Automated Reasoning

Rewriting-Based Deduction

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The Equality Relation

Equality $\doteq$: A very important relation

- Reflexive
- Symmetric
- Transitive
- Substitute equals by equals
- When equality is used in a theorem, we need extra axioms which describe the properties of equality

The Equality Relation: Example

Theorem: Let G be a group with the binary operation $\cdot$, the inverse $^{-1}$, and the identity e. If $x \cdot x = e$ for all $x \in G$, then G is commutative.

Axioms:

1. For all $x, y \in G$, $x \cdot y \in G$.
2. For all $x, y, z \in G$, $(x \cdot y) \cdot z \doteq x \cdot (y \cdot z)$.
3. For all $x \in G$, $x \cdot e \doteq x$.
4. For all $x \in G$, $x \cdot x^{-1} \doteq e$.

The Equality Relation: Example (Cont.)

Express the axioms and the theorem in first-order logic with equality:

(A1) $\forall x, y.\ \exists z.\ x \cdot y \doteq z$.

(A2) $\forall x, y, z.\ (x \cdot y) \cdot z \doteq x \cdot (y \cdot z)$.

(A3) $\forall x.\ x \cdot e \doteq x$.

(A4) $\forall x.\ x \cdot i(x) \doteq e$.

(T) $\forall x.\ x \cdot x \doteq e \Rightarrow \forall u, v.\ u \cdot v \doteq v \cdot u$.

The Equality Relation: Example (Cont.)

Take the conjunction of axioms and the negation of the theorem and bring it to the Skolem normal form. We obtain the set consisting of the clauses:

1. $x \cdot y \doteq f(x, y)$.
2. $(x \cdot y) \cdot z \doteq x \cdot (y \cdot z)$.
3. $x \cdot e \doteq x$.
4. $x \cdot i(x) \doteq e$.
5. $x \cdot x \doteq e$
6. $\neg(a \cdot b \doteq b \cdot a)$.

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6. $a \cdot b \not\doteq b \cdot a$.

By resolution alone, we can not derive the contradiction here.

The Equality Relation: Example (Cont.)

We need extra axioms to describe the properties of equality.

Let S be a set of clauses. The set of the equality axioms for S is the set consisting of the following clauses:

1. $x \doteq x$.
2. $x \not\doteq y \vee y \doteq x$.
3. $x \not\doteq y \vee y \not\doteq z \vee x \doteq z$.
4. $x \not\doteq y \vee \neg p(x_1, \ldots, x, \ldots, x_n) \vee p(x_1, \ldots, y, \ldots, x_n)$, where x and y appear in the same position i, for all $1 \leqslant i \leqslant n$, for every n-ary predicate symbol p appearing in S.
5. $x \not\doteq y \vee f(x_1, \ldots, x, \ldots, x_n) \doteq f(x_1, \ldots, y, \ldots, x_n)$, where x and y appear in the same position i, for all $1 \leqslant i \leqslant n$, for every n-ary function symbol f appearing in S.

The Equality Relation: Example (Cont.)

We add extra axioms:

$S:$	$x \cdot y \doteq f(x, y).$	$x \not\doteq y \vee y \not\doteq z \vee x \doteq z.$
	$(x \cdot y) \cdot z \doteq x \cdot (y \cdot z).$	$x \not\doteq y \vee x \not\doteq u \vee y \doteq u.$
	$x \cdot e \doteq x.$	$y \not\doteq x \vee u \not\doteq x \vee y \doteq u.$
	$x \cdot i(x) \doteq e.$	$x \not\doteq y \vee f(z, x) \doteq f(z, y).$
	$x \cdot x \doteq e.$	$x \not\doteq y \vee f(x, z) \doteq f(y, z).$
	$a \cdot b \not\doteq b \cdot a.$	$x \not\doteq y \vee x \cdot z \doteq y \cdot z.$
$K:$	$x \doteq x.$	$x \not\doteq y \vee z \cdot x \doteq z \cdot y.$
	$x \not\doteq y \vee y \doteq x.$	$x \not\doteq y \vee i(x) \doteq i(y).$

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$$
\begin{array}{lll}
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 & (x \cdot y) \cdot z \doteq x \cdot (y \cdot z). & x \not\doteq y \vee x \not\doteq u \vee y \doteq u. \\
 & x \cdot e \doteq x. & y \not\doteq x \vee u \not\doteq x \vee y \doteq u. \\
 & x \cdot i(x) \doteq e. & x \not\doteq y \vee f(z, x) \doteq f(z, y). \\
 & x \cdot x \doteq e. & x \not\doteq y \vee f(x, z) \doteq f(y, z). \\
 & a \cdot b \not\doteq b \cdot a. & x \not\doteq y \vee x \cdot z \doteq y \cdot z. \\
K: & x \doteq x. & x \not\doteq y \vee z \cdot x \doteq z \cdot y. \\
 & x \not\doteq y \vee y \doteq x. & x \not\doteq y \vee i(x) \doteq i(y).
\end{array}
$$

Unsatisfiability of this set can be proved by resolution.

The Equality Relation

The described approach has several drawbacks:

- Every time equality is used, one has to provide axioms that specify reflexive, symmetric, transitive, substitutive properties of equality.
- Clumsy approach.
- Generates large search space.
- Hopelessly inefficient.

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Requires a special approach.

Rewriting-Based Deduction for Unit Equalities

We assume that the given set of clauses consists of unit equalities and one ground inequality.

Goal: Design a calculus which works on such sets, is more efficient than the described approach, and is complete.

Later this calculus can be extended to general clauses.

Equational Theory

- E: A set of equations.
- Ax: The set of equality axioms for E.
- $E \models s \doteq t$ iff $S \models s \doteq t$ for all structures S which is a model of $E \cup Ax$.
- Equational theory of E:

$$\doteq_E := \{(s, t) \mid E \models s \doteq t\}$$

- Notation: $s \doteq_E t$ iff $(s, t) \in \; \doteq_E$.

Basic Concepts in Term Rewriting

- A rewrite rule is an ordered pair of terms, written $l \to r$.
- Term rewriting system (TRS): a set of rewrite rules.

Problem

Given: A set of equations $\mathtt{E}$ and two terms s and $\mathtt{t}$.

Decide: $s \doteq_{\mathtt{E}} \mathtt{t}$ holds or not.

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What's this?

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Solving Idea

Refute and skolemize the goal, obtaining the ground disequation $s' \not\doteq_{E} t'$.

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If yes, stop. You obtained a contradiction, which proves $s \doteq_E t$.

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If yes, stop. You obtained a contradiction, which proves $s \doteq_E t$.

If not, continue with completion. If this is not possible, then report: $s \doteq_E t$ does not hold.

What We Need To Know

- What is rewriting?
- What is a ground convergent set of equations and rewrite rules?
- What is completion?

Positions

The set of positions of a term t, $\mathrm{Pos}(t)$, is a set of strings of positive integers:

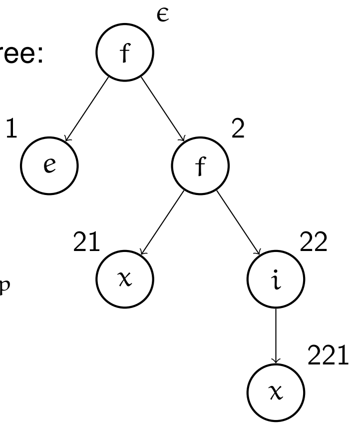
- If $t = x$, then $\mathrm{Pos}(t) := \{\epsilon\}$,
- If $t = f(t_1, \ldots, t_n)$, then

$$\mathrm{Pos}(t) := \{\epsilon\} \cup \{ip \mid 1 \leqslant i \leqslant n,\ p \in \mathrm{Pos}(t_i)\}.$$

More Notions about Terms

Term: $t = f(e, f(x, i(x)))$

Tree:



Subterm of t at position p: $t|_p$

$$\begin{aligned} t|_2 &= f(x, i(x)) \\ t|_{21} &= x \\ t|_{22} &= i(x) \end{aligned}$$

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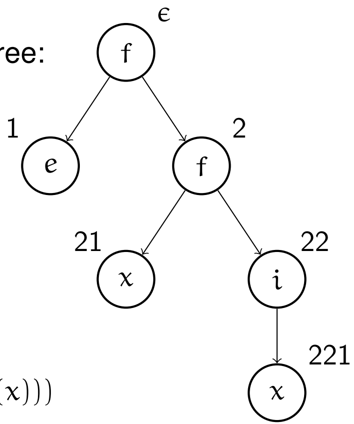

Replacing a subterm
at position p by s: $t[s]_p$

$$\begin{aligned} t[a]_\epsilon &= a \\ t[g(a,a)]_{21} &= f(e, f(g(a,a), i(x))) \\ t[i(y)]_{22} &= f(e, f(x, i(y))) \end{aligned}$$

More Notions about Terms

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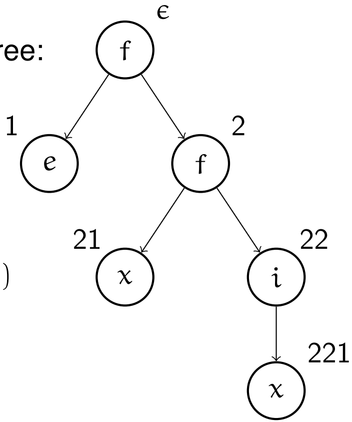


A size of t: $|t| = card(Pos(t))$

$$
\begin{aligned}
|t| &= 6 \\
|t[a]_2| &= 3 \\
|t|_{22}| &= 2
\end{aligned}
$$

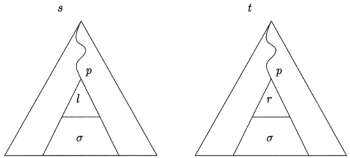
Basic Concepts in Term Rewriting

R: a term rewriting system.

- The rewrite relation induced by R, denoted $\to_R$, is a binary relation on terms defined as:

  $s \to_R t$ iff

  there exist $l \to r \in R$, a position p in s, a substitution σ such that $s|_p = \sigma(l)$ and $t = s[\sigma(r)]_p$.



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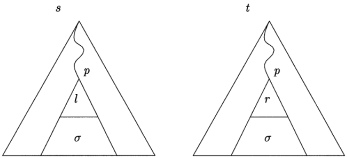

- $R \subseteq \to_R$. We may omit R when it is obvious.

Basic Concepts in Term Rewriting

- s reduces to t by R iff $s \to_R t$.
- s is reducible by R iff there is a t such that $s \to_R t$.
- s is irreducible (is in normal form) by R iff s is not reducible.
- $\leftarrow_R$ stands for the inverse and $\to_R^*$ for reflexive-transitive closure of $\to_R$.
- t is a normal form of s by R iff $s \to_R^* t$ and t is irreducible by R.
- R is terminating iff $\to_R$ is well-founded, i.e., there is no infinite sequence of rewrite steps $s_1 \to_R s_2 \to_R s_3 \to_R \cdots$.

Basic Concepts in Term Rewriting

R is confluent iff for all terms s, t_1, t_2, if

$$s \to_R^* t_1 \text{ and } s \to_R^* t_2,$$

then there exists a term r such that

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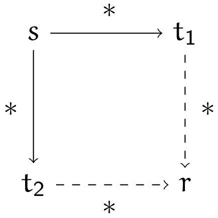
Graphically:



Basic Concepts in Term Rewriting

t_1 and t_2 are joinable by R if there exists a term r such that

$$t_1 \to_R^* r \text{ and } t_2 \to_R^* r.$$

Notation: $t_1 \downarrow_R t_2$.

Basic Concepts in Term Rewriting

Example

Let $+$ be a binary (infix) function symbol, s a unary function symbol, 0 a constant.

$$R := \{0 + x \to x, \quad s(x) + y \to s(x + y)\}.$$

Then:

- $s(0) + s(s(0)) \to_R s(0 + s(s(0))) \to_R s(s(s(0)))$.
- $s(0) + s(s(0)) \to_R^* s(s(s(0)))$.
- $s(s(s(0)))$ is irreducible by R and, hence, is a normal form of $s(0) + s(s(0))$, of $s(0 + s(s(0)))$, and of $s(s(s(0)))$.

Basic Concepts in Term Rewriting

A TRS $\mathcal{R}$ is convergent iff it is confluent and terminating.

A convergent TRS provides a decision procedure for the underlying equational theory: Two terms are equivalent iff they reduce to the same normal form.

Computation of normal forms by repeated reduction is a don't care non-deterministic process for convergent TRSs.

Basic Concepts in Term Rewriting

A strict order $>$ on terms is called a reduction order iff it is

1. monotonic: If $s > t$, then $r[s] > r[t]$ for all terms s, t, r;
2. stable: If $s > t$, then $\sigma(s) > \sigma(t)$ for all terms s, t and a substitution σ;
3. well-founded.

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Why are reduction orders interesting?

Theorem

A TRS R terminates iff there exists a reduction order $>$ that satisfies $l > r$ for all $l \to r \in R$.

Reduction Orders

- $|t|$: The size of the term t.
- The order $>_1$: $s >_1 t$ iff $|s| > |t|$.

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- $|t|$: The size of the term t.
- The order $>_1$: $s >_1 t$ iff $|s| > |t|$.
- $>_1$ is monotonic and well-founded.
- However, $>_1$ is not a reduction order because it is not stable:

$$|f(f(x,x),y)| = 5 > 3 = |f(y,y)|$$

For $\sigma = \{y \mapsto f(x,x)\}$:

$$|\sigma(f(f(x,x),y))| = |f(f(x,x),f(x,x))| = 7,$$
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Reduction Orders

- $|t|_x$: The number of occurrences of x in t.
- The order $>_2$: $s >_2 t$ iff $|s| > |t|$ and $|s|_x \geqslant |t|_x$ for all x.

Reduction Orders

- $|t|_x$: The number of occurrences of x in t.
- The order $>_2$: $s >_2 t$ iff $|s| > |t|$ and $|s|_x \geqslant |t|_x$ for all x.
- $>_2$ is a reduction order.

Methods for Construction Reduction Orders

- Polynomial orders
- Simplification orders:
  - Recursive path orders
  - Knuth-Bendix orders

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Goal: Provide a variety of different reduction orders that can be used to show termination; not only by hand, but also automatically.

Lexicographic Path Order

Main idea behind recursive path orders:

- Two terms are compared by first comparing their root symbols.
- Then recursively comparing the collections of their immediate subterms.

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- Collections seen as tuples yields the lexicographic path order.

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- Collections seen as multisets yields the multiset path order. (Not considered in this course.)
- Collections seen as tuples yields the lexicographic path order.
- Combination of multisets and tuples yields the recursive path order with status. (Not considered in this course.)

Lexicographic Path Order

Let $\mathcal{F}$ be a finite signature and $>$ be a strict order on $\mathcal{F}$ (called the precedence). The lexicographic path order $>_{lpo}$ on $\mathsf{T}(\mathcal{F}, \mathcal{V})$ induced by $>$ is defined as follows:

$s >_{lpo} t$ iff

(1) $t \in Var(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

- (2a) $s_i \geqslant_{lpo} t$ for some i, $1 \leqslant i \leqslant m$, or
- (2b) $f > g$ and $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, or
- (2c) $f = g$, $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, and there exists i, $1 \leqslant i \leqslant m$ such that $s_1 = t_1, \ldots s_{i-1} = t_{i-1}$ and $s_i >_{lpo} t_i$.

$\geqslant_{lpo}$ stands for the reflexive closure of $>_{lpo}$.

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Example

$\mathcal{F} = \{f, i, e\}$, f is binary, i is unary, e is constant, with $i > f > e$.

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- $f(x, e) >_{lpo} x$ by (1)

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- $f(x, e) >_{lpo} x$ by (1)
- $i(e) >_{lpo} e$ by (2a), because $e \geqslant_{lpo} e$.

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(2a) $s_i \geqslant_{lpo} t$ for some i, $1 \leqslant i \leqslant m$, or

(2b) $f > g$ and $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, or

(2c) $f = g$, $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, and there exists i, $1 \leqslant i \leqslant m$ such that $s_1 = t_1, \ldots s_{i-1} = t_{i-1}$ and $s_i >_{lpo} t_i$.

Example (Cont.)

$\mathcal{F} = \{f, i, e\}$, f is binary, i is unary, e is constant, with $i > f > e$.

- $i(f(x, y)) >^{?}_{lpo} f(i(x), i(y))$:

Lexicographic Path Order

$s >_{lpo} t$ iff

(1) $t \in Var(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

(2a) $s_i \geqslant_{lpo} t$ for some i, $1 \leqslant i \leqslant m$, or

(2b) $f > g$ and $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, or

(2c) $f = g$, $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, and there exists i, $1 \leqslant i \leqslant m$ such that $s_1 = t_1, \ldots s_{i-1} = t_{i-1}$ and $s_i >_{lpo} t_i$.

Example (Cont.)

$\mathcal{F} = \{f, i, e\}$, f is binary, i is unary, e is constant, with $i > f > e$.

- $i(f(x,y)) >^?_{lpo} f(i(x), i(y))$:
  - Since $i > f$, (2b) reduces it to the problems: $i(f(x,y)) >^?_{lpo} i(x)$ and $i(f(x,y)) >^?_{lpo} i(y)$.

Lexicographic Path Order

$s >_{lpo} t$ iff

(1) $t \in Var(s)$ and $t \neq s$, or

(2) $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$, and

(2a) $s_i \geqslant_{lpo} t$ for some i, $1 \leqslant i \leqslant m$, or

(2b) $f > g$ and $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, or

(2c) $f = g$, $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, and there exists i, $1 \leqslant i \leqslant m$ such that $s_1 = t_1, \dots s_{i-1} = t_{i-1}$ and $s_i >_{lpo} t_i$.

Example (Cont.)

$\mathcal{F} = \{f, i, e\}$, f is binary, i is unary, e is constant, with $i > f > e$.

- $i(f(x, y)) >^?_{lpo} i(x)$ is reduced by (2c) to $i(f(x, y)) >^?_{lpo} x$ and $f(x, y) >^?_{lpo} x$, which hold by (1).

Lexicographic Path Order

$s >_{lpo} t$ iff

(1) $t \in Var(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

(2a) $s_i \geqslant_{lpo} t$ for some i, $1 \leqslant i \leqslant m$, or

(2b) $f > g$ and $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, or

(2c) $f = g$, $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, and there exists i, $1 \leqslant i \leqslant m$ such that $s_1 = t_1, \ldots s_{i-1} = t_{i-1}$ and $s_i >_{lpo} t_i$.

Example (Cont.)

$\mathcal{F} = \{f, i, e\}$, f is binary, i is unary, e is constant, with $i > f > e$.

- $i(f(x, y)) >^{?}_{lpo} i(x)$ is reduced by (2c) to $i(f(x, y)) >^{?}_{lpo} x$ and $f(x, y) >^{?}_{lpo} x$, which hold by (1).
- $i(f(x, y)) >_{lpo} i(y)$ is shown similarly.

Lexicographic Path Order

$s >_{lpo} \mathrm{t}$ iff

(1) $\mathrm{t} \in Var(s)$ and $\mathrm{t} \neq s$, or

(2) $s = \mathrm{f}(s_1, \ldots, s_m)$, $\mathrm{t} = \mathrm{g}(\mathrm{t}_1, \ldots, \mathrm{t}_n)$, and

(2a) $s_i \geqslant_{lpo} \mathrm{t}$ for some i, $1 \leqslant i \leqslant m$, or

(2b) $\mathrm{f} > \mathrm{g}$ and $s >_{lpo} \mathrm{t}_j$ for all j, $1 \leqslant j \leqslant n$, or

(2c) $\mathrm{f} = \mathrm{g}$, $s >_{lpo} \mathrm{t}_j$ for all j, $1 \leqslant j \leqslant n$, and there exists i,
$1 \leqslant i \leqslant m$ such that $s_1 = \mathrm{t}_1, \ldots s_{i-1} = \mathrm{t}_{i-1}$ and $s_i >_{lpo} \mathrm{t}_i$.

Example (Cont.)

$\mathcal{F} = \{\mathrm{f}, \mathrm{i}, e\}$, f is binary, i is unary, e is constant, with $\mathrm{i} > \mathrm{f} > e$.

Lexicographic Path Order

$s >_{lpo} t$ iff

(1) $t \in Var(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

(2a) $s_i \geqslant_{lpo} t$ for some i, $1 \leqslant i \leqslant m$, or

(2b) $f > g$ and $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, or

(2c) $f = g$, $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, and there exists i, $1 \leqslant i \leqslant m$ such that $s_1 = t_1, \ldots s_{i-1} = t_{i-1}$ and $s_i >_{lpo} t_i$.

Example (Cont.)

$\mathcal{F} = \{f, i, e\}$, f is binary, i is unary, e is constant, with $i > f > e$.

- $f(f(x, y), z) >^?_{lpo} f(x, f(y, z)))$. By (2c) with $i = 1$:

Lexicographic Path Order

$s >_{lpo} t$ iff

(1) $t \in Var(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

(2a) $s_i \geqslant_{lpo} t$ for some i, $1 \leqslant i \leqslant m$, or

(2b) $f > g$ and $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, or

(2c) $f = g$, $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, and there exists i, $1 \leqslant i \leqslant m$ such that $s_1 = t_1, \ldots s_{i-1} = t_{i-1}$ and $s_i >_{lpo} t_i$.

Example (Cont.)

$\mathcal{F} = \{f, i, e\}$, f is binary, i is unary, e is constant, with $i > f > e$.

- $f(f(x, y), z) >_{lpo} x$ because of (1).

Lexicographic Path Order

$s >_{lpo} t$ iff

(1) $t \in Var(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

(2a) $s_i \geqslant_{lpo} t$ for some i, $1 \leqslant i \leqslant m$, or

(2b) $f > g$ and $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, or

(2c) $f = g$, $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, and there exists i, $1 \leqslant i \leqslant m$ such that $s_1 = t_1, \ldots s_{i-1} = t_{i-1}$ and $s_i >_{lpo} t_i$.

Example (Cont.)

$\mathcal{F} = \{f, i, e\}$, f is binary, i is unary, e is constant, with $i > f > e$.

- $f(f(x,y),z) >^?_{lpo} f(y,z)$: By (2c) with $i = 1$:
  - $f(f(x,y),z) >_{lpo} y$ and $f(f(x,y),z) >_{lpo} z$ by (1).
  - $f(x,y) >_{lpo} y$ by (1).

Lexicographic Path Order

$s >_{lpo} t$ iff

(1) $t \in Var(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

(2a) $s_i \geqslant_{lpo} t$ for some i, $1 \leqslant i \leqslant m$, or

(2b) $f > g$ and $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, or

(2c) $f = g$, $s >_{lpo} t_j$ for all j, $1 \leqslant j \leqslant n$, and there exists i, $1 \leqslant i \leqslant m$ such that $s_1 = t_1, \ldots s_{i-1} = t_{i-1}$ and $s_i >_{lpo} t_i$.

Example (Cont.)

$\mathcal{F} = \{f, i, e\}$, f is binary, i is unary, e is constant, with $i > f > e$.

- $f(x, y) >_{lpo} x$ by (1).

Reduction Orders

Reduction orders are not total for terms with variables.

For instance, $f(x)$ and $f(y)$ can not be ordered.

$f(x, y)$ and $f(y, x)$ can not be ordered either.

However, many reduction orders are total on ground terms.

Fortunately, in theorem proving applications one can often reason about non-ground formulas by considering the corresponding ground instances.

In such situations, ordered rewriting techniques can be applied.

Ordered Rewriting

Given: A reduction order $>$ and a set of equations E.

The rewrite system $E^>$ is defined as

$$E^> := \{\sigma(s) \to \sigma(r) \mid (s \doteq t \in E \text{ or } t \doteq s \in E) \text{ and } \sigma(s) > \sigma(t)\}$$

The rewrite relation $\to_{E^>}$ induced by $E^>$ represents ordered rewriting with respect to E and $>$.

Ordered Rewriting

Example

- If $>$ is a lexicographic path ordering with precedence $+ > a > b > c$, then $b + c > c + b > c$.
- Let $E := \{x + y \doteq y + x\}$.
- We may use the commutativity equation for ordered rewriting.
- $(b + c) + c \rightarrow_{E^>} (c + b) + c \rightarrow_{E^>} c + (c + b)$.

Ordered Rewriting

If $>$ is a reduction ordering total on ground terms, then $E^>$ contains all (non-trivial) ground instances of an equation $s \doteq t \in E$, either as a rule $\sigma(s) \to \sigma(t)$ or a rule $\sigma(t) \to \sigma(s)$.

A rewrite system R is called ground convergent if the induced ground rewrite relation (that is, the rewrite relation $\to_R$ restricted to pairs of ground terms) is terminating and confluent.

A set of equations E is called ground convergent with respect to $>$ if $E^>$ is ground convergent.

Critical Pairs

Ordered rewriting leads to the inference rule, called superposition:

$$\frac{s \doteq t \qquad r[u] \doteq v}{\sigma(r[t] \doteq v)},$$

where $\sigma = mgu(s, u)$, $\sigma(t) \not\succcurlyeq \sigma(s)$, $\sigma(v) \not\succcurlyeq \sigma(r)$, and u is not a variable.

The equation $\sigma(r[t] \doteq v)$ is called an ordered critical pair (with overlapped term $\sigma(r[u])$) between $s \doteq t$ and $r[u] \doteq v$.

Critical Pairs

Lemma

Let $>$ be a ground total reduction ordering.

A set E of equations is ground convergent with respect to $>$

iff

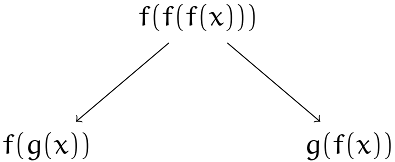
for all ordered critical pairs $\sigma(r[t] \doteq v)$ (with overlapped term $\sigma(r[u])$) between equations in E and for all ground substitutions φ,

if $\varphi(\sigma(r[u])) > \varphi(\sigma(r[t]))$ and $\varphi(\sigma(r[u])) > \varphi(\sigma(v))$, then $\varphi(\sigma(r[t])) \downarrow_{E^>} \varphi(\sigma(v))$.

Critical Pairs

Example

- Let $E := \{f(f(x)) \doteq g(x)\}$ and $>$ be the LPO with $f > g$.
- Take a critical pair between the equation and its renamed copy, $f(f(x)) \doteq g(x)$ and $f(f(y)) \doteq g(y)$.



Critical Pairs

Example

- Let $E := \{f(f(x)) \doteq g(x)\}$ and $>$ be the LPO with $f > g$.
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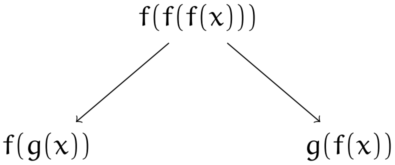
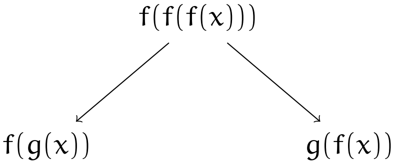


- $f(f(f(x))) > f(g(x))$ and $f(f(f(x))) > g(f(x))$, but $f(g(x)) \not\downarrow_{E_>} g(f(x))$.

Critical Pairs

Example

- Let $E := \{f(f(x)) \doteq g(x)\}$ and $>$ be the LPO with $f > g$.
- Take a critical pair between the equation and its renamed copy, $f(f(x)) \doteq g(x)$ and $f(f(y)) \doteq g(y)$.



- $f(f(f(x))) > f(g(x))$ and $f(f(f(x))) > g(f(x))$, but $f(g(x)) \not\downarrow_{E_>} g(f(x))$.
- E is not ground convergent with respect to $>$.

Adding Critical Pairs to Equations

Since critical pairs are equational consequences, adding a critical pair to the set of equations does not change the induced equational theory.

If $\mathtt{E}'$ is obtained from $\mathtt{E}$ by adding a critical pair, then $\doteq_{\mathtt{E}} = \doteq_{\mathtt{E}'}$.

The idea of adding a critical pair as a new equation is called “completion”.

Convergence

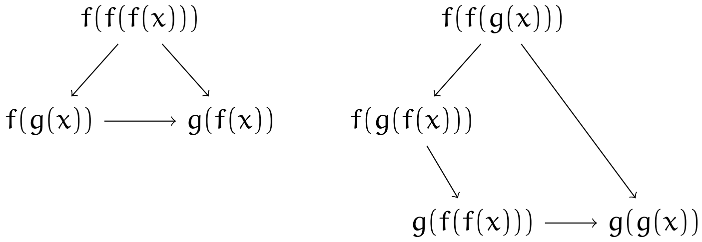
Example

- Let $E' := \{f(f(x)) \doteq g(x), f(g(x)) \doteq g(f(x))\}$
- Let $>$ be the LPO with $f > g$.

Convergence

Example

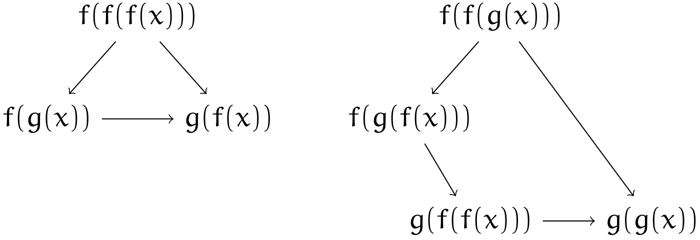
- Let $E' := \{f(f(x)) \doteq g(x), f(g(x)) \doteq g(f(x))\}$
- Let $>$ be the LPO with $f > g$.
- E' has two critical pairs. Both are joinable:



Convergence

Example

- Let $E' := \{f(f(x)) \doteq g(x), f(g(x)) \doteq g(f(x))\}$
- Let $>$ be the LPO with $f > g$.
- E' has two critical pairs. Both are joinable:



- E' is (ground) convergent.

Ordered Completion

Described as a set of inference rules.

Parametrized by a reduction ordering $>$.

Works on pairs (E, R), where E is a set of equations and R is a set of rewrite rules.

$E; R \vdash E'; R'$ means that $E'; R'$ can be obtained from $E; R$ by applying a completion inference.

Ordered Completion: Notions

Derivation: A (finite or countably infinite) sequence $(E_0; R_0) \vdash (E_1; R_1) \cdots$.

Usually, E_0 is the set of initial equations and $R_0 = \emptyset$.

The limit of a derivation: the pair $E_\omega; R_\omega$, where

$$E_\omega := \bigcup_{i \geqslant 0} \bigcap_{j \geqslant i} E_j \text{ and } R_\omega := \bigcup_{i \geqslant 0} \bigcap_{j \geqslant i} R_j.$$

Goal: to obtain a limit system that is ground convergent.

Ordered Completion: Notation

$\uplus$: Disjoint union

$s \rhd \mathtt{t}$: Strict encompassment relation. An instance of $\mathtt{t}$ is a subterm of s, but not vice versa.

$s \approxeq \mathtt{t}$ stands for $s \doteq \mathtt{t}$ or $\mathtt{t} \doteq s$.

$CP_{>}(\mathtt{E} \cup \mathtt{R})$: The set of all ordered critical pairs, with the ordering $>$, generated by equations in $\mathtt{E}$ and rewrite rules in $\mathtt{R}$ treated as equations.

Ordered Completion: Rules

DEDUCTION: $E; R \vdash E \cup \{s \doteq t\}; R$
if $s \doteq t \in CP_{>}(E \cup R)$.

ORIENTATION: $E \uplus \{s \approx t\}; R \vdash E; R \cup \{s \rightarrow t\}$, if $s > t$.

DELETION: $E \uplus \{s \doteq s\}; R \vdash E; R$.

Ordered Completion: Rules

COMPOSITION: $E; R \uplus \{s \to t\} \vdash E; R \cup \{s \to r\}$,

if $t \to_{R \cup E^{>}} r$.

SIMPLIFICATION: $E \cup \{s \approx t\}; R \vdash E \cup \{u \doteq t\}; R$,

if $s \to_R u$ or $s \to_{E^{>}} u$ with $\sigma(l) \to \sigma(r)$ for $l \approx r \in E$, $s \rhd l$.

COLLAPSE: $E; R \uplus \{s \to t\} \vdash E \cup \{u \doteq t\}; R$,

if $s \to_R u$ or $s \to_{E^{>}} u$ with $\sigma(l) \to \sigma(r)$ for $l \approx r \in E$, $s \rhd l$.

Ordered Completion: Properties

Theorem

Let $(E_0; R_0), (E_1; R_1), \ldots$ be an ordered completion derivation where all critical pairs are eventually generated (a fair derivation). Then these three properties are equivalent for all ground terms s and t:

(1) $E_0 \models s \doteq t$.

(2) $s \downarrow_{E_\omega^{>} \cup R_\omega} t$.

(3) $s \downarrow_{E_i^{>} \cup R_i} t$ for some $i \geqslant 0$.

This theorem, in particular, asserts the refutational completeness of ordered completion.

Proving by Ordered Completion: Example

Given:

1. $(x \cdot y) \cdot z \doteq x \cdot (y \cdot z)$.
2. $x \cdot e \doteq x$.
3. $x \cdot i(x) \doteq e$.
4. $x \cdot x \doteq e$.

Prove

Goal: $x \cdot y \doteq y \cdot x$.

Proving by Ordered Completion: Example

Proof by ordered completion:

- Skolemize the goal: $a \cdot b \doteq b \cdot a$.
- Take LPO as the reduction ordering with the precedence $i > f > e > a > b$
- $E_0 := \{(x \cdot y) \cdot z \doteq x \cdot (y \cdot z),\ x \cdot e \doteq x,\ x \cdot i(x) \doteq e,\ x \cdot x \doteq e\}$
- $R_0 := \emptyset$
- Start applying the rules.

Proving by Ordered Completion: Example

$$E_0 = \{(x \cdot y) \cdot z \doteq x \cdot (y \cdot z),\ x \cdot e \doteq x,\ x \cdot i(x) \doteq e,\ x \cdot x \doteq e\}$$

$$R_0 = \emptyset$$

Apply ORIENT 4 times:

$$E_4 = \emptyset$$

$$R_4 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e\}$$

Proving by Ordered Completion: Example

$$E_0 = \{(x \cdot y) \cdot z \doteq x \cdot (y \cdot z),\ x \cdot e \doteq x,\ x \cdot i(x) \doteq e,\ x \cdot x \doteq e\}$$
$$R_0 = \emptyset$$

Apply ORIENT 4 times:

$$E_4 = \emptyset$$
$$R_4 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e\}$$

Apply DEDUCE with the rules $(x \cdot y) \cdot z \to x \cdot (y \cdot z)$ and $x \cdot e \to x$ to the overlapping term $(x \cdot e) \cdot z$, and then ORIENT:

$$E_6 = \emptyset$$
$$R_6 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_6 = \emptyset$$
$$R_6 = \{(x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z),\ x \cdot e \rightarrow x,\ x \cdot i(x) \rightarrow e,\ x \cdot x \rightarrow e,$$
$$x_1 \cdot (e \cdot x_2) \rightarrow x_1 \cdot x_2\}$$

Apply DEDUCE with the rules $x_1 \cdot (e \cdot x_2) \rightarrow x_1 \cdot x_2$ and $x \cdot i(x) \rightarrow e$ to the overlapping term $x_1 \cdot (e \cdot i(e))$:

$$E_7 = \{x_1 \cdot i(e) \doteq x_1 \cdot e\}$$
$$R_7 = \{(x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z),\ x \cdot e \rightarrow x,\ x \cdot i(x) \rightarrow e,\ x \cdot x \rightarrow e,$$
$$x_1 \cdot (e \cdot x_2) \rightarrow x_1 \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_7 = \{x_1 \cdot i(e) \doteq x_1 \cdot e\}$$

$$R_7 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2\}$$

Apply ORIENT to $x_1 \cdot i(e) \doteq x_1 \cdot e$ and then COMPOSITION with the rule $x \cdot e \to x$:

$$E_9 = \emptyset$$

$$R_9 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ x \cdot i(e) \to x\}$$

Proving by Ordered Completion: Example

$$E_9 = \emptyset$$

$$R_9 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ x \cdot i(e) \to x\}$$

Apply DEDUCE with the rules $x \cdot x \to e$ and $x \cdot i(e) \to x$ to the overlapping term $i(e) \cdot i(e)$, and then ORIENT:

$$E_{11} = \emptyset$$

$$R_{11} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ x \cdot i(e) \to x,\ i(e) \to e\}$$

Proving by Ordered Completion: Example

$$E_{11} = \emptyset$$

$$R_{11} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ x \cdot i(e) \to x,\ i(e) \to e\}$$

Apply Collapse to $x \cdot i(e) \to x$ with $i(e) \to e$:

$$E_{12} = \{x \cdot e \doteq x\}$$

$$R_{12} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e\}$$

Proving by Ordered Completion: Example

$$E_{12} = \{x \cdot e \doteq x\}$$

$$R_{12} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e\}$$

Apply SIMPLIFICATION to $x \cdot e \doteq x$ with $x \cdot e \to x$ and then DELETE to the obtained $x \doteq x$:

$$E_{14} = \emptyset$$

$$R_{14} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e\}$$

Proving by Ordered Completion: Example

$$E_{14} = \emptyset$$

$$R_{14} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e\}$$

Apply DEDUCE to $(x \cdot y) \cdot z \to x \cdot (y \cdot z)$ and $x \cdot i(x) \to e$ with the overlapping term $(x \cdot i(x)) \cdot z$ and then ORIENT:

$$E_{16} = \emptyset$$

$$R_{16} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_{16} = \emptyset$$

$$R_{16} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2\}$$

Apply DEDUCE to $x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2$ and $x \cdot x \to e$ with the overlapping term $x_1 \cdot (i(x_1) \cdot i(x_1))$:

$$E_{17} = \{e \cdot i(x) \doteq x \cdot e\}$$

$$R_{17} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_{17} = \{e \cdot i(x) \doteq x \cdot e\}$$

$$R_{17} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2\}$$

Apply SIMPLIFICATION to $e \cdot i(x) \doteq x \cdot e$ with $x \cdot e \to x$ and then ORIENT:

$$E_{19} = \emptyset$$

$$R_{19} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,\ e \cdot i(x) \to x\}$$

Proving by Ordered Completion: Example

$$
\begin{aligned}
E_{19} &= \emptyset \\
R_{19} &= \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e, \\
&\quad x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\
&\quad e \cdot i(x) \to x\}
\end{aligned}
$$

Apply DEDUCE to $x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2$ and $e \cdot i(x) \to x$ with the overlapping term $x_1 \cdot (e \cdot i(x_2))$ and then ORIENT:

$$
\begin{aligned}
E_{21} &= \emptyset \\
R_{21} &= \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e, \\
&\quad x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\
&\quad e \cdot i(x) \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}
\end{aligned}
$$

Proving by Ordered Completion: Example

$$E_{21} = \emptyset$$

$$R_{21} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,$$
$$e \cdot i(x) \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}$$

Applying Collapse, Simplification, and Delete, we get rid of $x \cdot i(x) \to e$:

$$E_{24} = \emptyset$$

$$R_{24} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,$$
$$e \cdot i(x) \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_{24} = \emptyset$$

$$R_{24} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,$$
$$e \cdot i(x) \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}$$

Applying COLLAPSE and ORIENT, we replace $e \cdot i(x) \to x$ with $e \cdot x \to x$:

$$E_{26} = \emptyset$$

$$R_{26} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,$$
$$e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_{26} = \emptyset$$

$$R_{26} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,$$
$$e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}$$

Applying COLLAPSE and DELETE, we get rid of $x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2$:

$$E_{28} = \emptyset$$

$$R_{28} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,$$
$$e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_{28} = \emptyset$$

$$\begin{aligned} R_{28} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ &i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\ &e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\} \end{aligned}$$

Apply DEDUCE to $e \cdot x \to x$ and $x_1 \cdot i(x_2) \to x_1 \cdot x_2$ with the overlapping term $e \cdot i(x_2)$:

$$E_{29} = \{i(x_1) \doteq e \cdot x_2\}$$

$$\begin{aligned} R_{29} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ &i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\ &e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\} \end{aligned}$$

Proving by Ordered Completion: Example

$$E_{29} = \{i(x_2) \doteq e \cdot x_2\}$$

$$R_{29} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,$$
$$e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}$$

Apply SIMPLIFICATION to $i(x_1) \doteq e \cdot x_2$ with $e \cdot x \to x$ and then ORIENT:

$$E_{31} = \emptyset$$

$$R_{31} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,$$
$$e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x\}$$

Proving by Ordered Completion: Example

$$E_{31} = \emptyset$$

$$\begin{aligned} R_{31} = \{ & (x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ & i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\ & e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x \} \end{aligned}$$

Apply COLLAPSE and DELETE, we get rid of $i(e) \to e$:

$$E_{33} = \emptyset$$

$$\begin{aligned} R_{33} = \{ & (x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ & x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,\ e \cdot x \to x, \\ & x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x \} \end{aligned}$$

Proving by Ordered Completion: Example

$$
\begin{aligned}
E_{33} &= \emptyset \\
R_{33} &= \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\
&\quad x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,\ e \cdot x \to x, \\
&\quad x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x\}
\end{aligned}
$$

Applying COMPOSITION, we replace $x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2$ by $x_1 \cdot (i(x_1) \cdot x_2) \to x_2$:

$$
\begin{aligned}
E_{34} &= \emptyset \\
R_{34} &= \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\
&\quad x_1 \cdot (i(x_1) \cdot x_2) \to x_2,\ e \cdot x \to x, \\
&\quad x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x\}
\end{aligned}
$$

Proving by Ordered Completion: Example

$$E_{34} = \emptyset$$

$$\begin{aligned} R_{34} = \{ & (x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ & x_1 \cdot (i(x_1) \cdot x_2) \to x_2,\ e \cdot x \to x, \\ & x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x \} \end{aligned}$$

Applying SIMPLIFICATION and ORIENT, we replace $x_1 \cdot (i(x_1) \cdot x_2) \to x_2$ by $x_1 \cdot (x_1 \cdot x_2) \to x_2$:

$$E_{36} = \emptyset$$

$$\begin{aligned} R_{36} = \{ & (x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ & x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x, \\ & x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x \} \end{aligned}$$

Proving by Ordered Completion: Example

$$E_{36} = \emptyset$$

$$\begin{aligned} R_{36} = \{ & (x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ & x_1 \cdot (i(x_1) \cdot x_2) \to x_2,\ e \cdot x \to x, \\ & x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x \} \end{aligned}$$

Apply DEDUCE to $(x \cdot y) \cdot z \to x \cdot (y \cdot z)$ and $x \cdot x \to e$ with the overlapping term $(x_1 \cdot x_2) \cdot (x_1 \cdot x_2)$, then ORIENT:

$$E_{37} = \emptyset$$

$$\begin{aligned} R_{37} = \{ & (x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ & x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2, \\ & i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e \} \end{aligned}$$

Proving by Ordered Completion: Example

$$E_{37} = \emptyset$$

$$\begin{aligned} R_{37} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,\\ &x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,\\ &i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e\} \end{aligned}$$

Apply DEDUCE to $x_1 \cdot (x_1 \cdot x_2) \to x_2$ and $x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e$ with the overlapping term $x_1 \cdot (x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)))$, then ORIENT:

$$E_{39} = \emptyset$$

$$\begin{aligned} R_{39} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,\\ &x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,\\ &i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e,\ x_2 \cdot (x_1 \cdot x_2) \to x_1 \cdot e\} \end{aligned}$$

Proving by Ordered Completion: Example

$$E_{39} = \emptyset$$

$$\begin{aligned} R_{39} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ &x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2, \\ &i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e,\ x_2 \cdot (x_1 \cdot x_2) \to x_1 \cdot e\} \end{aligned}$$

Apply COMPOSITION to $x_2 \cdot (x_1 \cdot x_2) \to x_1 \cdot e$ with $x \cdot e \to x$:

$$E_{40} = \emptyset$$

$$\begin{aligned} R_{40} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ &x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2, \\ &i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e,\ x_2 \cdot (x_1 \cdot x_2) \to x_1\} \end{aligned}$$

Proving by Ordered Completion: Example

$$E_{41} = \emptyset$$

$$\begin{aligned} R_{41} = \{ & (x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ & x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2, \\ & i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e,\ x_2 \cdot (x_1 \cdot x_2) \to x_1 \} \end{aligned}$$

Apply DEDUCE to $x_1 \cdot (x_1 \cdot x_2) \to x_2$ and $x_2 \cdot (x_1 \cdot x_2) \to x_1$ with the overlapping term $x_2 \cdot (x_2 \cdot (x_1 \cdot x_2))$:

$$E_{42} = \{ x_1 \cdot x_2 \doteq x_2 \cdot x_1 \}$$

$$\begin{aligned} R_{42} = \{ & (x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ & x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2, \\ & i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e,\ x_2 \cdot (x_1 \cdot x_2) \to x_1 \cdot e \} \end{aligned}$$

Proving by Ordered Completion: Example

$$E_{42} = \{x_1 \cdot x_2 \doteq x_2 \cdot x_1\}$$

$$\begin{aligned} R_{42} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ &x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2, \\ &i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e,\ x_2 \cdot (x_1 \cdot x_2) \to x_1 \cdot e\} \end{aligned}$$

The equation $x_1 \cdot x_2 \doteq x_2 \cdot x_1$ joins the goal $a \cdot b \doteq b \cdot a$. Hence, the goal is proved.

Superposition Calculus with Ordering and Selection

Back to general clauses.

$\doteq$ the only predicate.

A well-behaves selection function wrt $\succ$:

- If only positive literals are selected in C, then all maximal (wrt $\succ$) literals in C are selected.

Superposition Calculus with Ordering and Selection

Back to general clauses.

$\doteq$ the only predicate.

A well-behaves selection function wrt $\succ$:

- If only positive literals are selected in C, then all maximal (wrt $\succ$) literals in C are selected.

Comparison between literals. Assume $l \succeq r$ and $s \succeq t$. Then

- If $l \succ s$, then $l \not\doteq r \succ l \doteq r \succ s \not\doteq t \succ s \doteq t$.
- If $l = s$, then $l \not\doteq r \succ s \doteq t$ and $s \not\doteq t \succ l \doteq r$,

Superposition Calculus with Ordering and Selection

Superposition:

$$\frac{\underline{l \doteq r} \vee C \qquad \underline{s[l'] \doteq t} \vee D}{\sigma(s[r] \doteq t \vee C \vee D)},$$

$$\frac{\underline{l \doteq r} \vee C \qquad \underline{s[l'] \not\doteq t} \vee D}{\sigma(s[r] \not\doteq t \vee C \vee D)}$$

Superposition Calculus with Ordering and Selection

Superposition:

$$\frac{\underline{l \doteq r} \lor C \qquad \underline{s[l'] \doteq t} \lor D}{\sigma(s[r] \doteq t \lor C \lor D)},$$

$$\frac{\underline{l \doteq r} \lor C \qquad \underline{s[l'] \not\doteq t} \lor D}{\sigma(s[r] \not\doteq t \lor C \lor D)}$$

where

- $\sigma = mgu(l, l')$,
- $l' \notin \mathcal{V}$,
- $\sigma(r) \not\succeq \sigma(l)$,
- $\sigma(t) \not\succeq \sigma(s[l'])$.

Superposition Calculus with Ordering and Selection

Equality resolution:

$$\frac{\underline{s \doteq t} \lor C}{\sigma(C)}, \qquad \text{where } \sigma = mgu(s,t).$$

Equality factoring:

$$\frac{\underline{l \doteq r} \lor l' \doteq r' \lor C}{\sigma(l \doteq r \lor r \not\doteq r' \lor C)},$$

Superposition Calculus with Ordering and Selection

Equality resolution:

$$\frac{\underline{s \doteq t} \lor C}{\sigma(C)}, \qquad \text{where } \sigma = mgu(s, t).$$

Equality factoring:

$$\frac{\underline{l \doteq r} \lor l' \doteq r' \lor C}{\sigma(l \doteq r \lor r \not\doteq r' \lor C)},$$

where

- $\sigma = mgu(l, l')$, $\sigma(r) \not\succeq \sigma(l)$, $\sigma(r') \not\succeq \sigma(l')$, $\sigma(r') \not\succeq \sigma(r)$,

The superposition calculus with ordering and selection is refutationally complete.