le j \le r_i$ *there exists a* $g \in \mathbb{E}_i^*$ *with* $a_{i1}\,\sigma(g) - a_{i2}\, g = f_{ij}$*. OD*
(5) IF $(\mathbb{E}_\lambda, \sigma) = (\mathbb{E}_0, \sigma)$ *RETURN* $((\mathbb{E}_0, \sigma), B)$ *FI*
(6) Compute a basis B' *of* $\mathrm{V}((1,-1), \boldsymbol{f}, \mathbb{E}_\lambda)$ *with dimension* d'*.*
(7) IF $d = d'$ *then RETURN* $((\mathbb{E}_0, \sigma), B)$ *else RETURN(*$(\mathbb{E}_\lambda, \sigma), B'$*) FI*

Theorem 7. *Let* $(\mathbb{E}_0, \sigma)$ *with* $\mathbb{E}_0 := \mathbb{F}(t_1)\dots(t_e)$ *be a* $\Pi\Sigma$*-field over a* σ*-computable* $\mathbb{K}$*,* $I = \{j_1 < \dots < j_\lambda\} \subseteq \{0, \dots, e\}$ *and* $\boldsymbol{f} \in \mathbb{E}_0^n$*. Then with Algorithm 3* Σ^**-extensions* $(\mathbb{E}_i, \sigma)$ *of* $(\mathbb{E}_{i-1}, \sigma)$ *over* $\mathbb{F}(t_1)\dots(t_{j_i})$ *can be computed which are* $\mathbb{F}(t_1)\dots(t_{j_i})$*-complete for* $\boldsymbol{f}$ *for* $1 \le i \le \lambda$*.*

The Σ^*-extension $(\mathbb{E}_\lambda, \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$ produced by Algorithm 3 can be reduced to a more compact extension that delivers the same solutions $\Pi_n(\mathrm{V}((1,-1), \boldsymbol{f}, \mathbb{E}_\lambda))$. Namely, if $\mathbb{E}_\lambda := \mathbb{E}(s_1)\dots(s_\epsilon)$, remove those s_i that do not occur in $\mathbb{W}_\lambda = \mathrm{V}((1,-1), \boldsymbol{f}, \mathbb{E}_\lambda)$. Moreover, join all those s_i's to one single Σ^*-extension which occur in a basis element of $\mathbb{W}_i$; see

Lemma 1. Furthermore, cancel constants from $\mathbb{K}$ that may occur in the summand $\sigma(s_i) - s_i$; see Example 6.

Observe that recursively applied indefinite summation can be treated more efficiently, if one reduces these extensions after each application of Algorithm 3.

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