

# Computation of the Topology of Algebraic Space Curves\*

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## Abstract

An algorithm for computing the topology of a real, algebraic space curve  $\mathcal{C}$  implicitly defined as the complete intersection of two surfaces is presented. Given  $\mathcal{C}$ , the algorithm generates a space graph which is topologically equivalent to the real variety on the Euclidean space. The algorithm is based on the computation of the graph of at most two birational projections of  $\mathcal{C}$ . For this purpose, we introduce the notion of space general position for space curves, we show that any curve in the above conditions can always be linearly transformed to be in general position, and we present effective methods to check whether space general position has been reached.

## 1 Introduction.

The problem of computing the topological graph of algebraic curves plays an important role in many applications as plotting (see [11]) or sectioning in computer aided geometric design (see [2], [10]). Many authors have addressed the problem for the plane curve case (see [1], [5], [9],[7]) and theoretical complexity analysis and practical improvements have been presented in several papers (see [8], [7], [12]). In addition, in [6] the computation of the topological type of a surface has been addressed for the non-singular case. However, the computation of the topological graph of algebraic space curves has not been treated so extensively.

One may approach this problem by considering two projections, and therefore reducing the problem to the plane case, to afterwards lift the corresponding plane graphs

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to obtain the space graph. In that case, one needs a theoretical and algorithmic analysis to ensure that the input curve is in a “sufficiently good” position; more precisely, we need to ensure that:

- (1) The projections of the curve are plane curves whose graphs can be computed by using well-known methods. In our case, we will use the method presented in [9].
- (2) All the significant topological information about the space curve can be recovered from the planar graphs of its projections, i.e. the space graph can be reconstructed from the planar graphs of the projections.

In this paper, we deal with these difficulties for the case of real, algebraic space curves implicitly defined as the complete intersection of two surfaces. We introduce the notion of space general position for denoting that a space curve is in that “sufficiently good position”, and we prove that for almost all linear transformations the curve is placed in space general position. Also, we show how to check algorithmically whether a given curve is already in space general position. Furthermore, in the case that the space curve is non-reduced, i.e. if it has infinitely many singular points, some details concerning general position and the computation of the space graph must be considered. Thus, we present a characterization for reducibility based on resultants. General position is introduced such that the projections are done over the  $XY$  plane and the  $XZ$  plane. The first projection, the one over the  $XY$  plane, is taken as principal projection, while the second works auxiliary and it is only necessary under the presence of points on the first projection with a multiple fiber. For the case of reduced curves, in order to improve the computation of the critical points of the second projection, we take advantage of certain properties verified by the points that, when projected, give rise to critical points. Finally, once that space general position is achieved, the algorithm for computing the space graph of the curve can be applied.

The paper is structured as follows. In Section 2 we introduce the terminology to be used throughout the paper and we briefly recall the general strategy for computing the topology of plane curves. Section 3 is devoted to the notion of space general position, and to develop algorithmic criteria to check it. In Section 4 we present the algorithm for computing the topology of the space curve, and a detailed example. In Section 5, we discuss some aspects of the non-reduced case.

## 2 Preliminaries

In this section, we fix the terminology and we introduce some notions and results that will be used later on the paper. More precisely, throughout this paper  $\mathcal{C}$  is a real, algebraic space curve implicitly defined as the complete intersection of two surfaces of equations  $f, g \in \mathbb{R}[x, y, z]$ ; i.e.  $\mathcal{C}$  is the algebraic set  $V(f, g)$  defined by  $f, g$ . Also, we

denote by  $\pi_z$  the projection  $\pi_z : \mathbb{C}^3 \rightarrow \mathbb{C}^2 : \pi_z(x, y, z) = (x, y)$ ; similarly for  $\pi_x$  and  $\pi_y$ . Furthermore, for  $P \in \mathcal{C}$  we denote by  $\mathcal{F}_{\pi_z}(P)$  the fiber of the restriction mapping  $\pi_z|_{\mathcal{C}}$  at  $\pi_z(P)$ ; i.e.  $\mathcal{F}_{\pi_z}(P) = \{Q \in \mathcal{C} \mid \pi_z(Q) = \pi_z(P)\}$ . Similarly, for  $\mathcal{F}_{\pi_y}(P)$  and  $\mathcal{F}_{\pi_x}(P)$ .

In addition, we denote by  $\nabla$  the gradient operator, and by  $\times$  the vectorial product. Moreover, let  $\{\vec{i}, \vec{j}, \vec{k}\}$  be the canonical basis of  $\mathbb{C}^3$ . Let  $\vec{T}(x, y, z) = \nabla f \times \nabla g$  and let  $U, V, W \in \mathbb{R}[x, y, z]$  be the coordinates of  $\vec{T}(x, y, z)$  w.r.t.  $\{\vec{i}, \vec{j}, \vec{k}\}$ ; i.e.  $\vec{T}(x, y, z) = U(x, y, z)\vec{i} + V(x, y, z)\vec{j} + W(x, y, z)\vec{k}$ . Also, we will denote the partial derivative of a polynomial  $g(x, y, z)$  w.r.t. the variable  $v \in \{x, y, z\}$  as  $g_v$ .

### TOPOLOGY OF A PLANE REAL CURVE

The topology of a real plane curve is approached by means of the so-called *graph associated* to the curve. This is a concept that has been addressed by several authors, see for instance [9], [7]. In this subsection, we briefly describe the standard strategy for computing it, and we also recall some aspects of the algorithm described in [9], which will be used later. Here, we assume that  $\mathcal{H}$  is a real plane curve defined by an square-free polynomial  $h(x, y) \in \mathbb{R}[x, y]$ . In this situation, one introduces the notion of critical points as follows.  $P \in \mathbb{R}^2$  is a *critical point* of  $\mathcal{H}$  if  $h(P) = 0$ , and  $\frac{\partial h}{\partial y}(P) = 0$ .  $P$  is a *ramification point* of  $\mathcal{H}$  if it is a non-singular critical point of  $\mathcal{H}$ .  $P$  is a *regular point* of  $\mathcal{H}$  if  $h(P) = 0$  and it is not critical. Then, the graph associated to  $\mathcal{H}$ , that we represent by  $\text{Graph}(\mathcal{H})$ , is essentially introduced as the graph which vertices are the critical points of  $\mathcal{H}$  and some additional real simple points on the curve, and where every edge of the graph corresponds to a branch of  $\mathcal{H}$  joining two vertices. In fact, this notion can be extended to algebraic plane curves with multiple components by defining the graph of such a curve as the graph of its square-free part.

In order to determine  $\text{Graph}(\mathcal{H})$ , one assumes that the curve is in *planar general position*; i.e. no component of  $\mathcal{H}$  is a real vertical line,  $\mathcal{H}$  has no vertical asymptotes, and the  $x$ -coordinates of the critical points of  $\mathcal{H}$  are different. To ensure the first two conditions, one requires that the leading coefficient of  $h(x, y)$  w.r.t.  $y$  has no real root. Also, note that almost all affine linear changes of coordinates transform  $\mathcal{H}$  in general position. Thus, one may always consider a random affine linear transformation, and use then the techniques provided in [9] for checking general position. An alternative approach is to apply the deterministic algorithm described in [3], that leads a plane curve to planar general position.

The graph of  $\mathcal{H}$  can be computed by performing the following steps (see [9] for further details):

- (S-1) [Critical Points] Compute the square-free part of the discriminant of  $h(x, y)$  w.r.t.  $y$ , and approximate its real roots,  $\alpha_1 < \dots < \alpha_r$  (i.e. the  $x$ -coordinates of the critical points). For each  $\alpha_i$ , compute the  $y$ -coordinates  $\beta_{i,j}$  of the points of  $\mathcal{H}$  lying on the line  $x = \alpha_i$ .

- (S-2) [In Out Edges] For each critical point  $(\alpha_i, \beta_i)$ , compute the number of half-branches to the right and to the left of  $(\alpha_i, \beta_i)$ . In order to do this, consider two auxiliary lines  $x = \delta_i$  and  $x = \delta_{i+1}$  verifying that  $\alpha_{i-1} < \delta_i < \alpha_i < \delta_{i+1} < \alpha_{i+1}$ , and let  $V(\delta_i)$ ,  $V(\alpha_i)$ ,  $V(\delta_{i+1})$  denote the number of real points of  $\mathcal{H}$  belonging to the lines  $x = \delta_i$ ,  $x = \alpha_i$ ,  $x = \delta_{i+1}$ , respectively. Then, the number of half-branches to the left of  $(\alpha_i, \beta_i)$  is  $V(\delta_i) - V(\alpha_i) + 1$ , and the number of half-branches to the right of  $(\alpha_i, \beta_i)$  is  $V(\delta_{i+1}) - V(\alpha_i) + 1$ .
- (S-3) [Graph] Construct  $\text{Graph}(\mathcal{H})$  by appropriately joining the points obtained in (S-1); more precisely, note that the way to join the points is uniquely determined since any incorrect way to join them leads to at least one intersection of two edges at a point that is not critical.

An alternative to computing the number of half-branches to the right and to the left of each critical point, is the following: for constructing the part of the graph lying between  $x = \alpha_{i-1}$  and  $x = \alpha_i$ , we explicitly compute the points of  $\mathcal{H}$  belonging to the line  $x = \delta_i$ ; then, in order to connect the points in the lines  $x = \alpha_{i-1}$  and  $x = \alpha_i$ , we first join the points of  $x = \alpha_{i-1}$  and  $x = \delta_i$ , and then we join the points of  $x = \delta_i$  and  $x = \alpha_i$ . Repeating the same process for all the critical points, the graph is constructed. Then, the auxiliary points may be kept as vertices of the graph, or they may be cleaned. This alternative process will be useful when we approach the space case (see section 4). Figure 1 illustrates these ideas.

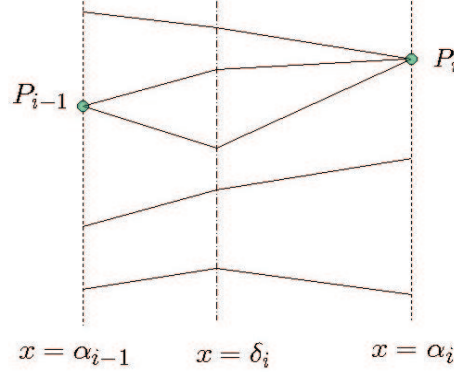


Figure 1: Construction of the planar graph

### 3 Generality of the method

The method described above requires that the curve is in planar general position, and therefore the curve has to be previously prepared under a linear change of coordinates.

In our algorithm, in order to analyze the topology of an space curve, we also need that the input curve satisfies certain properties that will lead to the notion of *space general position*. In this section, we introduce these properties, we provide computational techniques to test whether they are fulfilled by  $\mathcal{C}$ , and we show that almost all linear changes of coordinates transform  $\mathcal{C}$  onto general position. Furthermore, as we will see later, in some steps of our algorithm we need to know whether the space curve we are working with is reduced, or not. That is, we need to check whether  $\mathcal{C}$  has, or not, finitely many singular points. For that purpose, in this section we also present a result characterizing reduced curves, that allows to algorithmically certify that a curve is reduced. We begin with the characterization of space reduced curves, and then we will consider space general position.

### 3.1. CHECKING REDUCED SPACE CURVES

As we will see later, in some steps of our algorithm we need to know whether the space curve we are working with is reduced, or not. In this context, we say that  $P \in \mathcal{C}$  is a *singular point* of  $\mathcal{C}$ , if  $\vec{T}(P) = \vec{0}$ , and we say that  $\mathcal{C}$  is a *reduced curve*, if it has finitely many singular points; otherwise, we say that  $\mathcal{C}$  is *non-reduced*.

In the plane case one may check whether the curve is reduced or not by simply analyzing whether the defining polynomial is square-free or not. In the space case, one may check the existence of non-reduced components by means of Gröbner basis techniques. However, this may be costly. Thus, we show how to reduce the problem for space curves in complete intersection to the plane case, and therefore to resultant computations.

In order to do that, we first observe that any component  $\mathcal{D}$  of  $\mathcal{C}$  has dimension 1 (see for instance [16] Vol.I, page 74), and that  $\mathcal{D}$  is non-reduced if and only if  $\nabla f \times \nabla g$  vanishes at infinitely many points of  $\mathcal{D}$ . Note that, in that case, if  $\mathcal{D}$  is irreducible, then  $\nabla f \times \nabla g$  vanishes at every point of  $\mathcal{D}$ . Moreover, an irreducible component  $\mathcal{D}$  of  $\mathcal{C}$  is non-reduced if and only if either one of the gradients  $\nabla f, \nabla g$  vanishes at every point of  $\mathcal{D}$ , or both gradients  $\nabla f, \nabla g$  vanish at finitely many points of  $\mathcal{D}$  but  $\nabla f$  and  $\nabla g$  are parallel at every point of  $\mathcal{D}$ .

In the following theorem, a characterization for non-reduced curves by means of resultants is given. For the theorem, we will use the notation  $f^x = f, g^x = g$  if the leading coefficient of  $f$  w.r.t.  $x$  is constant and the restriction of  $\pi_x$  to any irreducible component of  $\mathcal{C}$  is birational, else  $f^x, g^x$  will denote the result of applying an affine linear transformation to  $f, g$  so that the above two conditions are verified (similarly for  $f^y, f^z, g^y, g^z$ ). Note that almost all affine linear transformations provide the required conditions.

**Theorem 1:** Let  $f, g \in \mathbb{C}[x, y, z]$  be square-free polynomials. Then,  $\mathcal{C}$  is non-reduced if and only if either  $\text{Res}_z(f^z, g^z)$  or  $\text{Res}_y(f^y, g^y)$  or  $\text{Res}_x(f^x, g^x)$  is not square-free.

**Proof.** Let  $\mathcal{C}^x, \mathcal{C}^y, \mathcal{C}^z$  be the space curves defined by  $\{f^x, g^x\}, \{f^y, g^y\}, \{f^z, g^z\}$ , respectively. Note that the property of being non-reduced is preserved by affine linear transformations, hence  $\mathcal{C}$  is non-reduced iff  $\mathcal{C}^x, \mathcal{C}^y, \mathcal{C}^z$  are non-reduced. Now, let  $K^z = \text{Res}_z(f^z, g^z)$ ,  $K^y = \text{Res}_y(f^y, g^y)$ , and  $K^x = \text{Res}_x(f^x, g^x)$ . Then, there exist polynomials  $A^x, A^y, A^z, B^x, B^y, B^z \in \mathbb{C}[x, y, z]$  such that  $K^v = A^v f^v + B^v g^v$  with  $v \in \{x, y, z\}$ . In this situation, let  $\mathcal{D}$  be an irreducible, non-reduced component of  $\mathcal{C}$ . Then, one of the following statements is true: (1)  $\nabla f$  or  $\nabla g$  vanishes at every point of  $\mathcal{D}$ ; (2)  $\nabla f$  and  $\nabla g$  vanish at finitely many points of  $\mathcal{D}$  but  $\nabla f$  and  $\nabla g$  are parallel at every point of  $\mathcal{D}$ . We distinguish several cases:

(i) Assume that (1) is true, and that  $\nabla f, \nabla g$  vanish at every point of  $\mathcal{D}$ . Then,  $\nabla f^z$  and  $\nabla g^z$  vanish at every point of the corresponding component  $\mathcal{D}^z$  of  $\mathcal{C}^z$ . Therefore, for every point  $P \in \mathcal{D}^z$ , it holds that:

$$K_x^z(\pi_z(P)) = A_x^z(P)f^z(P) + A^z(P)f_x^z(P) + B_x^z(P)g^z(P) + B^z(P)g_x^z(P) = 0,$$

and similarly  $K_y^z(\pi_z(P)) = 0$ . Furthermore, since  $P \in \mathcal{D}^z$ , it holds that  $\pi_z(P)$  is on the curve defined by  $K^z$  (i.e., on  $\pi_z(\mathcal{C}^z)$ ), so  $\pi_z(P)$  is a singularity of  $\pi_z(\mathcal{C}^z)$ . Moreover, the projection of  $\mathcal{C}^z$  onto  $x, y$  is birational, so  $\pi_z(\mathcal{C}^z)$  has infinitely many singularities. Therefore,  $\pi_z(\mathcal{C}^z)$  has a multiple component, so  $K^z$  is not square-free.

(ii) Assume that (1) is true, and that only one of the gradients  $\nabla f, \nabla g$  vanishes at every point of  $\mathcal{D}$ . Assume that the vanishing vector is  $\nabla g$ ; similarly for  $\nabla f$ . Then, at least one of the partial derivatives of  $f$ , say  $f_z$ , vanishes only at a finite number of points of  $\mathcal{D}^z$ ; similarly for  $f_x$  and  $f_y$ . This implies that  $\nabla g^z$  vanishes at every point of  $\mathcal{D}^z$ , and  $f_z^z$  vanishes only on a finite subset  $\Sigma^z$  of  $\mathcal{D}^z$ . Now, for every  $P \in \mathcal{D}^z \setminus \Sigma^z$ , it holds that  $K_v^z(\pi_z(P)) = A^z(P)f_v^z(P)$ , for  $v \in \{x, y, z\}$ . Since  $K^z \in \mathbb{C}[x, y]$ , we have that, for every  $P \in \mathcal{D}^z \setminus \Sigma^z$ ,  $K_z^z(\pi_z(P)) = 0$ . Thus, taking into account that  $f_z^z(P) \neq 0$ , it holds that  $A^z(P) = 0$ , so  $K_x^z(\pi_z(P)) = K_y^z(\pi_z(P)) = 0$ . Reasoning like in case (i) one concludes that  $K^z$  is not square-free.

(iii) Assume that (2) is true. Then, at least one of the partial derivatives of  $f$ , say  $f_z$ , vanishes only at a finite number of points of  $\mathcal{D}$ ; similarly for  $f_x$  and  $f_y$ . Thus,  $f_z^z$  vanishes only on a finite subset  $\Sigma^z$  of  $\mathcal{D}^z$ . Then, for every  $P \in \mathcal{D}^z \setminus \Sigma^z$ , it holds that  $f_y^z(P)g_z^z(P) - f_z^z(P)g_y^z(P) = 0$ ,  $-f_x^z(P)g_z^z(P) + f_z^z(P)g_x^z(P) = 0$ ,  $f_x^z(P)g_y^z(P) - f_y^z(P)g_x^z(P) = 0$ , and that

$$\begin{aligned} K_x^z(\pi_z(P)) &= A^z(P)f_x^z(P) + B^z(P)g_x^z(P) \\ K_y^z(\pi_z(P)) &= A^z(P)f_y^z(P) + B^z(P)g_y^z(P) \\ K_z^z(\pi_z(P)) &= A^z(P)f_z^z(P) + B^z(P)g_z^z(P). \end{aligned}$$

Since for every  $P \in \mathcal{D}^z \setminus \Sigma^z$ ,  $K_z^z(\pi_z(P)) = 0$ , and  $f_z^z(P) \neq 0$ , from the third equality one has that:

$$A^z(P) = -B^z(P) \frac{g_z^z(P)}{f_z^z(P)}.$$

Substituting in  $K_x^z(\pi_z(P))$ , and  $K_y^z(\pi_z(P))$  and using the equalities above, we get that  $K^z$  is not square-free.

Now we proceed to prove the converse statement. We assume w.l.o.g. that  $K^z$  is not square-free. Then, the plane curve defined by  $K^z$  has a multiple component  $\mathcal{L}$  and therefore  $K_x^z, K_y^z$  vanish at the points of  $\mathcal{L}$ . In this situation, for every  $P \in \pi_z^{-1}(\mathcal{L})$ , where  $\pi_z : \mathcal{C} \rightarrow \mathbb{C}^2$ , it holds that  $A^z(P)f_x^z(P) + B^z(P)g_x^z(P) = 0$ , and similarly  $A^z(P)f_y^z(P) + B^z(P)g_y^z(P) = 0$ . Furthermore,  $A^z(P)f_z^z(P) + B^z(P)g_z^z(P) = 0$ . Thus, for every  $P \in \pi_z^{-1}(\mathcal{L})$  we get that:

$$A^z(P)\nabla f^z(P) = -B^z(P)\nabla g^z(P)$$

Hence,  $\nabla f^z \times \nabla g^z$  vanishes at  $\pi_z^{-1}(\mathcal{L})$ . Since the projection is birational,  $\pi_z^{-1}(\mathcal{L})$  contains infinitely many points of  $\mathcal{C}^z$ , so its algebraic closure is a non-reduced component of  $\mathcal{C}^z$ . Thus,  $\mathcal{C}$  is non-reduced.  $\square$

### 3.2. SPACE GENERAL POSITION

Now we describe the properties we impose to the curve  $\mathcal{C}$  in order to be in general position, and we show that these requirements can always be achieved after a suitable linear change of coordinates. More precisely, the required properties are the following:

**Property  $\mathcal{P}_1$  :** The leading coefficient of either  $f$  or  $g$  w.r.t.  $z$  and the leading coefficient of either  $f$  or  $g$  w.r.t.  $y$  are constant.

Note that, by well-known properties on resultants,  $\mathcal{P}_1$  implies that the implicit equations of the plane curves  $\pi_z(\mathcal{C})$  and  $\pi_y(\mathcal{C})$  are  $\text{Res}_z(f, g)$  and  $\text{Res}_y(f, g)$ , respectively. Moreover, it also ensures that no component of  $\mathcal{C}$  is a perpendicular line to the plane  $z = 0$  or to the plane  $y = 0$ .

**Property  $\mathcal{P}_2$  :** The set of points  $P \in \mathcal{C}$  such that  $\mathcal{F}_{\pi_z}(P) \neq \{P\}$  and the set of points  $P \in \mathcal{C}$  such that  $\mathcal{F}_{\pi_y}(P) \neq \{P\}$  are either empty or 0-dimensional.

Observe that  $\mathcal{P}_2$  is equivalent to the fact that the restriction of  $\pi_z$  and  $\pi_y$  to every irreducible component of  $\mathcal{C}$  is birational.

For determining the graph of the space curve  $\mathcal{C}$  we need, in the worst case, to know the behavior of two projections (see subsection 4.1). For this reason, in addition to the above properties we also require that the projected curves are in planar general position. This motivates the following definition.

**Definition 1.** We say that  $\mathcal{C}$  is in *space general position* if it satisfies  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , and  $\pi_z(\mathcal{C})$  and  $\pi_y(\mathcal{C})$  are in planar general position.

Now, we analyze whether one can assume w.l.o.g. that  $\mathcal{C}$  is in space general position. For property  $\mathcal{P}_1$ , we observe that for every  $a, b, c, d \in \mathbb{R}$  such that  $c \neq d$  and such that

$(b, d, 1)$  and  $(a, c, 1)$  are not roots of the homogenous forms of maximum degree of  $f$  and  $g$  respectively, then the linear change of coordinates  $\{x = x' + ay' + bz', y = cy' + dz', z = y' + z'\}$  transforms  $\mathcal{C}$  onto a curve satisfying  $\mathcal{P}_1$ .

On the other hand, since  $\mathcal{P}_2$  is equivalent to the fact that the restriction of  $\pi_z$  and  $\pi_y$  to every irreducible component of  $\mathcal{C}$  is birational, one has that for almost all affine linear transformation  $L$  the curve  $L(\mathcal{C})$  verifies  $\mathcal{P}_2$ . Thus, one may choose a random linear change to force  $\mathcal{C}$  to verify  $\mathcal{P}_2$ . Furthermore, since there exist algorithms to check whether  $\mathcal{C}$  satisfies  $\mathcal{P}_2$  (see for instance [13]) one may also proceed deterministically. However, these algorithms require the use of Gröbner basis. Thus, we present two results which provide alternative ways to carry out the birationality test. For this purpose, we assume that  $\mathcal{P}_1$  is already satisfied.

For the first result we need the following technical lemma which extends Theorem I.10.9 of [17] to our purposes:

**Lemma 1:** Let  $D$  be a unique factorization domain, and let  $F, G \in D[x_1, \dots, x_s]$  be two homogeneous polynomials whose degrees are  $m$  and  $n$ , respectively. Assume that  $F$  has the form:

$$F(x_1, \dots, x_s) = A_0 x_s^m + A_1(\bar{x}) x_s^{m-1} + \dots + A_m(\bar{x})$$

where  $A_0$  is a non-vanishing constant and  $\bar{x} = (x_1, \dots, x_{s-1})$ . Then, the resultant  $R(\bar{x})$  of  $F$  and  $G$  w.r.t.  $x_s$ , is either  $R = 0$  or homogeneous of degree  $m \cdot n$ .

**Proof:** We can write  $G = B_0 x_s^n + B_1(\bar{x}) x_s^{n-1} + \dots + B_n(\bar{x})$ , where  $B_0$  is a constant that might be 0. If  $B_0 \neq 0$ , the result is Theorem I.10.9 of [17]. Thus, assume that  $B_0 = 0$ . Then, there exists  $k \in \{1, \dots, n\}$  such that  $B_k(\bar{x})$  is not zero; since we can similarly reason for any value of  $k$ , let us assume that  $k = 1$ , i.e. that  $B_1(\bar{x})$  is the first coefficient of  $G$  which is not 0. Then, using the Sylvester form of the resultant,  $R(\bar{x})$  can be expressed as:

$$R(\bar{x}) = \begin{vmatrix} A_0 & \dots & \dots & A_m & & \\ & \ddots & & & \ddots & \\ & & A_0 & \dots & \dots & A_m \\ B_1 & \dots & \dots & B_n & & \\ & \ddots & & & \ddots & \\ & & B_1 & \dots & \dots & B_n \end{vmatrix}$$

where, for sake of simplicity, we have written  $A_i$  instead of  $A_i(\bar{x})$ , and similarly for the  $B_j$ . Let  $\Delta(\bar{x}) = A_0 \cdot R(\bar{x})$ ; then,  $\Delta(\bar{x})$  can be expressed as a determinant in the following way:



$$\Delta(\bar{x}) = \begin{vmatrix} A_0 & \cdots & \cdots & A_m & & & \\ 0 & A_0 & \cdots & \cdots & A_m & & \\ \vdots & & & \ddots & & & \ddots \\ 0 & \cdots & \cdots & A_0 & \cdots & \cdots & A_m \\ 0 & B_1 & \cdots & \cdots & B_n & & \\ \vdots & & & \ddots & & & \ddots \\ 0 & \cdots & \cdots & B_1 & \cdots & \cdots & B_n \end{vmatrix}$$

In order to prove this identity, note that expanding the determinant above by the first column yields  $A_0 \cdot R(\bar{x})$ . Then, applying in this determinant the technique used for proving Theorem I.10.9 in [17], we get that  $\Delta(\bar{x})$  is either 0, or an homogeneous polynomial of degree  $m \cdot n$ . Since  $A_0 \neq 0$ , the same holds for  $R(\bar{x})$ .  $\square$

Now, let  $\hat{f}, \hat{g}$  denote the homogenizations of  $f, g$ , respectively, with respect to a variable  $w$ . Also, let  $\hat{\mathcal{C}} = V(\hat{f}, \hat{g})$ , let  $\hat{h} = \text{Res}_z(\hat{f}, \hat{g})$ , and let  $\hat{\mathcal{H}}$  be the curve defined by  $\hat{h}$ . Note that since  $f$  and  $g$  define a real space curve, we can assume that  $f$  and  $g$  have no common factor, so  $\hat{h} \neq 0$ . Furthermore, if we denote by  $m, n$  the degrees of  $\hat{f}, \hat{g}$ , respectively, since  $\mathcal{P}_1$  is satisfied, by Lemma 1 we get that the total degree of  $\hat{h}$  is  $r = n \cdot m$ . We still need another previous lemma:

**Lemma 2.** Let  $\hat{h}$  be square-free and let  $(a : b : c) \in \hat{\mathcal{H}}$  be regular. Then, there are finitely many projective lines  $L^*$  passing through  $(a : b : c)$  and verifying at least one of the following conditions:

- i.  $L^*$  is a component of  $\hat{\mathcal{H}}$ .
- ii.  $L^*$  is a factor of the homogeneous form of maximum degree of  $f$  or  $g$ .
- iii.  $L^*$  contains at least one singular point of  $\hat{\mathcal{H}}$ .
- iv.  $L^*$  is tangent to  $\hat{\mathcal{H}}$ .

**Proof:** The number of irreducible components of an algebraic plane curve is finite, so it is clear that there are finitely many lines satisfying i. Similarly for ii. Furthermore, since  $\hat{h}$  is square-free, the number of singularities of  $\hat{\mathcal{H}}$  is finite, so there are only finitely many lines passing through  $(a : b : c)$  that also pass through a singularity of  $\hat{\mathcal{H}}$ . For condition iv. we refer to Lemma VI.3 in [14].  $\square$

**Theorem 2.** Let  $\mathcal{C}$  satisfy  $\mathcal{P}_1$ . If the projection  $\pi_z$  is not birational, then the polynomial  $\hat{h} = \text{Res}_z(\hat{f}, \hat{g})$  is not square-free.

**Proof:** Let  $\hat{h}(x, y, w)$  be square-free. Then the curve  $\hat{\mathcal{H}}$  has finitely many singular points. Let  $(a : b : c) \in \hat{\mathcal{H}}$  be a simple point whose fiber  $\pi_z^{-1}(a : b : c)$  consists of at

least two points of  $\hat{\mathcal{C}}$  (note that, since by hypothesis  $\pi_z$  is not birational, there exist infinitely many points verifying this), and let  $L = (a : b : c)s + t(v_0 : v_1 : v_2)$  be the parametric expression of a projective line passing through  $(a : b : c)$  and not verifying any of the conditions i., ii., iii. and iv. of Lemma 2 (note that, by Lemma 2, there are infinitely many lines  $L$  fulfilling this). Then, by Bezout's Theorem, the number of intersections of  $\hat{\mathcal{H}}$  and  $L$ , is  $r$ ; in fact, since  $L$  does not verify any of the conditions i., ii., iii., iv. in Lemma 2, these intersections correspond to  $r$  distinct simple points of  $\hat{\mathcal{H}}$ . Therefore, the polynomial  $\hat{R}(t, s) = \hat{h}(as + tv_0, bs + tv_1, cs + tv_2)$  has  $r$  simple distinct roots  $(t_1 : s_1), \dots, (t_r : s_r) \in \mathbb{P}(\mathbb{C})$ , and each root  $(t_i : s_i)$  corresponds to a point  $P_i$  of  $\hat{\mathcal{H}}$ . Since  $\mathcal{P}_1$  holds, the fiber of each  $P_i$  consists of finitely many points. Thus, for each  $(t_i : s_i)$  there are only finitely many  $z$  such that  $\hat{f}(as_i + t_i v_0, bs_i + t_i v_1, cs_i + t_i v_2, z) = 0$ ,  $\hat{g}(as_i + t_i v_0, bs_i + t_i v_1, cs_i + t_i v_2, z) = 0$ . Consequently, we get that the polynomial system:

$$\left. \begin{array}{l} \hat{f}(as + tv_0, bs + tv_1, cs + tv_2, z) = 0 \\ \hat{g}(as + tv_0, bs + tv_1, cs + tv_2, z) = 0 \end{array} \right\} (*_1)$$

has a finite number of solutions in  $\mathbb{P}^2(\mathbb{C})$ . More precisely, since  $L$  does not verify condition ii. in Lemma 2, the polynomials defining  $(*_1)$  have the same degrees as  $\hat{f}, \hat{g}$ , so by Bezout's Theorem the number of solutions of  $(*_1)$ , counting multiplicities, is  $r$ . Furthermore, every solution of  $(*_1)$  is of the type  $(t_i : s_i : z^{i, k_i})$ , where  $i = 1, \dots, r$ .

Now, since  $(a : b : c) \in \hat{\mathcal{H}}$ , it is clear that  $(0 : 1)$  is a root of the polynomial  $\hat{R}$ . Assume w.l.o.g. that  $(t_1 : s_1) = (0 : 1)$ . Then, since the fiber  $\pi_z^{-1}(a : b : c)$  consists of at least two points, there exist at least two distinct numbers  $z^{0,1}, z^{0,2}$ , such that  $(0 : 1 : z^{0,1})$  and  $(0 : 1 : z^{0,2})$  are two distinct solutions of  $(*_1)$ . Since for all  $i \in \{2, \dots, r\}$ , the corresponding  $(t_i : s_i)$  has at least an associated  $z^{i, k_i}$  such that  $(t_i : s_i : z^{i, k_i})$  is a solution of  $(*_1)$ , it follows that  $(*_1)$  has at least  $r + 1$  solutions, which is a contradiction.

Therefore, the curve  $\hat{\mathcal{H}}$  has no simple point whose fiber consists of more than one point. Therefore, either  $\pi_z$  is birational, or  $\hat{\mathcal{H}}$  has infinitely many singular points. Since by hypothesis  $\pi_z$  is not birational, we get that  $\hat{\mathcal{H}}$  has infinitely many singular points, so it is not square-free.  $\square$

**Remark 1:** Using a similar strategy, one might also prove the following result: let  $\mathcal{C}$  verify  $\mathcal{P}_1$ , let  $\pi_z|_{\mathcal{C}}$  be birational, and let  $\pi_z(\mathcal{C})$  be square-free; then, if the cardinal of the fiber of  $(x_0, y_0) \in \pi_z(\mathcal{C})$  is greater than 1,  $(x_0, y_0)$  is a singular point of  $\pi_z(\mathcal{C})$ . In order to prove it, one assumes that  $(x_0, y_0)$  is non-singular, and argues in a similar way to Theorem 2 to show a contradiction.

An analogous theorem can be established for the projection  $\pi_y$ . Thus, one has the following corollary:

**Corollary 1:** If  $\mathcal{C}$  verifies  $\mathcal{P}_1$  and  $\text{Res}_z(\hat{f}, \hat{g})$  is square-free, then  $\pi_z$  is birational. Similarly for  $\text{Res}_y(\hat{f}, \hat{g})$  and  $\pi_y$ .

Note that, by Theorem 1, the converse of Theorem 2 is not true, since the fact that some resultant is not square-free may also be caused by non-reducibility of the curve. Hence, Theorem 2 is *not* a characterization for birationality; nevertheless, it provides a fast test that allows to certify the birationality of a projection in many cases. In fact, this result can be adapted to check the birationality of a projection over any plane different from the coordinate planes by first applying a linear change of coordinates that transforms the plane into a coordinate plane.

Since in the cases where there exists a resultant with a multiple factor we cannot decide the birationality of the projection by means of Theorem 2, we need another result for checking the birationality in those cases. For that purpose, let  $\mathcal{M}$  be an irreducible component of  $\pi_z(\mathcal{C})$  over  $\mathbb{C}$ , let  $m$  be the polynomial defining  $\mathcal{M}$ , and let  $\tilde{\mathcal{M}}$  denote the irreducible component of  $\mathcal{C}$  corresponding to  $\mathcal{M}$ . We denote by  $\mathbb{C}(\mathcal{M})$  the field of rational functions of  $\mathcal{M}$ . Then, we can consider  $f$  and  $g$  as elements of  $\mathbb{C}(\mathcal{M})[z]$ , which is an Euclidean domain. Hence, the  $\text{gcd}(f, g)$  in  $\mathbb{C}(\mathcal{M})[z]$ , that we represent by  $G_m(z)$ , can be computed by means of Euclides algorithm. Then, we have the following result:

**Proposition 1.** The restriction  $\pi_z|_{\tilde{\mathcal{M}}}$  is birational  $\Leftrightarrow G_m(z)$  has only one different root.

**Proof:** If  $\pi_z|_{\tilde{\mathcal{M}}}$  is birational, for almost all  $(x_0, y_0) \in \mathcal{M}$ , we get that  $\text{gcd}(f(x_0, y_0, z), g(x_0, y_0, z))$  has only one different root, so the implication ( $\Rightarrow$ ) follows. Conversely, if  $G_m(z)$  has only one root in  $\mathbb{C}(\mathcal{M})$ , then  $G_m(z) = (a(x, y)z - b(x, y))^s$ , where  $a, b \in \mathbb{C}(\mathcal{M})$ . Therefore,  $\frac{b(x, y)}{a(x, y)}$  is the inverse of  $\pi_z|_{\tilde{\mathcal{M}}}$  and thus  $\pi_z|_{\tilde{\mathcal{M}}}$  is birational.  $\square$

Proposition 1 requires the previous computation of the factors of  $\text{Res}_z(f, g)$  over the complex. In order to do that, we can use the algorithm provided in [4], where the factors of  $\text{Res}_z(f, g)$  can be obtained without a previous computation of  $\text{Res}_z(f, g)$  (in fact, using this algorithm,  $\text{Res}_z(f, g)$  may be computed *a fortiori* by multiplying its factors).

To finish our analysis of generality it only remains to study when  $\pi_z(\mathcal{C})$  and  $\pi_y(\mathcal{C})$  are in planar general position. For this purpose, let us assume that  $\mathcal{C}$  already satisfies  $\mathcal{P}_1$  and  $\mathcal{P}_2$ . First, we observe that when  $\mathcal{C}$  is reduced, the implicit equations of  $\pi_z(\mathcal{C})$  and  $\pi_y(\mathcal{C})$  are square-free (see Theorem 1); this is not the case when  $\mathcal{C}$  is non-reduced, but in that case we can consider their square-free part. As we mentioned in section 1, there are deterministic algorithms for computing a change of coordinates that leads the curve to general planar position; however, since for almost all linear changes in  $x, y$ , the projection  $\pi_z(\mathcal{C})$  is placed in general planar position, it is cheaper to choose a

random linear change, and then check whether  $\pi_z(\mathcal{C})$  is in general planar position, or not. Furthermore, leaving the  $z$ -coordinate invariant we get a linear change in  $x, y, z$  such that  $\pi_z(\mathcal{C})$  is in general planar position. Similarly for  $\pi_y(\mathcal{C})$ . In order to check whether general planar position for  $\pi_z(\mathcal{C})$  has been achieved, we first check that the leading coefficient w.r.t.  $y$  of  $h(x, y) = \text{Res}_z(f, g)$  does not have real roots (similarly for  $\text{Res}_y(f, g)$ ); if this condition is not fulfilled, it is easy to prove that almost all linear changes in  $x, y$  make that the leading coefficient of  $\pi_z(\mathcal{C})$  w.r.t.  $y$  is constant; then, one has to check that different critical points have different  $x$ -coordinates; this can be done using the technique in [9].

Summarizing, we can conclude with the following result.

**Theorem 3.** For almost all linear affine transformations  $L$  the curve  $L(\mathcal{C})$  is in space general position. Furthermore, algorithms for checking space general position are available.

### 3.3. EXAMPLES

In the following we illustrate the previous ideas by two examples. In Example 1 we consider a curve not verifying  $\mathcal{P}_1$ . After a linear change of coordinates, the transformed curve satisfies both  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , though it is non-reduced. In Example 2, we consider a curve whose projection over  $XY$  is not birational. After applying a linear change of coordinates, we get that the transformed curve is reduced, and verifies both  $\mathcal{P}_1$  and  $\mathcal{P}_2$ .

**Example 1.** Let  $\mathcal{C} = V(f, g)$ , where  $f = 18z^2y + 32z + 126xyz - 80z^2 - 194xz^2$  and  $g = 18z^2xy + 32xz + 5y^4 + 8z^2y^2 + 63zyx^2 - 80xz^2 - 97z^2x^2$ . The leading coefficient of  $f$  w.r.t. the variable  $z$  is constant; however, the leading coefficient of  $g$  w.r.t. the variable  $y$  is  $18y - 80 - 194x$ , so  $\mathcal{P}_1$  is not fulfilled. Thus, we apply the change  $x = x - y + z$ ,  $y = y - z$ ,  $z = y + z$ , where for sake of simplicity we use the same notation  $x, y, z$  also for the new variables; we also use the same notation  $f, g$  for the transformed polynomials. Hence, we get:

$$f = -920zy^2 - 1384z^2y - 658z^3 + 32y + 32z - 514xyz - 320xz^2 - 80y^2 - 160zy - 80z^2 - 194xy^2 - 194y^3$$

$$g = 32xy + 32xz + 96zy - 97x^2y^2 - 1742z^3y - 658z^3x - 80xz^2 - 320zy^2 - 400z^2y - 160z^2x^2 - 1640z^2y^2 - 80xy^2 - 663zy^3 - 194xy^3 - 257zyx^2 - 160xyz - 1384z^2xy - 920zy^2x + 64z^2 - 160z^3 - 80y^3 + 32y^2 - 663z^4 - 97y^4$$

Now, the leading coefficients of  $f, g$  w.r.t. the variables  $z$  and  $y$  are  $-658$  and  $-97$ , respectively; therefore, the transformed  $\mathcal{C}$  verifies  $\mathcal{P}_1$ . In order to check  $\mathcal{P}_2$ , we compute the resultant  $\text{Res}_z(f, g)$ ; it has two factors,  $m_1$  and  $m_2$ , that define two components  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , corresponding to two space components  $\tilde{\mathcal{M}}_1$  and  $\tilde{\mathcal{M}}_2$ . More precisely,  $m_1(x, y) = y^4$ , which is not square-free, while  $m_2(x, y)$  is a square-free polynomial of

degree 8. Therefore, we cannot decide the birationality of the projection by means of Theorem 2, so we have to compute  $G_{m_1}(z)$ ,  $G_{m_2}(z)$ . In order to do this, we apply Euclidean algorithm; we need 4 iterations of the algorithm, to finally get that:

$$G_{m_1}(z) = ((216482(4279500800x + 2893987840x^2 - 163840000x^3 + 885145600x^4 + 1746571264))/((-64000x + 345760x^2 + 4482192)^2)) \cdot z$$

$$G_{m_2}(z) = (216482(4279500800x - 6205276160y - 8186446592xy + 2965386976x^2y^2 + 2198658560x^3y - 2520481920yx^2 + 3481765120xy^2 + 3335679872xy^3 + 2893987840x^2 - 163840000x^3 - 11866514560y^3 + 9621182208y^2 + 885145600x^4 + 13270363216y^4 - 2027544120x^3y^2 - 871315200x^4y - 2845839120x^2y^3 - 269982512x^3y^3 + 118852839x^4y^2 + 172739967xy^5 - 401957135x^2y^4 - 5656028880y^5 + 137867476y^6 - 5352103480xy^4 + 1746571264))/((942799xy - 64000x - 3198320y + 345760x^2 + 1934004y^2 + 4482192)^2) \cdot z + (216482y(4279500800x - 6205276160y - 8186446592xy + 2965386976x^2y^2 + 2198658560x^3y - 2520481920yx^2 + 3481765120xy^2 + 4116607872xy^3 + 2893987840x^2 - 163840000x^3 - 10282597760y^3 + 9621182208y^2 + 885145600x^4 + 8985124656y^4 - 2027544120x^3y^2 - 871315200x^4y - 2827887120x^2y^3 + 6625488x^3y^3 + 118852839x^4y^2 + 565084267xy^5 + 66241300x^2y^4 - 3258652280y^5 + 386615616y^6 - 4920853480xy^4 + 1746571264))/((942799xy - 64000x - 3198320y + 345760x^2 + 1934004y^2 + 4482192)^2))$$

Since the expressions above are linear in  $z$ , by Proposition 1 we conclude that the  $\pi_z$  projections of  $\tilde{\mathcal{M}}_1$  and  $\tilde{\mathcal{M}}_2$  are birational. Similar computations for  $\text{Res}_y(f, g)$  provide two factors, one of them not square-free. In both cases the corresponding gcd are linear w.r.t.  $y$ , so we get that  $\mathcal{C}$  satisfies  $\mathcal{P}_2$ , though it is non-reduced.

**Example 2.** Let  $\mathcal{C} = V(f, g)$ , where  $f = x^4 + yx^2 - zx^2 + x^2y^2 + y^3 - zy^2 + z^2x^2 + z^2y - z^3 - x^2 - y + z$  and  $g = x^3 - 2yx^2 + 4zx^2 - 5x^2 + xy^2 - 2y^3 + 4zy^2 - 5y^2 - xz^2 + 2z^2y - 4z^3 + 5z^2$ . The leading coefficient of  $f$  w.r.t.  $z$  and of  $g$  w.r.t.  $y$  are  $-1$ ,  $-2$ , respectively, so  $\mathcal{P}_1$  is fulfilled. The resultant of  $f$  and  $g$  w.r.t. the variable  $z$  is:

$$\text{Res}_z(f, g) = -x^2(x + 2y - 5 + 4x^2)(2y - 1 + x^2)(20y^2 + 20y + 9 - 4xy - 10x + 17x^2)(2x^2 + 2y^2 - 1)^2$$

that is not square-free. Consider  $m_5 = 2x^2 + 2y^2 - 1$ . The first iteration of Euclides algorithm for the computation of  $G_{m_5}(z)$ , gives

$$(1/2y + 1/4x + x^2 - 5/4)z^2 + (-2x^2 - 2y^2 + 1)z - 1/4x^3 + 3/2yx^2 - y + 1/4x^2 - 1/4xy^2 + 3/2y^3 + 5/4y^2 + x^2y^2 + x^4$$

and the second iteration yields a polynomial that is equal to 0 mod  $\langle 2x^2 + 2y^2 - 1 \rangle$ . Thus,  $G_{m_5}(z)$  is the above polynomial, whose degree in  $z$  is 2; furthermore, we check that it is square-free, so by Proposition 1 the projection of the corresponding component is not birational. Therefore, we need to apply a linear change of coordinates; we use the following transformation:  $x = x, y = y - z, z = y + z$ . The resulting  $f$  and  $g$  are:

$$f = x^4 - 2zx^2 - yx^2 + 2z^2x^2 - 4z^3 - 6z^2y + x^2y^2 + 2zyx^2 - 4zy^2 - y^3 - x^2 + y + 2z$$

$$g = x^4 - 2yx^2 - zx^2 + 2x^2y^2 + 2zyx^2 + z^2x^2 - 4y^3 - 6zy^2 - 4z^2y - z^3 - x^2 + 2y + z$$

The leading coefficients of  $f, g$  w.r.t.  $z$  and  $y$  respectively, are  $-1, -1$ . We homogenize them to obtain  $\hat{f}, \hat{g}$ , and then we compute the resultants of  $\hat{f}$  and  $\hat{g}$  w.r.t. the variable  $z$  and w.r.t. the variable  $y$ , respectively; thus, we get:

$$\text{Res}_z(\hat{f}, \hat{g}) = -w(-3yw^2 - x^2 + 2wx^2)(5w^4y^2 - 4w^6 + 2yx^2w^2 + 4x^2w^4 + x^4)(5y^4 + 6x^2y^2 - 4x^2w^2 + 4w^4 - 4w^6 + 2x^2w^4 - 8w^2y^2 + w^8 + 2w^4y^2 + x^4)(5w^2y^2 - w^6 - 6yx^2w + x^2w^2 + 2x^4)$$

$$\text{Res}_y(\hat{f}, \hat{g}) = -w(3zw^2 - 2x^2 + wx^2)(5z^2w^4 + 2x^4 - 6x^2zw^2 + x^2w^4 - w^6)(x^4 + 6x^2z^2 + 2x^2w^2 - 4x^2w^4 - 8z^2w^4 + 5z^4 + 4w^8 + w^4 - 4w^6 + 2w^2z^2)(5w^2z^2 + x^4 + 2x^2zw + 4x^2w^2 - 4w^6)$$

Both of them are square-free, so applying Theorem 2 we get that the corresponding  $\mathcal{C}$  verifies  $\mathcal{P}_1$  and  $\mathcal{P}_2$ .

## 4 Topology of a Reduced Space Curve

In this section, we finally deal with the problem of computing the topology of an space curve  $\mathcal{C}$  that is reduced. Furthermore, in the sequel we assume that  $\mathcal{C}$  is in space general position. Hence, our aim is to compute an “space graph”  $\text{Graph}(\mathcal{C})$  from where one can derive the shape of  $\mathcal{C}$ , the number of singular points of it, the local behavior of  $\mathcal{C}$  around its singular points, the number of connected components of  $\mathcal{C}$ , and which of them are bounded. More precisely, we consider a graph verifying the following properties:

- (1) In case that  $\pi_z(\mathcal{C})$  has no critical points, the only vertices of  $\text{Graph}(\mathcal{C})$  are the points of  $\mathcal{C}$  obtained by intersecting it with two distinct planes of equations  $x = l_1, x = l_2$ . In this case, the obtained points are terminal vertices of  $\text{Graph}(\mathcal{C})$  (i.e., vertices that belong only to one edge of the graph). Note that since  $\pi_z(\mathcal{C})$  is in planar general position, the planes  $x = l_1$  and  $x = l_2$  do not contain any component of  $\mathcal{C}$ .
- (2) In case that  $\pi_z(\mathcal{C})$  has at least one critical point, the non-terminal vertices of  $\text{Graph}(\mathcal{C})$  are the points of  $\mathcal{C}$  whose projections onto the XY plane are critical points of  $\pi_z(\mathcal{C})$ . The terminal vertices of  $\text{Graph}(\mathcal{C})$  are points of  $\mathcal{C}$  which lie on two planes of equations  $x = k_1, x = k_2$ , such that the intersection of each of these two planes with the XY plane is a vertical line placed at the left of the leftmost critical point, and at the right of the rightmost critical point of  $\pi_z(\mathcal{C})$ , respectively.
- (3) Every edge of  $\text{Graph}(\mathcal{C})$  corresponds to a branch of  $\mathcal{C}$  connecting the points of  $\mathcal{C}$  associated to the vertices of  $\text{Graph}(\mathcal{C})$ .

In order to compute  $\text{Graph}(\mathcal{C})$  we may also introduce some additional vertices of  $\text{Graph}(\mathcal{C})$  that may be kept in the final graph, or that may be cleaned once the computation has finished. Also, note that every terminal vertex of  $\text{Graph}(\mathcal{C})$  corresponds to a non-bounded branch of  $\mathcal{C}$ .

The main ideas of the strategy for the computation of  $\text{Graph}(\mathcal{C})$  are exposed in the next subsection, where we show that in general it is necessary to compute two projections of the curve. Then, we will examine the relationship between the critical points of the projections over both planes; and finally, we will present the whole algorithm for computing the space graph. The section ends with a complete example that illustrates the algorithm. Furthermore, along this section, we assume that  $\mathcal{C}$  is a real, reduced, algebraic space curve defined as complete intersection of two surfaces  $f, g \in \mathbb{R}[x, y, z]$ , and that it is placed in space general position.

#### 4.1. THE STRATEGY

In order to compute  $\text{Graph}(\mathcal{C})$  we will use at most two projections. Let us see that, in general, only one projection is not enough (see Figure 2).

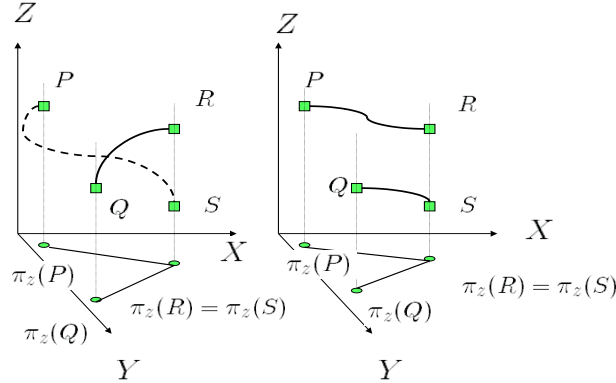


Figure 2: Only one projection is not enough

Figure 2 shows that two different space graphs may give rise to the same planar graph over the  $XY$  plane. In that case, since we derive the spacial graph from the planar graph, we cannot decide how the space graph looks like (for example, in Figure 2, we cannot decide how to connect  $P$ ,  $Q$  and  $R$ ,  $S$ ). Note that this happens when two points of the curve are projected over the *same* point of the  $XY$  plane (the points  $R$  and  $S$  in Figure 2), i.e. when there exists any line  $\tilde{L}$ , parallel to the  $OZ$  axe, which intersects  $\mathcal{C}$  in more than one point. In fact, in that case we cannot construct the edges of  $\text{Graph}(\mathcal{C})$  which enter or leave the intersection points of  $\tilde{L}$  and  $\mathcal{C}$ . However, the problem disappears if, in addition, one looks also at the projection of  $\mathcal{C}$  over the  $XZ$  plane, since the points of  $\tilde{L} \cap \mathcal{C}$  obviously have different  $z$ -coordinates, and so their

projections over the XZ plane are different. More precisely, in that case, we join the space points according to the way their projections over XZ are connected.

**Remark 2:** It may happen that there are also points of  $\mathcal{C}$  with same projection onto the XZ plane of some points of  $\tilde{L} \cap \mathcal{C}$ . However, a random affine transformation will avoid that problem in almost all cases.

The most complicated part in the computation of  $\text{Graph}(\mathcal{C})$  is the determination of the non-terminal vertices. For this purpose, the strategy is the following: we compute the projection  $\pi_z(\mathcal{C})$ , and we determine its critical points. If there is no critical point of the projection, all the vertices of  $\text{Graph}(\mathcal{C})$  are terminal, and they can be easily determined. Otherwise, let  $P_1, \dots, P_r$  be the critical points of  $\pi_z(\mathcal{C})$ , where  $P_i = (x_i, y_i)$ . For each  $P_i$ , the fiber  $\mathcal{F}_{P_i}$  consists of the points  $P_i^{(i_j)} = (x_i, y_i, z_i^{(i_j)})$ , where the  $z_i^{(i_j)}$  are the real roots of  $\gcd(f(x_i, y_i, z), g(x_i, y_i, z))$ . Note that  $\{P_i^{(i_j)}\}$  are the non-terminal vertices of  $\text{Graph}(\mathcal{C})$ . We distinguish two types of critical points of the projection, equivalently two types of vertices of  $\text{Graph}(\pi_z(\mathcal{C}))$ :

- A-type: if the corresponding fiber consists of only one point.
- B-type: if the corresponding fiber consists of more than one point.

It is clear that if the vertices of an edge of the graph associated to  $\pi_z(\mathcal{C})$  are A-type, the edge can be “lifted” to an space edge of  $\text{Graph}(\mathcal{C})$  whose vertices are the elements of their simple fibers. Now, let us consider an edge  $e$  of  $\text{Graph}(\pi_z(\mathcal{C}))$  whose vertices correspond to the points  $P_a = (x_a, y_a)$  and  $P_b = (x_b, y_b)$ , where at least one of them, say  $P_b$ , is B-type. The projections over the XZ plane of the points in  $\mathcal{F}_{P_b}$  are distinct points that lie on the line  $x = x_b$ , and similarly for  $\mathcal{F}_{P_a}$ ; taking into account Remark 2, we can assume that the projections over XZ of the fibers of  $P_a$  and  $P_b$  are A-type. Then, once we know how those projected points are connected in the planar graph of  $\pi_y(\mathcal{C})$ , we join in the same way the corresponding points of  $\mathcal{F}_{P_a}$  and  $\mathcal{F}_{P_b}$ . More precisely, we have the following cases:

- In the case that both lines  $x = x_a$ ,  $x = x_b$  contain critical points of  $\pi_y(\mathcal{C})$ , the computation of  $\text{Graph}(\pi_y(\mathcal{C}))$  shows how to join the points of  $\pi_y(\mathcal{C})$  belonging to them.
- In the case that one of the lines, say  $x = x_b$ , does not contain any critical point of  $\pi_y(\mathcal{C})$ , we force  $x_b$  to play the role of  $\delta_i$  (see section 2), i.e. we use  $x = x_b$  as an auxiliary line that helps to construct  $\text{Graph}(\pi_y(\mathcal{C}))$ .
- If none of the lines  $x = x_a$ ,  $x = x_b$  contain any critical point of  $\pi_y(\mathcal{C})$ , we apply the preceding consideration to both lines. Furthermore, if both lines lie in the same cylinder between two critical points, or they both are placed to the right of the rightmost critical point, or to the left of the leftmost critical point, we



include the points of  $\pi_y(\mathcal{C})$  belonging to both lines as vertices of  $\text{Graph}(\pi_y(\mathcal{C}))$  (i.e., we use not one, but two lines of intermediate points for the construction of the corresponding part of the graph, what does not affect to the correctness of the results).

Note that once the graph of a projection has been constructed, the edges, defined by their vertices, can be stored; therefore, we have a method to decide how to connect the points of the fibers of  $P_a$  and  $P_b$ , that can be automatically performed.

Once the vertices of  $\text{Graph}(\mathcal{C})$  have been computed, we proceed to determine the edges. For this purpose, let us prove that the edges of  $\text{Graph}(\pi_z(\mathcal{C}))$  are in one-to-one correspondence with the edges of the space graph. We need first the following lemma:

**Lemma 3:** Let  $P = (x_p, y_p, z_p)$  be an isolated point of  $\mathcal{C}$ . Then,  $\pi_z(P)$  is a singular point of  $\pi_z(\mathcal{C})$ , and similarly for  $\pi_y$ .

**Proof:** We will proof the result for the projection  $\pi_z$ . Similar ideas can be applied for the projection  $\pi_y$ . Now, if  $\pi_z(P)$  is an isolated point of  $\pi_z(\mathcal{C})$ , then it is a singular point of  $\pi_z(\mathcal{C})$  and the statement is true. So let us assume that  $\pi_z(P)$  is not an isolated point of  $\pi_z(\mathcal{C})$ . Then, there are only two possibilities:

- i. There exists a non-isolated point  $P' \in \mathcal{C}$  such that  $\pi_z(P) = \pi_z(P')$ .
- ii. The case i. does not occur and the space line  $L_p$  defined by the intersection of the planes  $x - x_p = 0$  and  $y - y_p = 0$  is an asymptote of  $\mathcal{C}$ .

Note that since all the components of  $\mathcal{C}$  are 1-dimensional over  $\mathbb{C}$  (see subsection 3.1), through every point of  $\mathcal{C}$  there exists at least one (real or complex) branch of  $\mathcal{C}$  passing through it. In particular, there exists a complex branch of  $\mathcal{C}$  passing through  $P$ . Furthermore, since  $\mathcal{P}_1$  is fulfilled, the projection of any branch of  $\mathcal{C}$  passing through a point  $Q$  of  $\mathcal{C}$  is a branch of  $\pi_z(\mathcal{C})$  that passes through  $\pi_z(Q)$ . Thus, in the case i., the projections of the branches of  $\mathcal{C}$  passing through  $P$  and  $P'$  respectively are both branches of  $\pi_z(\mathcal{C})$  passing through  $\pi_z(P)$ . Furthermore, since  $\mathcal{P}_2$  is satisfied, branches of  $\pi_z(\mathcal{C})$  are distinct. Since there are at least two different branches of  $\pi_z(\mathcal{C})$  passing through  $\pi_z(P)$ , we get that  $\pi_z(P)$  is a singular point of  $\pi_z(\mathcal{C})$ .

In the case ii., we also have two different branches of  $\pi_z(\mathcal{C})$  passing through  $\pi_z(P)$ , which are the projections of the branch of  $\mathcal{C}$  passing through  $P$ , and of the branches of  $\mathcal{C}$  that has the line  $L_p$  as an asymptote, respectively. By the birationality of  $\pi_z|_{\mathcal{C}}$ , the projections over  $XY$  of these two branches of  $\mathcal{C}$  are different. Therefore, reasoning like in the previous case, we get that  $\pi_z(P)$  is a singular point of  $\pi_z(\mathcal{C})$ .  $\square$

Now, we can prove the result about the edges of  $\text{Graph}(\pi_z(\mathcal{C}))$ .

**Theorem 4.** The edges of  $\text{Graph}(\pi_z(\mathcal{C}))$  correspond to the projection of the real branches of  $\mathcal{C}$ .

**Proof:** Let  $(t_0, t_1)$  be any point on an edge  $e$  of  $\text{Graph}(\pi_z(\mathcal{C}))$  that is not a vertex, and let  $(x_0, y_0)$  be the associated point to  $(t_0, t_1)$  on the corresponding real branch of  $\pi_z(\mathcal{C})$ . Then,  $(x_0, y_0)$  is not a critical point of  $\pi_z(\mathcal{C})$  so, in particular, it is not B-type. Therefore, since  $\mathcal{P}_2$  is satisfied, there exists only one point on  $\mathcal{C}$  that projects on  $(x_0, y_0)$ . Thus, the square-free part of  $\gcd(f(x_0, y_0, z), g(x_0, y_0, z))$  is linear, and since gcds do not extend the ground field, it is real. Hence, if  $z_0$  is its root, then  $(x_0, y_0, z_0) \in \mathbb{R}^3$  is the point on  $\mathcal{C}$  that is projected as  $(x_0, y_0)$ . Repeating this process for all the points on the edge that are not vertices, we obtain an infinite number of real space points. Let us see that all of them belong to a same branch of  $\mathcal{C}$ . For that purpose, note that, since the projection  $\pi_z$  is birational over every irreducible component of  $\mathcal{C}$ ,  $\pi_z^{-1}$  is a rational real function and therefore it is continuous at almost all points of the corresponding component of  $\pi_z(\mathcal{C})$ . Now, let  $A$  be the (open in the usual Euclidean topology induced over  $\pi_z(\mathcal{C})$ ) set formed by all the points of  $\pi_z(\mathcal{C})$  that correspond to the points of the edge  $e$ , except for the vertices. Then, we have that  $A$  can be expressed as  $A = A_1 \cup \{Q_1\} \cup \dots \cup \{Q_{p-1}\} \cup A_p$ , where the  $Q_j = (x_{Q_j}, y_{Q_j})$  are points of  $A$  where  $\pi_z^{-1}$  is not defined, and the  $A_j$  are connected subsets of  $A$ . Since  $\pi_z^{-1}$  is continuous over each  $A_j$ , and  $A_j$  is connected, then for all  $j$  we have that  $\pi_z^{-1}(A_j)$  is connected. Therefore, for all  $j$ ,  $\pi_z^{-1}(A_j)$  corresponds to a real branch of  $\mathcal{C}$ . Furthermore, since  $\mathcal{C}$  is an algebraic curve, no  $Q_j$  is B-type, and there is only one branch of  $\mathcal{C}$  passing through  $\pi_z^{-1}(Q_j)$  (otherwise  $Q_j$  would have been a vertex), for each pair  $\pi_z^{-1}(A_j), \pi_z^{-1}(A_{j+1})$  with  $j = 1, \dots, p-1$ , we only have two possibilities: (a)  $\pi_z^{-1}(A_j)$  and  $\pi_z^{-1}(A_{j+1})$  correspond to a same real branch of  $\mathcal{C}$ ; (b) the line defined as the intersection of the planes  $x - x_{Q_j} = 0, y - y_{Q_j} = 0$  is an asymptote of  $\mathcal{C}$ . However, let us see that in our case (b) cannot happen. Indeed, assume that case (b) occurs, and let  $Q_j^* \in \mathcal{C}$  be the real point such that  $\pi_z(Q_j^*) = Q_j$ . Then,  $Q_j^*$  has to be an isolated point of  $\mathcal{C}$ , since otherwise reasoning like in Lemma 3 we would get that there are two branches of  $\pi_z(\mathcal{C})$  passing through  $Q_j$ , so  $Q_j$  would be a singular point of  $\pi_z(\mathcal{C})$ . Nevertheless, if  $Q_j^*$  is an isolated point of  $\mathcal{C}$ , by Lemma 3 we also conclude that  $Q_j$  is a singular point of  $\pi_z(\mathcal{C})$ , and this is a contradiction because  $Q_j$  is not a vertex of the graph associated to  $\pi_z(\mathcal{C})$ . Therefore, the case (b) cannot happen and hence we get that the  $\pi_z^{-1}(A_j)$  all correspond to the same real branch of  $\mathcal{C}$ . Furthermore, since  $\mathcal{C}$  is an algebraic curve, we deduce that the  $Q_j^*$  also belong to this branch, and therefore all the points corresponding to the edge  $e$  belong to the same real branch of  $\mathcal{C}$ .  $\square$

From this theorem one deduces the following corollary.

**Corollary 2.**  $\mathcal{C}$  is real if and only if  $\pi_z(\mathcal{C})$  is real.  $\square$

Observe that Corollary 2 provides a criterion for checking the reality of the space curve  $\mathcal{C}$ . In order to do that one needs to check the reality of a plane algebraic curve; this can be done, for instance, applying results in [15].

Note that in the proof of Theorem 4 we have not used whether  $\mathcal{C}$  is reduced or not; in fact, it is also valid for non-reduced curves. Note also that Theorem 4 is not true in

the case that  $\mathcal{C}$  does not verify  $\mathcal{P}_2$ . For example, consider the following curve:

$$\mathcal{D} \equiv \begin{cases} x^2 + y^2 + z^2 + 5 & = 0 \\ 2x^2 + 2y^2 + z^2 + 1 & = 0 \end{cases}$$

It is easy to check that  $\mathcal{D}$  consists of two circles contained in the complex planes  $z = 3i$ ,  $z = -3i$ , so  $\mathcal{D}$  is not real. Nevertheless, its projection onto  $x, y$  is the curve  $(x^2 + y^2 - 4)^2 = 0$ , which is a real curve.

Furthermore, observe that in the proof of Theorem 4 the point to be lifted  $(x_0, y_0)$  belongs to an edge of the graph of  $\pi_z(\mathcal{C})$ , so it is not an isolated singularity of  $\pi_z(\mathcal{C})$ ; in fact, an isolated vertex of the graph associated to  $\pi_z(\mathcal{C})$  is not necessarily lifted to a real vertex of  $\text{Graph}(\mathcal{C})$  (see the example in section 4.4).

#### 4.2. RELATIONSHIP BETWEEN THE CRITICAL POINTS OF THE PROJECTIONS

Since in the case that  $\pi_z(\mathcal{C})$  has some vertex whose fiber is multiple we need to compute the planar graphs of both  $\pi_z(\mathcal{C})$  and  $\pi_y(\mathcal{C})$ , and since the computation of the critical points is essential for the construction of the corresponding graphs, our aim now is to find some relationships between the critical points of both projections. In general, we may not relate all the critical points of both projections; in fact, in the preceding subsection we have seen that the space points which give rise to B-type critical points of  $\pi_z(\mathcal{C})$  usually give rise to regular points of  $\pi_y(\mathcal{C})$ . Nevertheless, the results of this subsection make easier the computation of the critical points of  $\pi_y(\mathcal{C})$  once the critical points of  $\pi_z(\mathcal{C})$  have been computed.

Along this subsection, we will denote by  $h$  the resultant  $\text{Res}_z(f, g)$ . We start with the following lemma:

**Lemma 4.** Let  $P = (x_0, y_0, z_0) \in \mathcal{C}$  be a non-singular point. If  $U(P), V(P)$  do not simultaneously vanish, then  $\pi_z(\vec{T}(P))$  is a non-zero tangent vector to  $\pi_z(\mathcal{C})$  at  $(x_0, y_0)$ .

**Proof:** Since  $P$  is non-singular,  $\vec{T}(P)$  is a non-zero tangent vector to  $\mathcal{C}$  at  $P$ . Let us assume that  $U(P) \neq 0$  (similarly for  $V(P)$ ). By the Implicit Function Theorem,  $y$  and  $z$  are defined as differentiable functions of  $x$  around  $P$ . Therefore, there exists a local parameterization  $C(x) = (x + x_0, y(x), z(x))$  of  $\mathcal{C}$  around  $P$ , where  $C(0) = P$ , and  $\nabla C(0) = \vec{T}(P)$ . Furthermore, since there exist  $M, N \in \mathbb{R}[x, y, z]$  such that  $h = Mf + Ng$ , we get that  $\pi_z(C(x))$  is also a local parameterization of  $\pi_z(\mathcal{C})$  around  $\pi_z(P)$ . Thus,  $\pi_z(C'(0)) = \pi_z(\vec{T}(P))$  is tangent to  $\pi_z(\mathcal{C})$  at  $\pi_z(C(0)) = (x_0, y_0)$ . Furthermore, since  $C'(x) = (1, y'(x), z'(x))$ , we get that  $\pi_z(C'(0)) = (1, y'(0)) \neq \vec{0}$ .  $\square$

Substituting  $V$  by  $W$  in Lemma 4, we can also prove a similar result for  $\pi_y(\mathcal{C})$ ; abusing language, we will refer to this result as Lemma 4, too.

**Lemma 5.** Let  $P \in \mathcal{C}$  be such that  $U(P) = 0$  and  $V(P) = 0$ , and assume that  $f_z, g_z$  do not simultaneously vanish at  $P$ . Then, the projection  $\pi_z(P)$  is a singular point of  $\pi_z(\mathcal{C})$ .

**Proof:** We have that  $U = f_y g_z - g_y f_z$ , and  $V = -f_x g_z + g_x f_z$ . Thus, if  $U(P) = 0$  and  $V(P) = 0$ , making easy calculations we get that

$$(f_x(P), f_y(P)) \cdot g_z(P) = (g_x(P), g_y(P)) \cdot f_z(P) \quad (*_2)$$

Let  $M, N \in \mathbb{R}[x, y, z]$  such that  $h = Mf + Ng$ . Then, differentiating  $h$  w.r.t. the variables  $x, y, z$  respectively, and evaluating the derivatives in  $P$ , we get that:

$$\begin{aligned} h_x(P) &= M(P)f_x(P) + N(P)g_x(P) \\ h_y(P) &= M(P)f_y(P) + N(P)g_y(P) \\ h_z(P) &= M(P)f_z(P) + N(P)g_z(P) \end{aligned}$$

Note that, since  $h$  does not depend on  $z$ , it holds that  $h_z = 0$ . Therefore,  $M(P)f_z(P) = -N(P)g_z(P)$ . Now, assume w.l.o.g. that  $g_z(P) \neq 0$  (similarly for  $f_z$ ), and multiply by  $M(P)$  the expression  $(*_2)$ ; then, using the last equality, we get that

$$[(f_x(P), f_y(P)) \cdot M(P) + (g_x(P), g_y(P)) \cdot N(P)] \cdot g_z(P) = 0$$

Since  $g_z(P) \neq 0$ , we conclude that  $(f_x(P), f_y(P)) \cdot M(P) + (g_x(P), g_y(P)) \cdot N(P) = 0$ , and consequently  $h_x(P) = 0$ ,  $h_y(P) = 0$ . Since  $h$  does not depend on the variable  $z$ ,  $h_x(P) = h_x(\pi_z(P))$  and similarly for  $h_y$ . Thus,  $h_x, h_y$  vanish at  $\pi_z(P)$ , so it is a singular point of  $\pi_z(\mathcal{C})$ .  $\square$

In the same way, we can prove that if  $U(P) = 0$ ,  $W(P) = 0$  and  $f_y, g_y$  do not simultaneously vanish at  $P$ , then  $\pi_y(P)$  is a singular point of  $\pi_y(\mathcal{C})$ ; abusing language, we will refer to this result as Lemma 5, too. Now, we are ready to relate the critical points of both projections. In order to do this, consider the critical points  $P_{(z)}$  of  $\pi_z(\mathcal{C})$  that verify one of the following conditions:

- I.  $P_{(z)}$  is a ramification point, and there exists  $P \in \mathcal{F}_{P_{(z)}}$  satisfying one of the following properties:
  - a.  $W(P) \neq 0$
  - b.  $W(P) = 0$  and  $f_y, g_y$  do not simultaneously vanish at  $P$
- II.  $P_{(z)}$  is a singular point, and there exists  $P \in \mathcal{F}_{P_{(z)}}$  satisfying one of the following properties:
  - a.  $U(P) = 0$ ,  $W(P) \neq 0$
  - b.  $U(P) = 0$ ,  $W(P) = 0$  and  $f_y, g_y$  do not simultaneously vanish at  $P$

- III.  $P_{(z)}$  is a singular point and there are at least two different branches of  $\mathcal{C}$  passing through a point  $P \in \mathcal{F}_{P_{(z)}}$ .
- IV.  $P_{(z)}$  is a singular point and there exists  $P \in \mathcal{F}_{P_{(z)}}$  such that  $P$  is an isolated point of  $\mathcal{C}$ .

If  $P_{(z)}$  verifies one of the above conditions, we will refer to the corresponding point  $P \in \mathcal{F}_{P_{(z)}}$  as an *associated space point* to  $P_{(z)}$ .

**Theorem 5:** Let  $P_{(z)}$  be a critical point of  $\pi_z(\mathcal{C})$  satisfying one of the conditions I, II, III, IV. Then, the projection over the XZ plane of an associated space point to  $P_{(z)}$  is a critical point of  $\pi_y(\mathcal{C})$ .

**Proof:** Assume that  $P_{(z)}$  verifies I and let  $P \in \mathcal{F}_{P_{(z)}}$  be associated to  $P_{(z)}$ . Since  $P_{(z)}$  is a ramification point, the tangent  $L$  to  $\pi_z(\mathcal{C})$  at  $P_{(z)}$ , is vertical. We distinguish two situations: if  $V(P) \neq 0$ , since  $L$  is vertical, by Lemma 4 we have that  $U(P) = 0$ . Therefore, in the case I.a., since  $W(P) \neq 0$ , by Lemma 4 we get that  $\pi_y(P)$  is a ramification point of  $\pi_y(\mathcal{C})$  and hence it is critical. In the case I.b., since  $W(P) = 0$  and  $f_y, g_y$  do not simultaneously vanish, by Lemma 5 we get that  $\pi_y(P)$  is a singular point of  $\pi_y(\mathcal{C})$  and hence it is critical. On the other hand, if  $V(P) = 0$ , we distinguish two cases depending on whether  $P$  is regular or not. If it is regular, then  $U(P) = 0$ , since by Lemma 4 the fact that  $U(P) \neq 0$  implies that the vector  $(U(P), 0)$  is tangent to  $\pi_z(\mathcal{C})$  at  $P_{(z)}$ , and so the tangent to  $\pi_z(\mathcal{C})$  at  $P_{(z)}$  would not be vertical. Then, if  $P$  is regular we apply the corresponding reasoning that we have done before. If  $P$  is singular, then  $U(P) = V(P) = W(P) = 0$  and the only case that may arise is I.b; in that case, we also reason as before.

In the cases II.a and II.b, the result follows from Lemma 4, and Lemma 5, respectively. In the case III, since  $\pi_y|_{\mathcal{C}}$  is birational, every branch of  $\mathcal{C}$  through  $P$  is projected as a different branch of  $\pi_y(\mathcal{C})$  through  $\pi_y(P)$ . Hence, the number of branches of  $\mathcal{C}$  passing through  $P$  and the number of branches of  $\pi_y(\mathcal{C})$  passing through  $\pi_y(P)$ , are the same. Hence, there are at least two branches of  $\pi_y(\mathcal{C})$  passing through  $\pi_y(P)$ , and  $\pi_y(P)$  is singular. In the case IV, the result follows from Lemma 3.  $\square$

In order to algorithmically analyze the conditions I,II, III and IV, we may proceed as follows. The points verifying the cases I.a, I.b, II.a and II.b can be found by directly checking the conditions over the points belonging to the fibers of the critical points of  $\pi_z(\mathcal{C})$ . In order to detect III and IV, let  $P_{(z)}$  be a critical point of  $\pi_z(\mathcal{C})$  verifying III or IV, let  $r^*$  denote the number of real branches of  $\pi_z(\mathcal{C})$  passing through  $P_{(z)}$ , and let  $q_1, q_2$  denote the number of regular and singular points of  $\mathcal{C}$  in  $\mathcal{F}_{P_{(z)}}$  respectively. Hence,  $q_1 + q_2$  is the cardinal of  $\mathcal{F}_{P_{(z)}}$ . Now, note the following:

- (1) If  $P_{(z)}$  verifies III or IV, there must be at least one singular point  $P \in \mathcal{F}_{P_{(z)}}$ . In fact, if  $\mathcal{F}_{P_{(z)}}$  contains only one singular point, then  $P_{(z)}$  verifies III if and only if  $r^*$  is greater than  $q_1 + q_2$ , and  $P_{(z)}$  verifies IV if and only if  $r^*$  is equal to  $q_1 + q_2 - 1$ .

- (2) If  $q_2 > 1$  and  $r^* - q_1 > q_2$ , then  $P_{(z)}$  satisfies III. The converse is not necessarily true since there may be isolated singular points among the singular points of  $\mathcal{F}_{P_{(z)}}$ .
- (3) If  $q_2 > 1$  and  $r^* < q_1 + q_2$ , then  $P_{(z)}$  satisfies IV. In fact, in this case there may be several isolated points belonging to  $\mathcal{F}_{P_{(z)}}$ .

**Remark 3:** If  $q_2 > 1$ , there may be singular points in  $\mathcal{F}_{P_{(z)}}$  that are not associated to  $P_{(z)}$ . These non-associated singular points are difficult to distinguish from the associated ones. Thus, in this case, instead of determining the associated points to  $P_{(z)}$  and then projecting them over XZ, we directly compute the critical points of  $\pi_y(\mathcal{C})$  belonging to the vertical line  $L_{(z)}$  of the XZ plane defined by the  $x$ -coordinate of  $P_{(z)}$ . Note that if there are several critical points of  $\pi_y(\mathcal{C})$  belonging to  $L_{(z)}$ , then  $\pi_y(\mathcal{C})$  is not in general planar position, so it is necessary to apply a linear transformation to the curve  $\mathcal{C}$ . An alternative approach would be to apply a linear transformation in order to avoid that two singular points of  $\mathcal{C}$  are projected over the same point of  $\pi_z(\mathcal{C})$ .

We can take advantage of Theorem 5 for efficiently computing the critical points of  $\pi_y(\mathcal{C})$ . In order to do this, we first compute the critical points  $P_1, \dots, P_s, P_{s+1}, \dots, P_r$  of  $\pi_z(\mathcal{C})$ , with  $P_i = (\alpha_i, \beta_i)$ , where the  $s$  first points verify some of the conditions I, II, III, IV; then, we compute their fibers. For each  $P_i$  with  $1 \leq i \leq s$ , we decide the points whose projections over the XZ plane are critical points of  $\pi_y(\mathcal{C})$  and then we compute the projections of those points over the XZ plane, or we apply Remark 3. Of course there may be some critical points of  $\pi_y(\mathcal{C})$  that we do not obtain this way. In order to compute them, we first study whether there are other points of  $\pi_y(\mathcal{C})$  with the same  $x$ -coordinates than the critical points of  $\pi_y(\mathcal{C})$  previously determined. If we find some points of this kind, then  $\pi_y(\mathcal{C})$  is not in general planar position, and a linear transformation must be applied. Then, we compute the square-free part  $m(x)$  of the discriminant of  $j(x, z) = \text{Res}_y(f, g)$ , and we approach the roots of

$$\tilde{m}(x) = \frac{m(x)}{(x - \alpha_1) \cdots (x - \alpha_s)}$$

Once the roots of  $\tilde{m}(x)$  have been computed, we can determine the corresponding  $z$ -coordinates. Finally, we check whether  $\pi_y(\mathcal{C})$  is in general planar position, or not. If it is not, a random affine transformation will lead it to general planar position in almost all cases.

#### 4.3. THE ALGORITHM

Putting together the preceding results and ideas, we present the algorithm for the computation of  $\text{Graph}(\mathcal{C})$  in the reduced case:

**Algorithm:** REDTOPSPACE

**Input:**  $\mathcal{C} = V(f, g)$ , with  $f, g \in \mathbb{R}[x, y, z]$ . **Output:**  $\text{Graph}(\mathcal{C})$

- (1) Check  $\mathcal{P}_1$ ; if it is not fulfilled, apply an appropriate linear change of coordinates, and start again.
- (2) Compute  $h(x, y) = \text{Res}_z(f, g)$ ; if it is square-free, by Theorem 2 we get that  $\pi_z|_{\mathcal{C}}$  is birational; then, go to (3). Otherwise, check the birationality of  $\pi_z|_{\mathcal{C}}$  by means of Proposition 1. If it is not birational, then apply a random linear affine transformation, and start again; if it is birational, then the curve is non-reduced, and we refer to section 5.
- (3) Check whether  $\pi_z(\mathcal{C})$  is in general planar position; if it is not, apply an appropriate linear transformation, and start again.
- (4) Compute  $\text{Graph}(\pi_z(\mathcal{C}))$ . By using Corollary 2 one may check the reality of  $\mathcal{C}$ .
- (5) Compute the fibers of the vertices of  $\text{Graph}(\pi_z(\mathcal{C}))$ ; if all of them are simple, then join the points of their fibers according to the way the vertices of  $\text{Graph}(\pi_z(\mathcal{C}))$  are connected, and the algorithm finishes; otherwise, there are some points of  $\pi_z(\mathcal{C})$  with multiple fiber, i.e. B-type, that we denote as  $R_1, \dots, R_u$ , where  $R_k = (\alpha_k, \beta_k)$ .
- (6) Compute  $j(x, y) = \text{Res}_y(f, g)$ . If it is square-free, then  $\mathcal{P}_2$  is fulfilled; if it is not, use Proposition 1 to check if it is due to non-birationality of the corresponding projection, or to the fact that  $\mathcal{C}$  is non-reduced. If the projection  $\pi_y|_{\mathcal{C}}$  is not birational, apply a random linear transformation, and start again. In case that  $\mathcal{C}$  is non-reduced, we refer to section 5.
- (7) Examine the critical points of  $\pi_z(\mathcal{C})$  to identify those ones that fulfill some of the conditions I, II, III, IV; for those points, determine the corresponding critical points of  $\pi_y(\mathcal{C})$ . If we find two critical points of  $\pi_y(\mathcal{C})$  with the same  $x$ -coordinate, then  $\pi_y(\mathcal{C})$  is not in general planar position, so apply an appropriate linear transformation, and start again.
- (8) Determine the square-free part of the discriminant  $m$  of  $j$  w.r.t. the variable  $z$ ; compute the polynomial  $\tilde{m}$  in section 4.2, and approach its roots. Then, check if  $\pi_y(\mathcal{C})$  is in general planar position; if it is not, apply an appropriate linear transformation, and start again.
- (9) Compute  $\text{Graph}(\pi_y(\mathcal{C}))$ ; in order to do this, if the line  $x = \alpha_k$ , where  $\alpha_k$  is the  $x$ -coordinate of one of the  $R_k$ , does not contain any critical point of  $\pi_y(\mathcal{C})$ , then use it as an auxiliary line for constructing  $\text{Graph}(\pi_y(\mathcal{C}))$ .
- (10) Lift the planar graphs. In order to do this, do the following:
  - (10.1) Edges of  $\pi_z(\mathcal{C})$  whose vertices are A-type: they can be directly lifted by connecting the points in the fibers of the vertices according to the way their projections are joined in  $\text{Graph}(\pi_z(\mathcal{C}))$ .

- (10.2) Edges of  $\pi_z(\mathcal{C})$  with at least a B-type vertex: we connect the points belonging to the fibers of the vertices according to the way their projections are connected in  $\text{Graph}(\pi_y(\mathcal{C}))$ .

#### 4.4. A DETAILED EXAMPLE

Let  $\mathcal{C}$  be the space curve defined by the polynomials  $f$  and  $g$ , where

$$f(x, y, z) = 16z - 10xz + 2zy - 3x^2 + xy - 8z^2$$

$$g(x, y, z) = -12x^2y^2 - 192z^3y - 1024z^3x - 34x^3y - 484x^3z - 704xz^2 + 40zy^2 - 64z^2y - 132yx^2 - 1048z^2x^2 - 40z^2y^2 - 808zx^2 - 24xy^2 + 4zy^3 + 2xy^3 + 4y^3 - 44zy^2x - 320z^2xy - 160x + 32y - 128xy - 392x^2 - 288xyz + 384z - 296x^3 + 8y^2 - 256 - 128z^2 - 128z^3 - 85x^4 - 384z^4 + y^4 - 448xz + 128zy - 180zyx^2.$$

Now, we will show how the different steps of REDTOPSPACE work in this example.

- (1) The leading coefficient of  $f$  w.r.t.  $z$  is  $-8$ , and the leading coefficient of  $g$  w.r.t.  $y$  is  $1$ , so  $\mathcal{P}_1$  holds.

- (2) The computation of  $\text{Res}_z(f, g)$  is:

$$\begin{aligned} \text{Res}_z(f, g) = & -939524096x + 4294967296y + 5695864832xy + 1146093568x^2y^2 + \\ & 1615855616x^3y + 4555014144yx^2 + 4951375872xy^2 + 1710227456xy^3 + 5813305344x^2 + \\ & 2388656128x^3 - 301989888y^3 + 486539264y^2 - 353370112x^4 - 167772160y^4 + \\ & 427032576x^3y^2 - 172097536x^4y + 325189632xy^4 + 60030976x^2y^3 + 52953088x^3y^3 - \\ & 27885568x^4y^2 + 36962304xy^5 - 15368192x^2y^4 + 2359296x^5y + 16384x^4y^4 - 49152x^5y^3 - \\ & 1916928x^2y^5 - 1490944x^4y^3 - 40960x^2y^6 + 2736128x^3y^4 + 16384x^3y^5 + 40960x^6y^2 - \\ & 245760x^5y^2 - 16384x^7y + 409600x^6y + 2277376xy^6 + 11141120x^5 - 47316992y^5 + \\ & 49152xy^7 + 819200x^6 - 49152x^7 - 7897088y^6 - 671744y^7 + 4096x^8 - 20480y^8 + 7516192768 \end{aligned}$$

Since its degree is 8, which is the product of the degrees of  $f$  and  $g$ , and it is square-free, we get that  $\text{Res}_z(\hat{f}, \hat{g})$  must also be square-free. Therefore, the projection of  $\mathcal{C}$  over the XY plane is birational.

- (3) The leading coefficient of  $\text{Res}_z(f, g)$  is 4096, so in particular it has no real root. The real roots of the discriminant w.r.t.  $y$  of  $\text{Res}_z(f, g)$  are:

$$\begin{aligned} & -24.6054617087, -9.54866281743, -7.68837448497, -4.81632514469, -4.56947500307, \\ & -2.47285430121, 0.0949595194882, 1.06094450089, 1.55593088693, 4.33782933304 \end{aligned}$$

Each of these values defines a vertical line of the type  $x = \alpha_i$ ; we check that in all those lines there is only one critical point of  $\pi_z(\mathcal{C})$ , so it is in general planar position. More precisely, the coordinates of the critical points are:  $A = (-24.6054617087, -14.63311299)$ ,  $B = (-9.54866281743, -5.939052811)$ ,  $C = (-7.68837448497, -7.668006658)$ ,  $D = (-4.81632514469, -5.748894895)$ ,



$E = (-4.56947500307, -10.79103826)$ ,  $F = (-2.47285430121, -0.9586190410)$ ,  
 $G = (0.0949595194882, -5.437764938)$ ,  $H = (1.06094450089, -5.430312902)$ ,  $I =$   
 $(1.55593088693, -3.037881273)$ ,  $J = (4.33782933304, -3.215005470)$ .

(4) In order to compute  $\text{Graph}(\pi_z(\mathcal{C}))$ , we determine the real points of  $\pi_z(\mathcal{C})$  belonging to the vertical lines of the type  $x = \alpha_i$ , where the  $\alpha_i$  are the  $x$ -coordinates of the critical points; for example, the real points belonging to the vertical line defined by  $C$ , are

$c_1 = (-7.68837448497, -13.29062946)$ ,  $c_2 = C$ ,  $c_3 = (-7.68837448497, -5.609966815)$

and the corresponding points for  $D$  are

$d_1 = (-4.81632514469, -11.02430721)$ ,  $d_2 = (-4.81632514469, -10.52457564)$ ,  $d_3 = D$

Finally, we use auxiliary lines to compute the graph, that is shown in Figure 3. Here, we see that the points  $H, I, J$  are isolated singular points of  $\pi_z(\mathcal{C})$ . Furthermore, by Corollary 2 we know that  $\mathcal{C}$  is real.

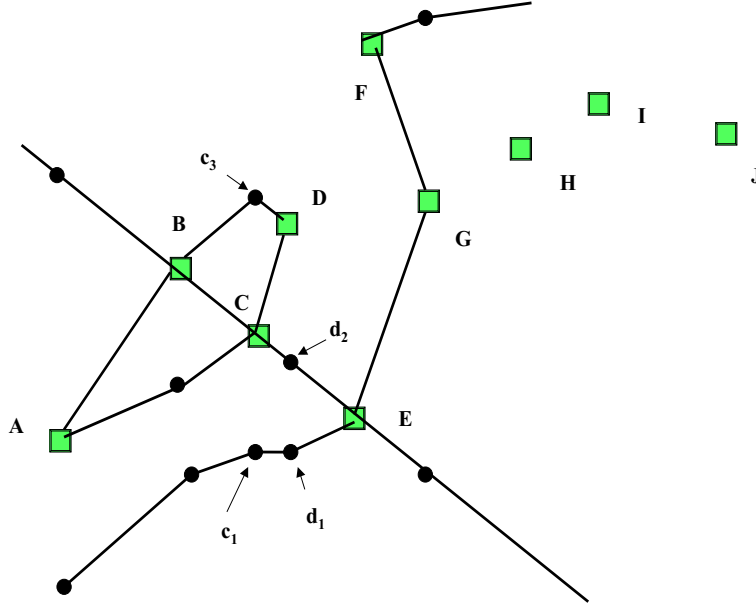


Figure 3: Graph of  $\pi_z(\mathcal{C})$

(5) Now, we proceed to compute the fibers of the vertices of  $\text{Graph}(\pi_z(\mathcal{C}))$ . For example, for the point  $C$ , the real roots of

$$\gcd(f(-7.68837448497, -7.668006658, z), g(-7.68837448497, -7.668006658, z))$$

are 1.898263633 and 7.795202813, so  $C$  is a B-type vertex of  $\text{Graph}(\pi_z(\mathcal{C}))$  whose fiber consists of the points  $C^1 = (-7.68837448497, -7.668006658, 1.898263633)$ , and  $C^2 =$

$(-7.68837448497, -7.668006658, 7.795202813)$ ; in the same way, we get:

$$\begin{aligned}\mathcal{F}_{c_1} &= \{(-7.68837448497, -13.290629466.932848797)\} \\ \mathcal{F}_{c_3} &= \{(-7.68837448497, -5.609966815, 8.149554250)\} \\ \mathcal{F}_{d_1} &= \{(-4.81632514469, -11.02430721, 4.838179808)\} \\ \mathcal{F}_{d_2} &= \{(-4.81632514469, -10.52457564, 0.4814006820)\} \\ \mathcal{F}_D &= \{(-4.81632514469, -5.748894895, 5.657341036)\}\end{aligned}$$

and the fibers of the rest of the points defining edges of  $\text{Graph}(\pi_z(\mathcal{C}))$ ; the points  $B$  and  $E$  are also B-type, and for the points  $H, I, J$  we find fibers consisting of two points whose  $z$ -coordinates are conjugate complex numbers. Hence, these points are not real, and consequently they do not appear in  $\text{Graph}(\mathcal{C})$ , and their projections over XZ do not appear in so in  $\text{Graph}(\pi_y(\mathcal{C}))$ .

(6) We compute  $\text{Res}_y(f, g)$ , which is a square-free polynomial of degree 8, so the projection over the XZ plane is birational, and  $\mathcal{P}_2$  is fulfilled.

(7) We examine the non-isolated critical points of  $\pi_z(\mathcal{C})$ . Their fibers consist of points where  $W$  does not vanish, so all of them give rise to ramification points of  $\pi_y(\mathcal{C})$ . More precisely, these points are  $Az = (-24.6054617087, 19.99467799)$ ,  $Dz = (-4.81632514469, 5.65734103)$ ,  $Fz = (-2.47285430121, 0.45409812)$ ,  $Gz = (0.0949595194882, 0.24840511)$ , that correspond to the projections over the XZ plane of the points in the fibers of  $A, D, F$  and  $G$ , respectively. Furthermore, we check that there are no other critical points of  $\pi_y(\mathcal{C})$  over the vertical lines of XZ defined by  $Az, Dz, Fz$  and  $Gz$ , respectively.

(8) In order to determine the rest of the critical points of  $\pi_y(\mathcal{C})$  we need to approximate the roots of

$$\tilde{m} = \frac{m}{(x - x_A)(x - x_D)(x - x_F)(x - x_G)}$$

where  $x_A, x_D, x_F, x_G$  are the  $x$ -coordinates of  $A, D, F, G$  respectively; in order to get good results, we need to compute these numbers with a bigger precision, in this case 30 digits. The resulting  $\tilde{m}$  has only one real root, which is 0.

The leading coefficient of  $\text{Res}_y(f, g)$  is constant, and we check that on every vertical line there exists only one critical point of  $\pi_y(\mathcal{C})$ , so  $\pi_y(\mathcal{C})$  is in general planar position; the critical points of  $\pi_y(\mathcal{C})$  are  $Az, Dz, Fz, Gz$ , and  $K = (0, 0)$ , which is a B-type point of  $\pi_y(\mathcal{C})$ .

(9) In order to compute  $\text{Graph}(\pi_y(\mathcal{C}))$ , note that the  $x$ -coordinates of  $B$  and  $C$ , that are  $-9.54866281743, -7.68837448497$  respectively, lie between the  $x$ -coordinates of  $Az$  and  $Dz$ ; thus, we use both lines  $x = -9.54866281743, x = -7.68837448497$  as auxiliary lines for constructing the graph between  $Az$  and  $Dz$ ; furthermore, the  $x$ -coordinate of  $E$ , that is equal to  $-4.56947500307$ , lies between the  $x$ -coordinates of  $Dz$  and  $Fz$ , so the line  $x = -4.56947500307$  is used as an auxiliary line to construct that part of the

graph. Now, we can compute  $\text{Graph}(\pi_y(\mathcal{C}))$  (see Figure 4). There, we can locate the points  $Bz^1, Bz^2$ , which are the projections over XZ of the points in the fiber of  $B$ , and similarly for the points  $C$  and  $E$ .

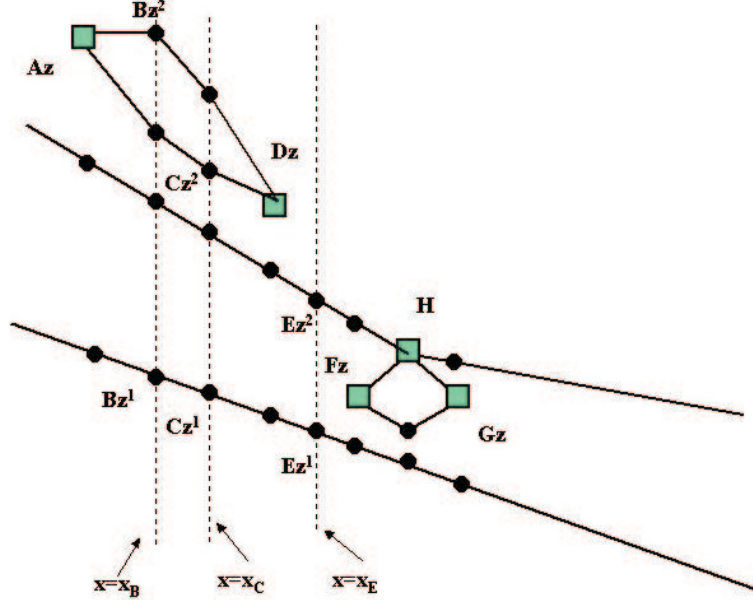


Figure 4: Graph of  $\pi_y(\mathcal{C})$

(10) The edges of  $\text{Graph}(\pi_z(\mathcal{C}))$  whose vertices are A-type can be directly lifted by connecting the points in their fibers; for example, the edge of vertices  $F$  and  $G$  is lifted to an edge connecting the space points  $(-2.47285430121, -0.9586190410, 0.45409812)$  and  $(0.0949595194882, -5.437764938, 0.2484051100)$ . For the edges of  $\text{Graph}(\pi_z(\mathcal{C}))$  with at least one B-type vertex, we use  $\text{Graph}(\pi_y(\mathcal{C}))$ ; for example, consider the edge defined by  $C$  and  $D$ ; the fiber of  $C$  consists of two points  $C^1, C^2$ , so from  $\text{Graph}(\pi_z(\mathcal{C}))$  we do not know which of them must be connected to the point  $D^1$  in the fiber of  $D$ ; then, we look at  $\text{Graph}(\pi_y(\mathcal{C}))$ , where we find an edge whose vertices are the projections of  $C^2$  and  $D^1$  over the XZ plane, so  $C^2$  and  $D^1$  must be connected, while  $C^1$  is connected to the point in the fiber of  $d_2$ . Similarly for the rest of the edges.

Finally,  $\text{Graph}(\mathcal{C})$  is computed (see Figure 5). Thus, we get that  $\mathcal{C}$  consists of 3 connected components, 2 of which are non-bounded (one of them a line).

## 5 Remarks on the non-reduced case

Now, we will consider some aspects that must be taken into account for the non-reduced case. First of all, note that in the case when  $\mathcal{C}$  is non-reduced but both  $\text{Res}_z(f, g)$ ,

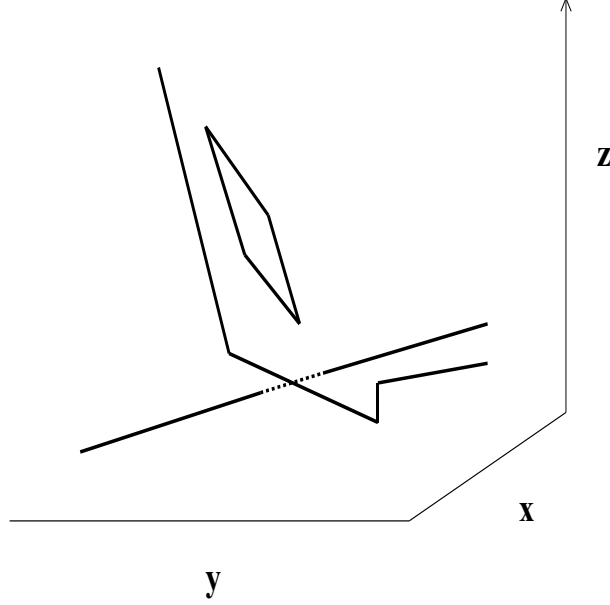


Figure 5: Graph of  $\mathcal{C}$

$\text{Res}_y(f, g)$  are square-free, the algorithm REDTOPSPACE also works. Indeed, in that case, by Theorem 1 we would get that  $\text{Res}_x(f, g)$  is not square-free, but  $\text{Res}_x(f, g)$  is not used in the algorithm REDTOPSPACE.

Thus, assume that any of the resultants  $\text{Res}_z(f, g)$ ,  $\text{Res}_y(f, g)$  is not square-free. In that case, any of the curves  $\pi_z(\mathcal{C})$ ,  $\pi_y(\mathcal{C})$  has multiple components. Then, since the graph of an algebraic plane curve is equal to the graph of its square-free part (see section 2), we compute the square-free part of  $\text{Res}_z(f, g)$  (resp.  $\text{Res}_y(f, g)$ ), and we force the corresponding plane curve to play the role of  $\pi_z(\mathcal{C})$  (resp.  $\pi_y(\mathcal{C})$ ).

On the other hand, note that the results of Theorem 5, that are used in the steps (7) and (8) of REDTOPSPACE for the computation of the critical points of  $\pi_y(\mathcal{C})$ , are only useful over the components of  $\mathcal{C}$  whose projections over XY and XZ have finitely many critical points, i.e. both projections are square-free. Hence, in order to use Theorem 5 in the non-reduced case, we need to previously determine this kind of components of  $\mathcal{C}$ . In order to do this, we can proceed as follows: in each square-free component of  $\pi_z(\mathcal{C})$ , we take a point  $Q$  that is not critical, and such that  $\pi_y(Q^*)$ , where  $Q^*$  is the point in the fiber of  $Q$ , is not a B-type point of  $\pi_y(\mathcal{C})$ . Note that since  $Q$  is not a critical point of  $\pi_z(\mathcal{C})$ , there is only one real branch of  $\pi_z(\mathcal{C})$  passing through it, and so there is also only one branch of  $\mathcal{C}$  passing through  $Q^*$ ; then, since  $\pi_y(Q^*)$  is not a B-type point of  $\pi_y(\mathcal{C})$ , there exists only one component of  $\pi_y(\mathcal{C})$  that contains  $\pi_y(Q^*)$ . Finally, we check if this component of  $\pi_y(\mathcal{C})$  is multiple, or not. If it is not, then we

can use Theorem 5 over the corresponding component of  $\mathcal{C}$ . In order to determine the critical points of  $\pi_y(\mathcal{C})$  that cannot be computed this way, we proceed as in subsection 4.2. An alternative approach to the use of Theorem 5 for the non-reduced case, is to compute the critical points of the square-free part of  $\pi_y(\mathcal{C})$  by directly approximating the roots of the square-free part of the discriminant of  $\text{Res}_y(f, g)$ , i.e. to proceed as in the computation of the critical points of the square-free part of  $\pi_z(\mathcal{C})$ .

Thus, bearing in mind the above considerations, and the ideas applied in the construction of the algorithm REDTOPSPACE, an algorithm that also works for the non-reduced case might be derived.

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