

# Different notions of conuity and intensional models for $\lambda$ -calculus

## (Brief Content of the report)

Alexandre Lyaletsky (Jr.)

Faculty of Cybernetics, Kyiv National Taras Shevchenko University,  
2, Glushkov avenue, building 6, 03680 Kyiv, Ukraine  
e-mail: aal@mail.ru

**I.** The functional paradigm as a basis for mathematics, logic, and computer science: the role of  $\lambda$ -calculus. The self-applicability principle.

**$\lambda$ -calculus:**

**Axioms:**

$$\begin{aligned}M &= M, \\(\lambda x.M)N &= M[x \rightarrow N], \\ \lambda x.M &= \lambda y.M[x \rightarrow y]\end{aligned}$$

**Rules:**

$$\frac{M = N}{N = M}$$

$$\frac{M = N, N = L}{N = L}$$

$$\frac{M = N}{\lambda x.M = \lambda x.N}$$

$$\frac{M = N}{MZ = NZ}$$

$$\frac{M = N}{ZM = ZN}$$

**II.** Models of  $\lambda$ . D.Scott's topological ("informational") and category-based models. The non-existence of a "naive" model: the Cantor's theorem. Discussion on the groundness axiom. The potential-informative natural interpretation of complete partial order relations between elements of topological models.

**Theorem** (D.Scott). Given function  $f : \langle A_1, \leq_1 \rangle \rightarrow \langle A_2, \leq_2 \rangle$ , where  $\langle A_i, \leq_i \rangle$  are complete partially ordered sets. Then  $f$  is Scott-continuous if, and only if, for every directed and non-empty subset  $X$  of  $A_1$ , the equality  $f(\bigvee X) = \bigvee f(X)$  holds.

**III.** Types (i.e. levels) of abstraction and the concept of positive (i.e. useful) and negative (i.e. useless) information: the pragmatism and goal-drivenness. Tasks - as objects, and reductions of describable classes of tasks with the smooth abstraction type property - as continuous functions and morphisms.

Let  $\langle A, \leq \rangle$  be any complete and co-complete poset (partially ordered set),  $P$  a generalized sequence of its elements (i.e. a sequence with directed set of indexes, not necessarily equal to the set of naturals). Then, in the case if  $P$  is increasing (decreasing), we define its limit as the least upper bound (the greatest lower bound) of its elements. In the general case, we put limit of  $P$  equal to such an element (if it exists) of  $A$  that is the limit of every cofinal monotone subsequence of  $P$  and the set of all such sequences is non-empty (otherwise, by definition,  $P$  is divergent).

For shortness, here we give a new definition of continuity in the case if  $f$  is an operation. An operation  $f$  on  $A$  is said to be  $\leq$ -continuous if, and only if, it preserves the limits of all the convergent sequences on  $A$ .

A "pictorial" comparison of the new notion of continuity (convergence) with (*o*)-continuity and Scott's continuity (convergence).

#### IV. Results

**Theorem 1.** Let  $\langle A_1, \leq_1 \rangle$  and  $\langle A_2, \leq_2 \rangle$  be complete and co-complete posets,  $f : A_1 \rightarrow A_2$  be a function. Then  $f$  is continuous (in the sense the new notion) if, and only if, the following conditions are satisfied:

- (i) if  $f(\lim P)$  is defined then  $\lim f(P)$  is defined, for every sequence  $P$  on  $A_1$ ;
- (ii) the restriction of the function  $f$  on each set, on which  $f$  is monotone, is continuous w.r.t. a corresponding Scott topology.

**Theorem 2.** It is not true that for every poset  $\langle A, \leq \rangle$ , there exists such a topology  $T$ , that the set of all  $\leq$ -continuous operations on  $A$  is equal to the set of all the  $T$ -continuous operations, and vice versa.

**Theorem 3.** Investigations of  $T$ -convergence and of  $T$ -continuity can be formally reduced to investigations of  $\leq$ -convergence and  $\leq$ -continuity, and vice versa.