An Implementation of

Karr's Summation Algorithm

in Mathematica

A Collection of the Transparencies presented in the talk at the
"Séminaire Lotharingien de Combinatoire 43"*

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Abstract

There are implementations of the celebrated Gosper algorithm (1978) on almost any computer algebra platform. Within my PhD thesis work I implemented **Karr's Summation Algorithm** (1981) in the Mathematica system. This is the first implementation of this algorithm in a major computer algebra system. Besides including all the difference-field machinery, this version already contains new extensions in order to handle also **definite** summation problems. The talk introduces this work according a collection of examples.

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1 A Bonus Problem in "Concrete Mathematics"

Chapter 6. Special Numbers, Bonus problem 69:

Find a closed form for

$$\sum_{k=1}^{n} k^2 H_{n+k},$$

where $H_n := \sum_{k=1}^{n} \frac{1}{k}$.

Knuth's answer to the problem is

$$\frac{1}{3}n\left(n+\frac{1}{2}\right)(n+1)\left(2H_{2n}-H_n\right)-\frac{1}{36}n\left(10n^2+9n-1\right)$$

with the remark

"It would be nice to automate the derivation of formulas such as this."

```
In[1]:= Problem69 = DefineSum[k^2DefineHNumber[n + k], {k, 1, n}]
Out[1]= Sum_{k=1}^{n} (k^2 H_{k+n})
```

```
In[2]:= KReduce[Problem69]//Simplify
Out[2]= -1/36 n (1 + n) (-1 + 10 n + 6 (1 + 2 n) H_n - 12 (1 + 2 n) H_{2 n})
```

2 A Difference Field Extension

```
In[3]:= tripleSum = DefineSum[
            DefineSum[DefineSum[1/h, {h, 1, j}]/(j + 1), {j, 5, k}],
            {k, 7, n}]
```

$$\text{Out[3]}= \sum_{k=7}^{n}\left(\sum_{j=5}^{k}\left(\frac{\sum_{h=1}^{j}\left(\frac{1}{h}\right)}{1+j}\right)\right)$$

```
In[4]:= KReduce[tripleSum]
```

$$\text{Out[4]}= \frac{191}{36}+n-\sum_{\iota_1=1}^{n}\left(\frac{1}{\iota_1}\right)-n\sum_{\iota_1=1}^{n}\left(\frac{1}{\iota_1}\right)+2\sum_{\iota_1=5}^{n}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{\iota_2}\right)}{1+\iota_1}\right)+$$

$$n\sum_{\iota_1=5}^{n}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{\iota_2}\right)}{1+\iota_1}\right)$$

An additional extension element

```
In[5]:= H2 = DefineSum[1/i^2, {i, 1, n}]
```

$$\text{Out[5]}= \sum_{i=1}^{n}\left(\frac{1}{i^2}\right)$$

```
In[6]:= KReduce[tripleSum, Tower- > {H2}]//Simplify
```

$$\text{Out[6]}= \frac{1}{72\ (1+n)}\left(112+49\ n-63\ n^2-36\ \left(2+3\ n+n^2\right)\ \sum_{\iota_1=1}^{n}\left(\frac{1}{\iota_1^2}\right)-\right.$$

$$\left.72\ \left(-1+n+n^2\right)\ \sum_{\iota_1=1}^{n}\left(\frac{1}{\iota_1}\right)+36\ \left(2+3\ n+n^2\right)\ \left(\sum_{\iota_1=1}^{n}\left(\frac{1}{\iota_1}\right)\right)^2\right)$$

3 The Power of Karr's Algorithm

```
In[7]:= example = DefineSum[1/DefineSum[1/i, {i, 1, k}]/
                    (1 - k * DefineSum[1/i, {i, 1, k}]),
                  {k, 2, n}]
```

$$\text{Out[7]}= \sum_{k=2}^{n} \left(\frac{1}{\left(\sum_{i=1}^{k} \left(\frac{1}{i} \right) \right) \left(1 - k \sum_{i=1}^{k} \left(\frac{1}{i} \right) \right)} \right)$$

```
In[8]:= KReduce[example]
```

$$\text{Out[8]}= -1 + \frac{1}{\sum_{\iota_1=1}^{n} \left(\frac{1}{\iota_1} \right)}$$

4 Automatic Extensions

4.1 Manual and Automatic Extension

```
In[9]:= cSum = DefineSum[DefineHNumber[k]/(k^2 - 1), {k, 2, n}]
```

Out[9]= $\sum_{k=2}^{n} \left(\frac{H_k}{-1+k^2}\right)$

Automatic Extension

```
In[10]:= KReduce[cSum, TowerSuggestion-> True]//Simplify
```

Out[10]= $\frac{1}{4\ n\ (1+n)} \left(-2\ (1+2\ n)\ H_n + n\ (1+n)\ \left(3+4\ \sum_{\iota_1=2}^{n} \left(\frac{-1+2\ \iota_1}{2\ (-1+\iota_1)\ \iota_1^2}\right)\right)\right)$

Manual Extension

In[11]:= KReduce$\left[\sum_{\iota_1=2}^{n} \left(\frac{-1+2\ \iota_1}{2\ (-1+\iota_1)\ \iota_1^2}\right),\right.$
Tower-> {{DefineHNumber[2, k], k}}]//
Simplify

Out[11]= $\frac{-1+n\ H_n^{(2)}}{2\ n}$

4.2 Example in Two Steps

```
In[12]:= cSum = DefineSum[
            DefineSum[DefineSum[1/DefineHNumber[i], {i, 1, k}], {k, 1, n}],
          {n, 2, m}]
```

$$\text{Out[12]}= \sum_{n=2}^{m}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\left(\frac{1}{H_i}\right)\right)\right)$$

First Step

```
In[13]:= f = KReduce[cSum, TowerSuggestion-> True]//Simplify
```

$$\text{Out[13]}= (1+m)\sum_{\iota_1=1}^{m}\left(\frac{1}{H_{\iota_1}}\right)+(1+m)\sum_{\iota_1=1}^{m}\left(-\frac{\iota_1}{H_{\iota_1}}\right)+\sum_{\iota_1=2}^{m}\left(\frac{\iota_1\left(-1+\iota_1+H_{\iota_1}\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{H_{\iota_2}}\right)\right)}{H_{\iota_1}}\right)$$

Second Step

```
In[14]:= KReduce[f, m, TowerSuggestion-> True, Level-> 2]
```

$$\text{Out[14]}= \frac{1}{2\ H_m}\left(2\ (-1+m)\ m+H_m\left(-2+(1+m)\ (2+m)\sum_{\iota_1=1}^{m}\left(\frac{1}{H_{\iota_1}}\right)+2\ (1+m)\sum_{\iota_1=1}^{m}\left(-\frac{\iota_1}{H_{\iota_1}}\right)-\sum_{\iota_1=2}^{m}\left(-\frac{(1+H_{\iota_1}\ (-4+\iota_1))\ (-1+\iota_1)\ \iota_1}{H_{\iota_1}\ (-1+H_{\iota_1}\ \iota_1)}\right)\right)\right)$$

5 Definite Summation

Example 1 (Goldberg)

```
In[15]:= mySum = DefineSum[DefinePower[-1, k](-1)^n*
              DefineFactorial[n + k - 1]/DefineFactorial[n - k]/
                DefineFactorial[k]^2DefineHNumber[n + k - 1]n^2,
            {k, 0, n}]
```

$$\text{Out[15]}= \sum_{k=0}^{n} \left(\frac{(-1)^n \; n^2 \; H_{-1+k+n} \; (-1+k+n)!_{.} \; (-1)^{k}_{.}}{(k)!_{.}^{2} \; (-k+n)!_{.}} \right)$$

```
In[16]:= GenerateRecurrence[mySum]
```

```
Out[16]= {n! SUM[n] - n! SUM[1 + n] == 0}
```

Example 2 (Panholzer)

```
In[17]:= mySum = DefineSum[DefineHNumber[n - k]/k, {k, 1, n}]
```

$$\text{Out[17]}= \sum_{k=1}^{n} \left(\frac{H_{-k+n}}{k} \right)$$

```
In[18]:= rec = GenerateRecurrence[mySum, RecOrder- > 2]//Simplify
Out[18]= {(1 + n) ((1 + n) SUM[n] - (3 + 2 n) SUM[1 + n] + (2 + n) SUM[2 +
n]) == 2}
```

Example 3 (Panholzer)

```
In[19]:= mySum = DefineSum[DefineHNumber[k]DefineHNumber[n-k], {k, 0, n}]
```

$$\text{Out[19]}= \sum_{k=0}^{n} (H_k \; H_{-k+n})$$

```
In[20]:= rec = GenerateRecurrence[mySum]//Simplify
```

```
Out[20]= {-(2 + n) SUM[n] + (1 + n) SUM[1 + n] == 2 (-n + (1 + n) H_n)}
```

Example 4 (Krattenthaler)

```
In[21]:= mySum = DefineSum[DefineFactorial[n + k + 3]DefinePower[-1, k]
             DefinePower[-1, n - 1]/(DefineFactorial[k + 1]
                  DefineFactorial[k + 2]DefineFactorial[n - k])
              (DefineHNumber[k]),
            {k, 1, n}]
```

$$\text{Out[21]}= \sum_{k=1}^{n} \left(\frac{H_k \ (3+k+n)! \ (-1)^k \ (-1)^{-1+n}}{(1+k)! \ (2+k)! \ (-k+n)!} \right)$$

```
In[22]:= rec = GenerateRecurrence[mySum]//Simplify
```

$$\begin{aligned}\text{Out[22]}= \{ & n \ (1+n) \ (2+n) \ (3+n) \ (4+n) \ (-1+n)! \\ & \Big(-(9+2\ n) \ \left(8+6\ n+n^2\right) \ \text{SUM}[n] + \\ & \quad (9+2\ n) \ \left(13+8\ n+n^2\right) \ \text{SUM}[1+n] + \\ & \quad \left(30+42\ n+17\ n^2+2\ n^3\right) \ \text{SUM}[2+n] - \\ & \quad (3+n) \ \left(25+15\ n+2\ n^2\right) \ \text{SUM}[3+n] \Big) == \\ & 2 \ (-1)^n \ (9+2\ n) \ \left(35+24\ n+4\ n^2\right) \ (4+n)! \}\end{aligned}$$

6 Karr's Method and an Example

Goal: Find a closed form for

$$\sum_{k=0}^{n} k \cdot k!.$$

A Difference Field for the Problem

Let t_1, t_2 be indeterminates where

$$\begin{aligned} k &\longleftrightarrow t_1 \\ k! &\longleftrightarrow t_2. \end{aligned}$$

Consider the field automorphism $\sigma : \mathbb{Q}(t_1, t_2) \to \mathbb{Q}(t_1, t_2)$ induced by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q} \\ \sigma(t_1) &= t_1 + 1 \\ \sigma(t_2) &= (t_1 + 1)t_2. \end{aligned}$$

$(\mathbb{Q}(t_1, t_2), \sigma)$ is our difference field.

The Telescoping Problem

$$\text{Find } g \in \mathbb{Q}(t_1, t_2): \quad \boxed{\sigma(g) - g = t_1 t_2}$$

$$\downarrow \text{ by Karr}$$

$$g = t_2.$$

The Closed Form

$$\boxed{(k+1)! - k! = k \cdot k!}$$

$$\downarrow$$

$$\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1.$$

7 Karr's Solution Space

Let

$\mathbb{K}$ be a field of characteristic 0 ($\mathbb{Q}$ or $\mathbb{Q}(n)$

$\mathbb{F} = \mathbb{K}(t_1)(t_2)\ldots(t_n)$ and $f, a, f_i, g \in \mathbb{F}$.

- Given f

  `Find g`:

$$\boxed{\sigma(g) - g = f}$$

$$\downarrow$$

- Given f, a

  `Find g`:

$$\boxed{\sigma(g) - \boxed{a} \cdot g = f}$$

$$\downarrow$$

- Given f, a

  `Find` $\boxed{c \in \mathbb{K}}$ `and` $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - a \cdot g = \boxed{f \cdot c}}$$

$$\downarrow$$

- Given $\langle f_1, \ldots, f_d \rangle$ and a and $W \leq \mathbb{F}$ specified by
  Karr's reduction process

  `Find all` $\langle c_1, \ldots, c_d \rangle \in \mathbb{K}^d$ `and all` $\boxed{g \in W}$:

$$\boxed{\sigma(g) - a \cdot g = \boxed{f_1 \cdot c_1 + \cdots f_d \cdot c_d}}$$

Remark: $f_i = \mathrm{N}^{i-1} f(n, k)$.

↓

- Given $\mathfrak{f} = \langle f_1, \ldots, f_d \rangle$ and a and $W \leq \mathbb{F}$ specified by Karr's reduction process

  Find basis of

$$\boxed{V(a, \mathfrak{f}, W) := \{\boxed{\langle c_1, \ldots, c_d, g \rangle} \in \mathbb{K}^d \times W \mid \sigma(g) - ag = f_1 \cdot c_1 + \cdots f_d \cdot c_d\}}$$

8 Back To Our Example: The Reduction Process

Consider the vectorspaces

$$\mathbb{Q}(t_1,t_2) := \{ \underbrace{p}_{poly.part} + \underbrace{\frac{q}{r}}_{fract.part} \mid p,q,r \in \boxed{\mathbb{Q}(t_1)[t_2]}, \deg q \prec \deg r\}$$

$$\mathbb{Q}(t_1,t_2)\boxed{d} := \{u \in \mathbb{Q}(t_1,t_2) \mid \text{the polynomial part of } u \text{ has } \deg \leq \boxed{d}\}$$

$$\mathbb{Q}(t_1,t_2)\boxed{-1} := \{u \in \mathbb{Q}(t_1,t_2) \mid \text{the polynomial part is } 0\}.$$

$$\text{Find } g \in \mathbb{Q}(t_1,t_2) : \boxed{\sigma(g) - g = t_1 \cdot t_2}$$

$$\downarrow$$

DEGREE ANALYSIS: The polynomial part of g has $\deg \leq 1$.

$$\downarrow$$

$$\text{Find basis of } \boxed{V(1, \langle t_1 t_2\rangle, \mathbb{Q}(t_1,t_2))}$$

$$\|$$

$$\boxed{V(1, \overbrace{\langle t_1 t_2\rangle}^{\mathbf{f_1}}, \mathbb{Q}(t_1,t_2)_1)}$$

REDUCTION-POLY

$$\text{Find } \mathbf{f_0}. \longrightarrow \text{Find basis of } \boxed{V(1, \mathbf{f_0}, \mathbb{Q}(t_1,t_2)_0)}$$

$$\downarrow$$

$$\vdots$$

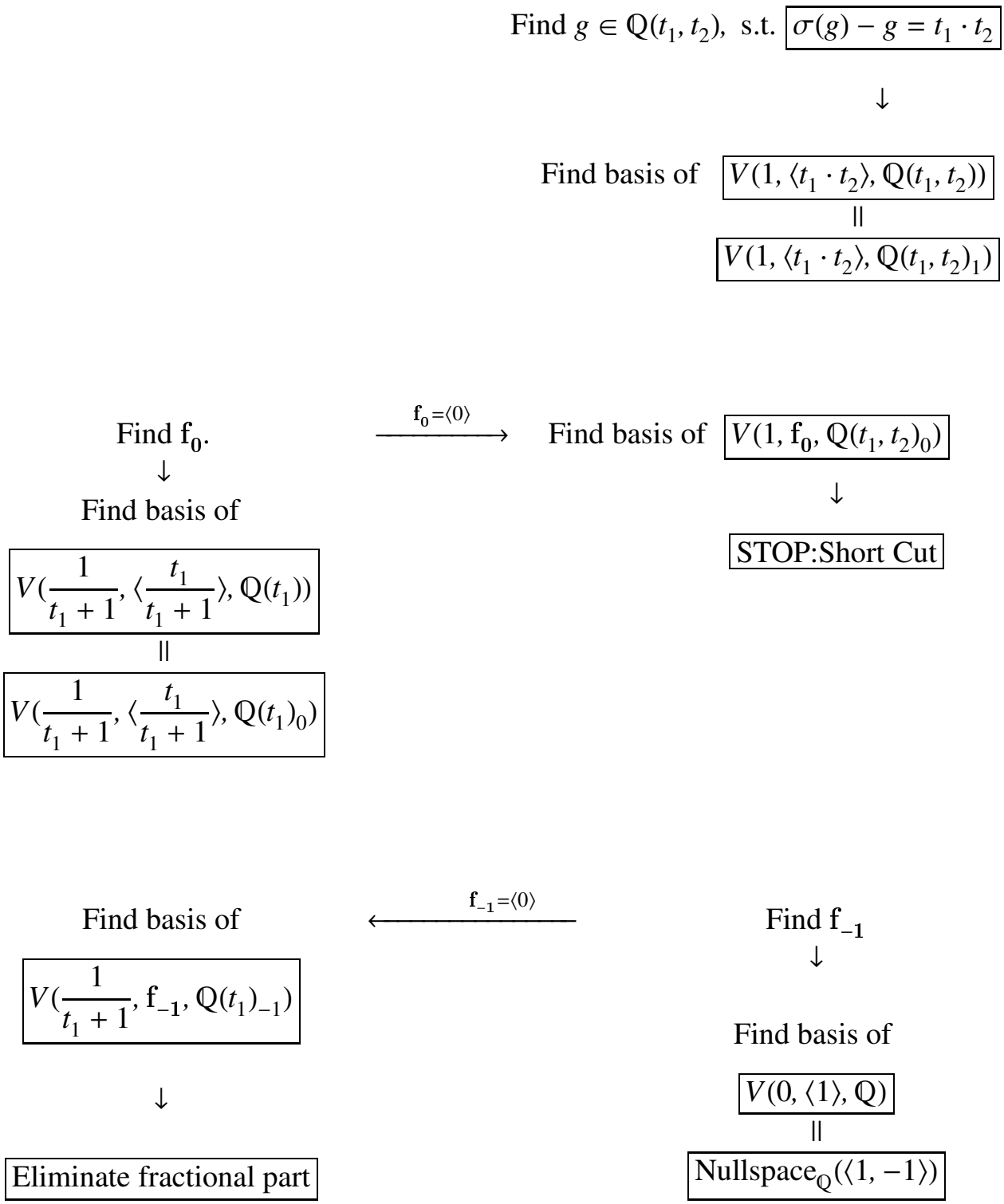
$$\downarrow$$

$$\text{Find basis of } \boxed{V(1, \mathbf{f_{-1}}, \mathbb{Q}(t_1,t_2)_{-1})}$$

$$\downarrow$$

REDUCTION-FRAC $\boxed{\cdots}$

9 The Recursion Process in Our Example



10 My Main Tasks/Topics of My PhD-Work

1. Extension and Optimization of Karr's Algorithm:
   - Avoidance of factorizations by using greatest factorial factorization.
   - Are algebraic extensions of the constant field possible in Karr's approach?
   - Automatic computation of "good" difference field extensions.
   - Embedding of Karr's theory in difference rings:
     E.g. we want to use
     $$(-1)^n := \prod_{i=1}^{n} (-1)$$
     as an extension element.
   - Which classes of definite summation can be handled by Karr?

2. Karr's approach & related algorithms:
   - Karr $\longleftrightarrow$ Gosper
   - Karr $\longleftrightarrow$ Risch

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