

# Solving Linear Constraints over Real

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# Talk outline

- ❖ Introduction
- ❖ Fourier-Motzkin elimination
- ❖ The main algorithm scheme
- ❖ Equalities elimination
- ❖ Redundancies elimination, simplification
- ❖ An example
- ❖ Conclusions

# Application areas

- Hardware and software verification
- Program analysis
- Compiler optimization
- Planning, scheduling

# Existing techniques

- Fourier's methods
  - Omega Test
- Special cases
  - Shostk's loop residue algorithm [4]
  - Bounding boxes, Bounding differences, Octagons
    - Nelson-Open Decision Procedure[5]
- Modified Simplex Methods [6]
- The algorithm of Motzkin-Chernikova-Le Verge (the PPL[3] library)

# Inputs

- Formula type:
  - Linear equality, inequality over Real (Rational)
- The input formula consists of
  - Boolean connectivity: Or( $\vee$ ), And ( $\wedge$ ), Not( $\sim, \neg$ )
  - Quantifies: Forall( $\forall$ ), Exists( $\exists$ )
  - Operators: plus (+), minus(-), multiplication on number ()
  - Predicates: <, <=, >=, >, =

# Projection of polytope

- Suppose we have a polytope

$$S = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

- We would like to construct the projection onto

$$\{x \in \mathbb{R}^n \mid x_1 = 0\}$$

- Call this projection  $P(S)$

# Projection

- We would like to find inequalities that define the projection  $P(S)$

$$P(S) = \{x_2 \mid \text{there exists } x_1 \text{ such that } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S\}$$

Some other way to say this:

- We would like to find *valid inequalities* do not depend on  $x_1$
- We would like to perform *quantifier elimination* to remove *there exists* and find a *basic semi algebraic*, i.e. defined by conjunction of polynomial inequalities, representation of  $P(S)$

# Fourier-Motzkin elimination

- **This procedure was invented by Fourier (1826) and rediscovered by Dines (1918) and Motzkin (1936)**
- **Similar to Gaussian elimination (1800)**

# Fourier-Motzkin elimination

- Let the source system  $Ax \leq b$  where

$$A = \begin{bmatrix} a_1 \\ \dots \\ a_m \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \dots \\ b_m \end{bmatrix}$$

- We can generate equalities of the form

$$(\lambda_1 a_1 + \dots + \lambda_m a_m) x \leq \lambda_1 b_1 + \dots + \lambda_m b_m$$

- The idea is to combine pairs of inequalities that cancel  $x_1$ . Since  $\lambda_i \geq 0$  member of each pair need opposite signed coefficients of  $x_1$

# Fourier-Motzkin theorem

- Take all pairs of inequalities with opposite coefficients of  $x_1$ , and for each generate a new valid inequality that eliminates  $x_1$
- Also take all inequalities from the original set which do not depend on  $x_1$

This collection of inequalities defines projection of  $S$  onto  $x_1 = 0$

# Algorithm complexity

- **Eliminating an existential over  $n$  constraints we may introduce  $n^2/4$  new constraints**
- **With  $k$  quantifiers to eliminate, we might end with**

$$n^{2^k} / 4^k$$

# The Solver main algorithm scheme

1. Normalize the formula tree
2. repeat {
  - 💣 Convert to DNF, handle ***disequalities***
  - Eliminate bounded variables from ***equalities***
  - 💣 Eliminate bounded variables from ***inequalities***} until the formula has bounded variables
3. Simplify result
4. End.

*Note. The 'bomb' marks operations which has exponential worst case complexity*

# Goals

- Handle full theory over real
- To be efficient enough for handling the real-life verification problems

But how to do that?

- Use efficient redundant elimination techniques
- Find case when we can avoid generics but apply faster special algorithms

# Tree transformation rules

$$\forall v : f \rightarrow \neg \exists v : \neg f$$

$$\exists v : (f_1 \vee \dots \vee f_n) \rightarrow \exists v : f_1 \vee \dots \vee \exists v : f_n$$

$$\neg (f_1 \vee \dots \vee f_n) \rightarrow \neg f_1 \wedge \dots \wedge \neg f_n$$

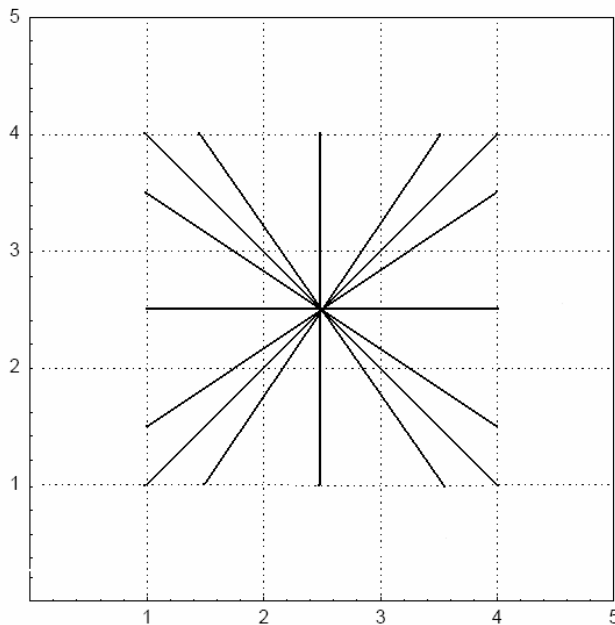
$$\neg \neg f \rightarrow f$$

# Equalities elimination

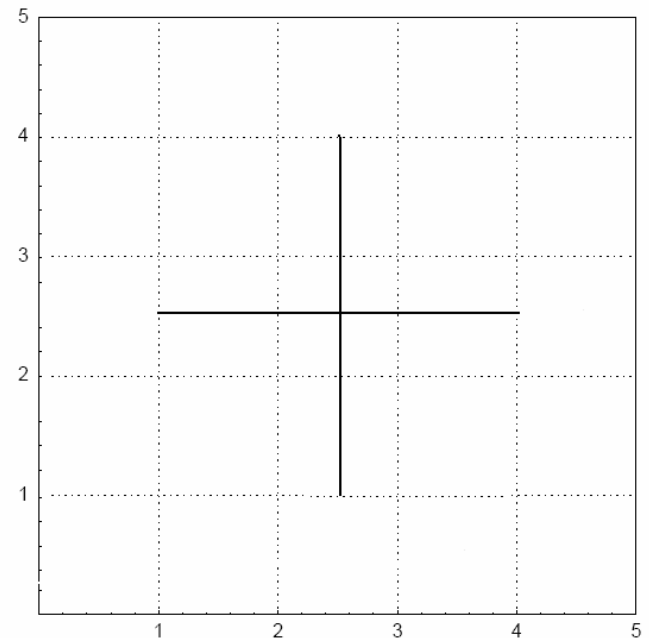
- If input system has both equalities and inequalities equalities are eliminated first
- We use Gaussian elimination procedure for the system of equalities and inequalities constraints[7]

# Equalities Simplification

- In practice systems of equalities may be highly redundant



Redundant system



Simplified system

# Elimination of the redundant equalities

- Redundant elimination similar to solving
- Available efficient solvers in different libraries for system where number of equalities equal to number of variables...  
But how to solve the redundant systems using the solvers?

# Redundant equalities elimination algorithm

Let  $M$  - number of equalities,  $N$  – number of variables

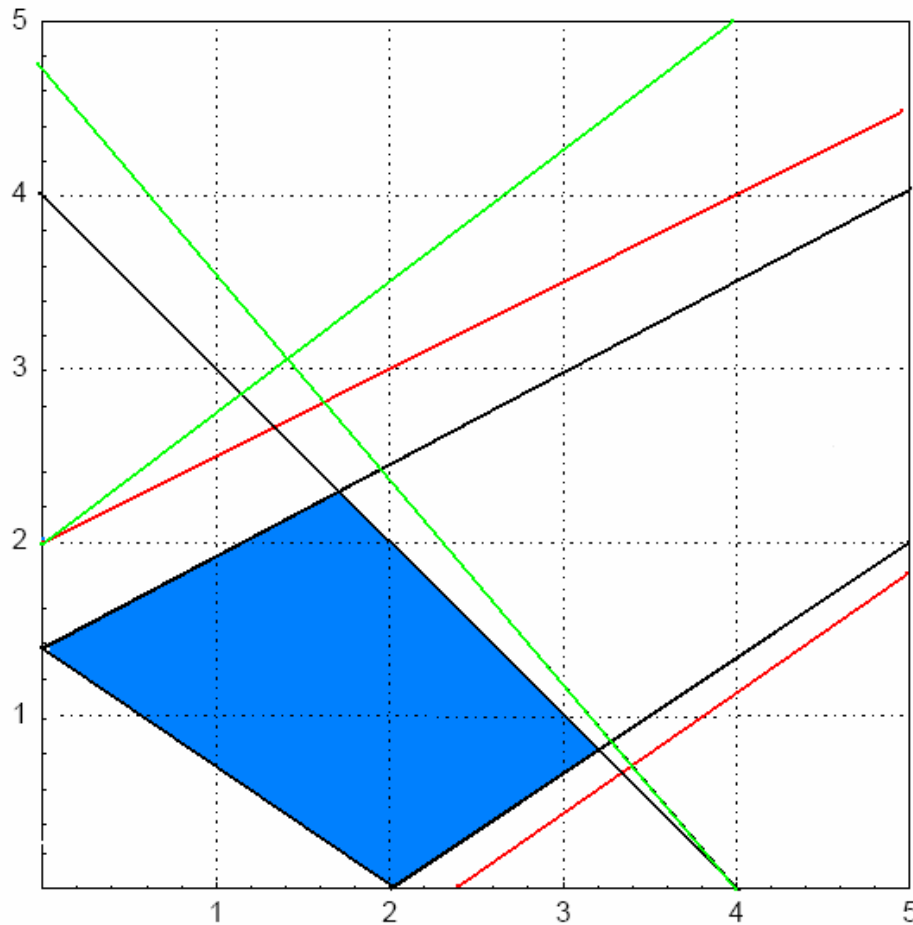
If  $M > N$

1. Normalize by removing of the trivially same equalities
2. Solve first  $N$  equalities
3. If result found propagate it to the rest of equalities
4. If the rest is satisfiable then replace all equalities by the founded solution

If  $M < N$  the system cannot be solved and simplified.

*What if there are several separate systems which depend on the different variables?*

# The Redundant Facets



-  Solutions
-  Redundant 1
-  Redundant 2

# Elimination of the Redundant 1 (parallel) facets

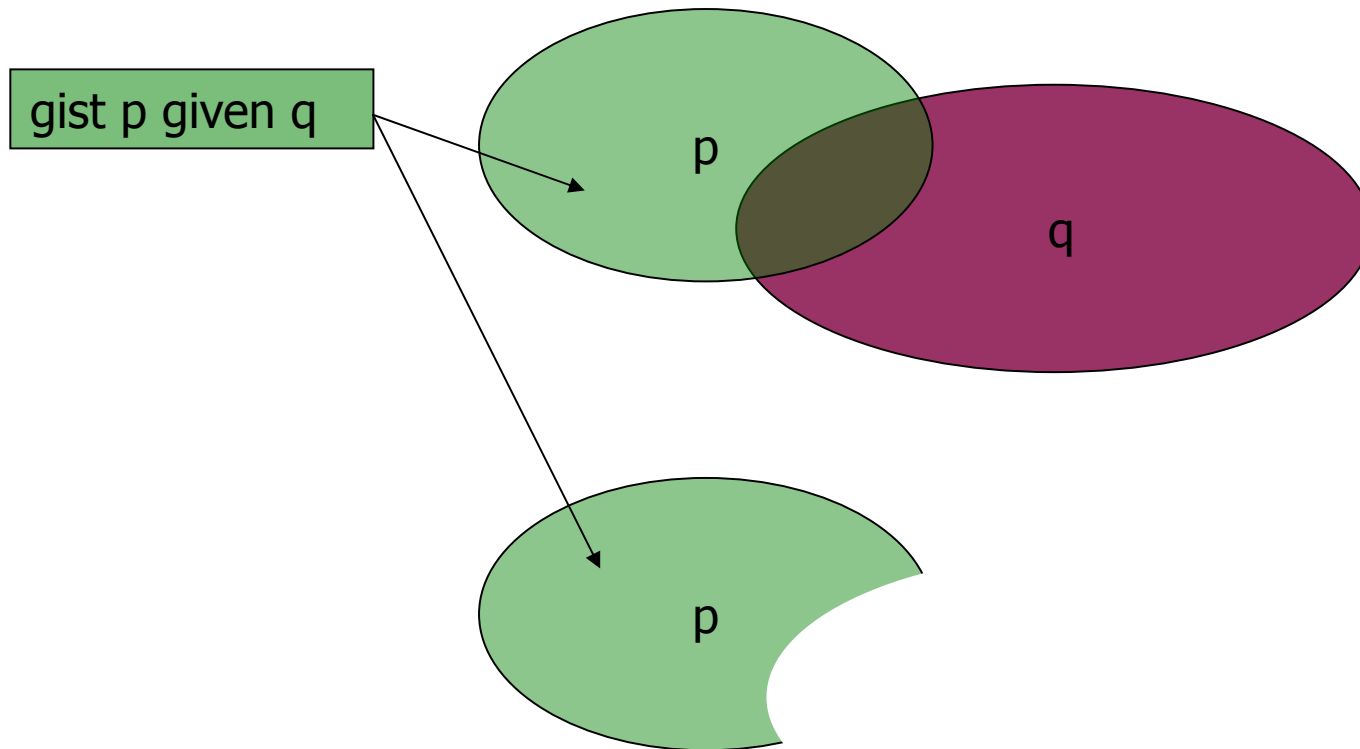
Let  $A, -A$  positive and negative system of inequalities, equalities respectively, which all are parallel between each other

1. Normalize the system
2. Find the upper bound inequality  $A_i \geq \text{MAX}(b_i)$
3. Find the lower bound inequality  $-A_j \geq \text{MIN}(b_j)$
4. Remove all other inequalities from  $A$  and  $-A$
5. Simplify and check for consistency including equalities which may be in  $A$  and  $-A$ :  
if  $-A_i \geq -b_2$  and  $A_i \geq b_1 \Leftrightarrow b_1 \leq A_i \leq b_2 \rightarrow$   
consistent if  $(b_1 \leq b_2)$   
if  $-A_i \geq -b$  and  $A_i \geq b \rightarrow A_i = b$   
...and similar
7. Do the same simplification for all group of the parallel constraint
8. Return the simplified system or inconsistency if found

# Elimination of redundant constraints using gist

- The gist operation was introduced by W. Pugh, D. Wonnacott in [1]
- *gist p given q* – conjunction of constraint a minimal subset of constraints of  $p$  such that  $((\text{gist } p \text{ given } q) \wedge q) = (p \wedge q)$
- Intuitively *gist p given q* the new information contained in  $p$ , given that we already know  $q$
- Gist always has not more constraints than an initial system of constraints

# Gist p given q



# Computing Gist Algorithm

- If  $q$  is satisfiable, we could compute gist  $p$  given  $q$  as follows:

*gist p given q =*

*if p=True return True*

*else let c be constraint in p*

*if  $p^c_{\sim(c)} \wedge q$  is satisfiable,*

*then return  $c \wedge (\text{gist } p^c_{True} \text{ given } (q \wedge c))$*

*else return gist  $p^c_{True}$  given  $q$*

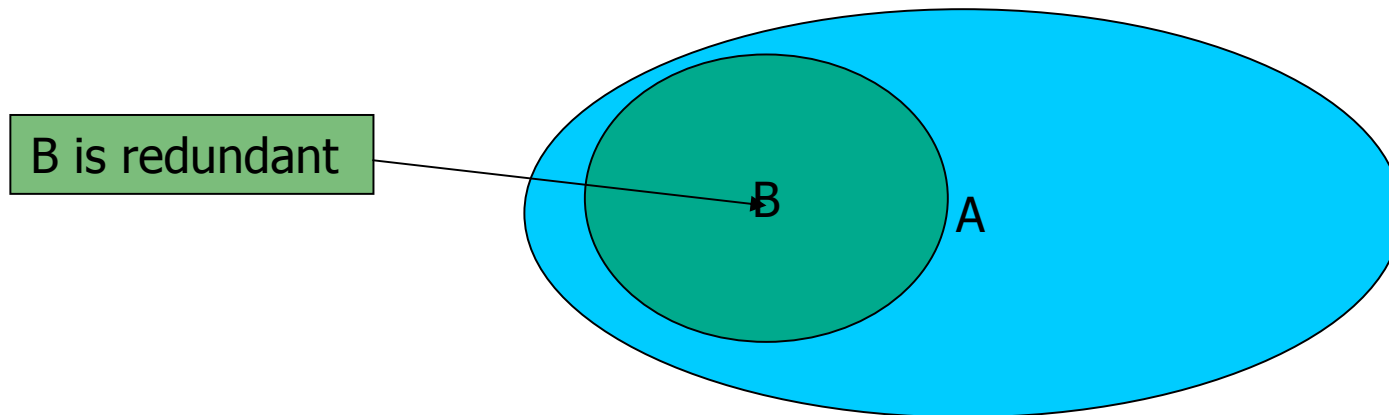
Where  $p^{\text{oldc}}_{\text{newc}}$  is  $p$  with *oldc* replaced by *newc*

# Checking tautologies with Gist

■ *Gist p given q = True*  $\Leftrightarrow q = (p \wedge q) \Leftrightarrow (q \Rightarrow p)$

■ We can simplify disjunction of conjuncts:

*Let  $A \vee B \Leftrightarrow A$  if  $A \Rightarrow B$*



# Using gist to simplify negations

- $A \wedge (\text{gist } B \text{ given } A) \equiv (A \wedge B)$

$$A \wedge \neg B \equiv A \wedge \neg(A \wedge B)$$

$$\equiv A \wedge \neg(A \wedge (\text{gist } B \text{ given } A))$$

$$\equiv A \wedge \neg (\text{gist } B \text{ given } A)$$

- $A \wedge \neg(\text{gist } B \text{ given } A)$  often have fewer clauses than  $A \wedge \neg B$ , since  $\text{gist } B \text{ given } A$  often have fewer clauses than  $B$  [2]

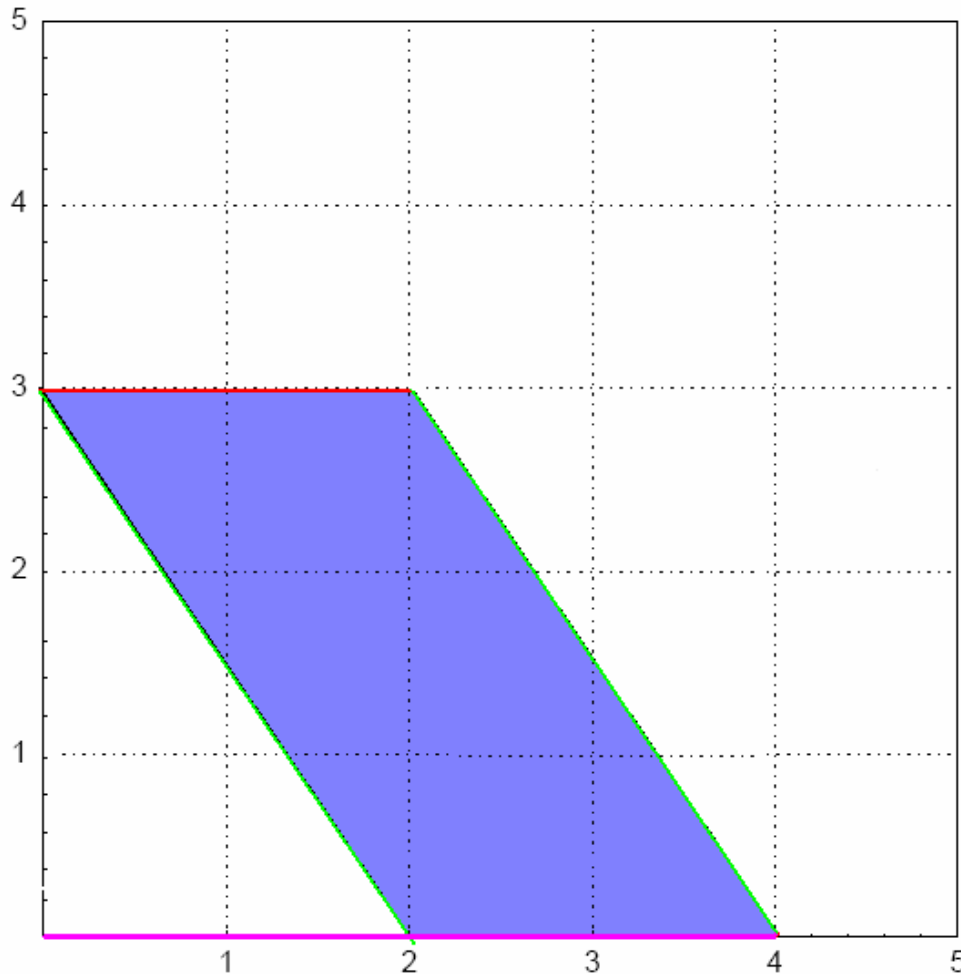
# Disequalities handling

- Generic disequality transformation

$$(x_1 \neq a_1) \Leftrightarrow (a_1 < x_1) \vee (x_1 < a_1)$$

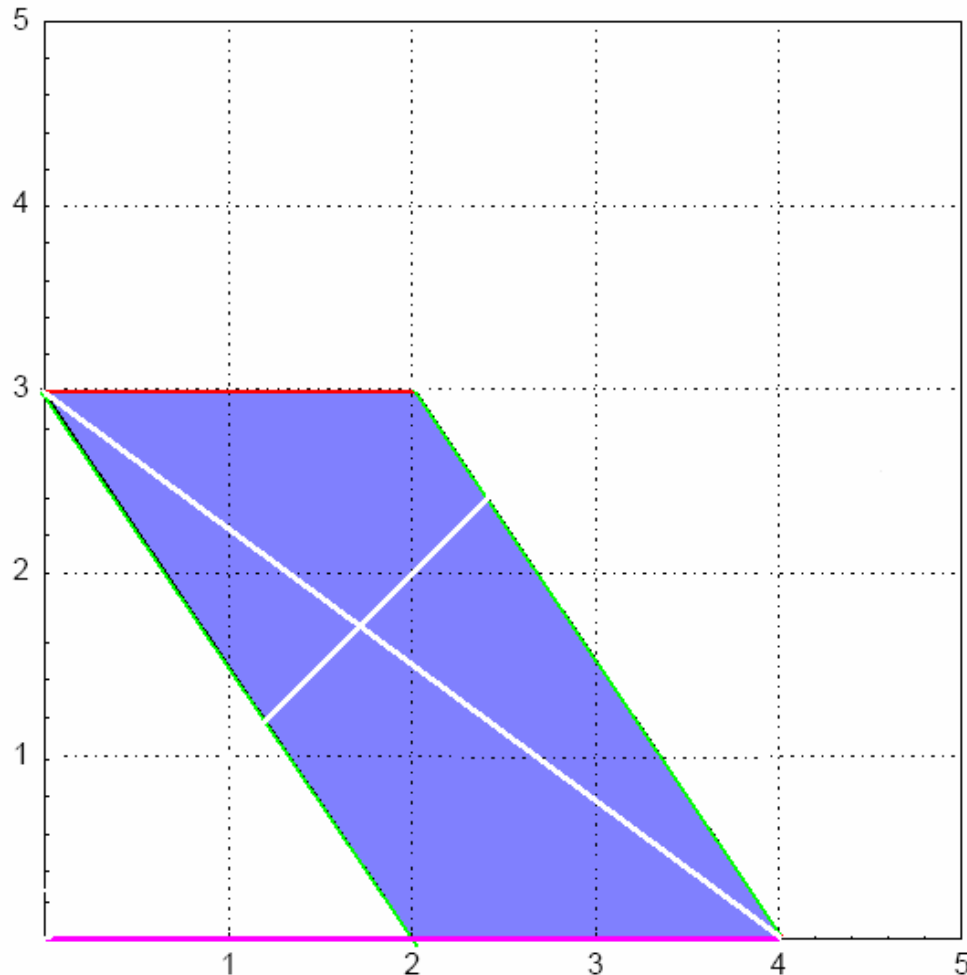
- If this transformation followed by conversion to DNF the size of problem increases exponentially
- Satisfiability test of a conjunction of  $m$  inequalities and  $k$  disequalities involves  $2^k$  satisfiability tests of conjunctions of  $m+k$  inequality constraints

# Disequalities (cont.)



$$\left. \begin{array}{l} y \leq 3 \\ y \geq 0 \\ 2y + 3x = 6 \\ 2y + 3x = 12 \end{array} \right\}$$

# Disequalities (cont.)



$$y \leq 3$$

$$y \geq 0$$

$$2y + 3x = 6$$

$$2y + 3x = 12$$

$$x \neq y$$

$$3x + 4y \neq 12$$

Disequalities are independent [8]. Finite number of disequalities cannot eliminate all solutions. For real only.

# Disequalities (cont.)

- Main key is *Independence of the System of Disequality Constraints* [8].
- There is no way for a finite number of disequalities to add up together make the system unsatisfiable
- Thus, satisfiability testing of a conjunction  $m$  inequalities and  $k$  disequalities on real variables can be treated  $2k$  satisfiability tests of  $m + 1$  inequalities
- The independence property of disequalities allows obtain DNF in polynomial time
- For the integer arithmetic the disequalities are not independent, so other approach may be used [9]

# Technologies used

- SUSE Linux 10.0
- GCC 4.0.2 – C++ compiler
- STL, BOOST (smart\_pointer, bind, date/time, multi\_index\_container) libraries
- CxxTest – unit tests framework
- KDevelop - IDE

# Example: TTP 3 nodes model

```
Solve (M)(
  forall (df12,df13,df21,df23,df31,df32, np1,np2,np3, nc1,nc2,nc3,
         p1,p2,p3, c1,c2,c3, d1,d2,d3)
  (
    (np1=p1+c1+d1)&
    (np2=p2+c2+d2)&
    (np3=p3+c3+d3)&
    (df12 = p1+(-1)*p2 + tau1*(c1+(-1)*c2+d1+(-1)*d2))& /* tau1 */
    (df13 = p1+(-1)*p3 + tau1*(c1+(-1)*c3+d1+(-1)*d3))& /* tau1 */
    (df21 = p2+(-1)*p1 + tau2*(c2+(-1)*c1+d2+(-1)*d1))& /* tau2 */
    (df23 = p2+(-1)*p3 + tau2*(c2+(-1)*c3+d2+(-1)*d3))& /* tau2 */
    (df31 = p3+(-1)*p1 + tau3*(c3+(-1)*c1+d3+(-1)*d1))& /* tau3 */
    (df32 = p3+(-1)*p2 + tau3*(c3+(-1)*c2+d3+(-1)*d2))& /* tau3 */
    (nc1= 0.5*(df21+df31))&
    (nc2= 0.5*(df12+df32))&
    (nc3= 0.5*(df23+df13))&
    (( -1 <= d1) & (d1 <=1))&
    (( -1 <= d2) & (d2 <=1))&
    (( -1 <= d3) & (d3 <=1))&
    ((-1)*M<=p1+(-1)*p2)&
    ((-1)*M<=p1+(-1)*p3)&
    ((-1)*M<=p3+(-1)*p2)&
    (p1 +(-1)* p2<=M)&
    (p1 +(-1)* p3<=M)&
    (p3 +(-1)* p2<=M)&
    (2+(-1)* M<=(p1+(-1)*p2+c1+(-1)*c2))&
    (2+(-1)* M<=(p1+(-1)*p3+c1+(-1)*c3))&
    (2+(-1)* M<=(p3+(-1)*p2+c3+(-1)*c2))&
    ((p1+(-1)*p2+c1+(-1)*c2)<=M +(-1)*2)&
    ((p1+(-1)*p3+c1+(-1)*c3)<=M +(-1)*2)&
    ((p3+(-1)*p2+c3+(-1)*c2)<=M +(-1)*2)
    ->
    (2+(-1)*M<=(np1+(-1)*np2+nc1+(-1)*nc2))&
    (2+(-1)*M<=(np1+(-1)*np3+nc1+(-1)*nc3))&
    (2+(-1)*M<=(np3+(-1)*np2+nc3+(-1)*nc2))&
    ((np1+(-1)*np2+nc1+(-1)*nc2)<=M+(-1)*2)&
    ((np1+(-1)*np3+nc1+(-1)*nc3)<=M+(-1)*2)&
    ((np3+(-1)*np2+nc3+(-1)*nc2)<=M+(-1)*2)
  ));
```

21 variables

12 implicit disequalities

Hard to solve by a  
generic Fourier-  
Motzkin solver

1..60 sec by the partly  
enhanced Solver. The  
time depends on the  
 $\tau_i$  values.

$\tau_i$ ; (0..1)

# Future Plans


- Implement ALL presented here
- Algorithms enhancement
- Compare speed of the variables projection to the PPL library
- Experiment with the Simplex based methods
- Experiment with the representation of polytope using vectors of points and rays
- Pick up a faster projection algorithm

# Conclusions

- Methods for implementing of an efficient solver over real (rational) numbers are presented
- Redundancy elimination and complexity handling techniques are outlined
- The Solver implemented using the presented techniques believed to be efficient enough to handle the real-life verification problems

# References

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9. R. Seater D. Wonnacott Efficient Manipulation of Disequalities During Dependence Analysis



Thank you.  
Questions?